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THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

CAMBRIDGE, MASSACHUSETTS

From August 30th to September 6, nineteen hundred and fifty, an International Congress of Mathematicians was held at Harvard University under the auspices of the American Mathematical Society. In addition to the principal host, Harvard University, the following acted as co-hosts: The American Academy of Arts and Sciences, Boston College, Boston University, the Massachusetts Institute of Technology, and Tufts College.

This was the first International Congress of Mathematicians held in the United States since that assembled in connection with the Chicago World's Fair in 1893. At that time an international gathering of mathematicians met at Northwestern University under the presidency of Professor William E. Story of Clark University. Professor Felix Klein of the University of Göttingen attended this Congress as the official representative of the German government and brought with him a number of papers by prominent foreign mathematicians. These papers formed an important part of the program of the meetings which were attended by twenty-five mathematicians. The 1950 Congress was the first International Congress of Mathematicians held on the North American Continent since that in Toronto in 1924.

At the International Congress in Oslo in 1936, the American delegation invited the mathematicians of the world to hold their next general gathering in the United States in 1940. Plans had been practically completed for such a gathering at Harvard University when the outbreak of World War II necessitated the cancellation of the Congress. At the close of the War in 1945, the Council of the Society at once began discussion concerning the possibility of a gathering of the mathematicians of the world. Those guiding the policies of the American Mathematical Society were insistent that there should be no international congress until such a time that the gathering could be truly international in the sense that mathematicians could be invited irrespective of national or geographic origins. On December 10, 1947, the Council of the Society voted unanimously to accept the recommendation of the Emergency Committee for the International Congress of Mathematicians that steps be taken at once to elect officers and committees for the next congress.

Early in 1948, the following officers of the Congress were elected: President Designate: Oswald Veblen, Institute for Advanced Study;

Secretary: J. R. Kline, University of Pennsylvania; Associate Secretary: R. P. Boas, Jr., Mathematical Reviews.

Garrett Birkhoff was elected chairman of the Organizing Committee with W. T. Martin as vice chairman. The other members of this committee, as finally constituted, were: A. A. Albert, J. L. Doob, G. C. Evans, T. H. Hildebrandt, Einar Hille, J. R. Kline, Solomon Lefschetz, Saunders MacLane, Marston Morse, John von Neumann, Oswald Veblen, J. L. Walsh, Hassler Whitney, D. V. Widder, and R. L. Wilder.[1]

Invitations to send delegates to the Congress were sent to national academies and royal societies, universities and colleges, and to the mathematical societies throughout the world. In the case of all national academies and royal societies of countries with which the United States Government maintained diplomatic relations, these invitations were transmitted through the diplomatic mail pouch of the State Department. The various mathematical societies cooperated by distributing to their members invitations to and literature about the Congress which was furnished to them by the Secretariat. In attempting to maintain the non-political nature of the Congress, many serious difficulties had to be overcome. In the solution of these problems, officers of the Congress found the various officials of the Department of State most sympathetic and helpful. As a part of the effort to keep the Congress apolitical, they tried to secure a visa for every mathematician who notified them about any visa difficulties before cancelling his passage. As far as they know only one mathematician from any independent nation was prevented from attending the Congress because he failed to pass a political test and this man did not notify the officers of the Congress about his difficulties. Only two mathematicians from occupied countries failed to secure visas. Mathematicians from behind the Iron Curtain were uniformly prevented from attending the Congress by their own governments who generally refused to issue passports to them for the trip to the Congress. Their non-attendance was not due to any action of the United States Government.

Just before the opening of the Congress there was received from the President of the Soviet Academy of Sciences the following cablegram:

> "USSR Academy of Sciences appreciates receiving kind invitation for Soviet scientist take part in International Congress of Mathematicians to be held in Cambridge. Soviet Mathematicians being very much occupied with their regular work

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[1] R. G. D. Richardson and J. L. Synge were also originally members of this Committee. Dean Richardson died July 17, 1949, while Professor Synge resigned from the Committee when he left the United States to assume his position at the Institute for Advanced Study in Dublin in August 1948.

unable attend congress. Hope that impending congress will be significant event in mathematical science. Wish success in congress activities.

S. Vavilov, President, USSR Academy of Sciences."

This cablegram was read at the opening plenary session of the Congress which was held on Wednesday afternoon in the Sanders Theatre of Harvard University. This session was opened by Professor Garrett Birkhoff as Chairman of the Organizing Committee of the Congress. After the short address of welcome by Professor Birkhoff, Professor T. A. Skolem of the University of Oslo presented the name of Professor Oswald Veblen for the Presidency of the Congress. Professor Veblen was unanimously elected and then delivered his presidential address. This address will appear in full in the Proceedings of the Congress; these the Editorial Committee hopes will be published within a year. Immediately after this address, Professor Harald Bohr of the University of Copenhagen spoke on behalf of the Committee to award the Field's Medals.[2] The medals were awarded to Professors Laurent Schwartz of the University of Nancy and to Atle Selberg of the Institute for Advanced Study. Professor Bohr gave an excellent résumé of the work of Schwartz on distributions and of the work of Selberg on the Riemann zeta function and his elementary proof of the celebrated prime number theorem. Professor Bohr's paper will appear in full in the Proceedings of the Congress.

There were in attendance at the Congress 2302 persons. These were distributed as follows:

American and Canadian members attending	1429
Americans and Canadians attending as associate members	539
Members outside United States and Canada attending	271
Associate members outside the United States and Canada attending	63
	2302

In addition there were a number of mathematicians who became members of the Congress but did not attend the gathering at Cambridge. Some of these joined fully expecting to be present but were prevented from attending either by illness or other unavoidable circumstances while others assumed membership to support an enterprise which they considered most worthy. The statistics in this connection are as follows:

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[2] At the International Congress at Toronto in 1924 it was decided that at each international mathematical congress two gold medals should be awarded. Professor J. C. Fields, the Secretary of the 1924 Congress, presented a fund to subsidize these medals. They were first awarded in Oslo in 1936. The Committee to select the winners of the 1950 medals was: Professor Harald Bohr (Chairman), Professors L. V. Ahlfors, Karol Borsuk, Maurice Frechet, W. V. D. Hodge, A. N. Kolmogoroff, D. Kosambi, Marston Morse.

American and Canadian Congress members who did not attend	158
Associate American and Canadian Congress members who did not attend	29
Congress members outside United States and Canada who did not attend	50
Associate members outside United States and Canada who did not attend	11
Total Congress members who did not attend	248
Total Congress membership	2550

The following countries outside the United States and Canada were represented:

Argentina, Australia, Austria, Belgium, Brazil, Burma, Chile, China, Columbia, Cuba, Denmark, Egypt, Eire, England, Finland, France, Germany, Greece, India, Iran, Israel, Italy, Japan, Mexico, Netherlands, Nigeria, Norway, Panama, Peru, Philippines, Scotland, South Africa, Spain, Sweden, Switzerland, Turkey, Uruguay, Venezuela, Yugoslavia.

Every state in the Union was represented except South Dakota.

THE SCIENTIFIC PROGRAM

In addition to the contributed ten-minute papers and the stated addresses, which were delivered on invitation of the Organizing Committee, the scientific program of the Congress included four conferences on the following topics: Algebra, Analysis, Applied Mathematics, and Topology. The chairman of each section for contributed papers was given the privilege of inviting not more than three persons to deliver thirty minute addresses in his section.

STATED ADDRESSES

The following stated addresses were delivered by invitation of the Organizing Committee:

A. A. Albert: *Power-associative algebras.*

Arne Beurling: *On null-sets in harmonic analysis and function theory.*

S. Bochner: *Laplace operator on manifolds.*

H. Cartan: *Sur les fonctions analytiques de variables complexes.*

S. S. Chern: *Differentiable geometry of fiber bundles.*

H. Davenport: *Recent work in the geometry of numbers.*

Kurt Gödel: *Rotating universes in general relativity theory.*

W. V. D. Hodge: *Topological invariants of algebraic varieties.*

Heinz Hopf: *Die n-dimensionalen Sphären und projektiven Räume in der Topologie.*

Witold Hurewicz: *Homology and homotopy theory.*

S. Kakutani: *Ergodic theory.*

M. Morse: *Recent advances in variational theory in the large.*

J. von Neumann: *Shock interaction and its mathematical aspects.*

J. F. Ritt: *Differential groups.*

A. Rome: *The calculation of an eclipse of the sun according to Theon of Alexandria.*

L. Schwartz: *Distributions and principal applications.*

A. Wald: *Basic ideas of a general theory of statistical decision rules.*

A. Weil: *Number-theory and algebraic geometry.*

H. Whitney: *r-dimensional integration in n-space.*

N. Wiener: *Comprehensive view of prediction theory.*

R. L. Wilder: *The cultural basis of mathematics.*

O. Zariski: *The fundamental ideas of abstract algebraic geometry.*

CONFERENCES

In recent years mathematicians have been much impressed by the success of the conference method of presenting research in various fields in which vigorous advances have been made or are in progress. In such conferences there is a well-coördinated program of formal lectures and open informal discussion. At the International Congress the Organizing Committee decided to hold four conferences in the following fields: Algebra, Analysis, Applied Mathematics, and Topology. A number of the Stated Addresses listed above were integrated into the work of the Conferences.

A. *Algebra Conference.* The Algebra Conference was under the Chairmanship of Professor A. A. Albert.

There were four sessions, devoted to:

1. Algebraic geometry. In this session Professors A. Weil and O. Zariski gave their Stated Addresses.

2. Groups and universal algebra. The speakers were G. Birkhoff, H. Zassenhaus, S. MacLane, R. Baer, and C. Chevalley.

3. Structure theory of rings and algebras. The Stated Address of A. A. Albert was part of this session. Other speakers in this conference were R. Brauer, N. Jacobson, J. Dieudonné, and T. Nakayama.

4. Arithmetic algebra. Principal speakers in this session were E. Artin, W. Krull, M. Deuring, and M. Krasner.

B. *Analysis Conference*. This Conference was under the Chairmanship of M. Morse and there were four sessions devoted solely to analysis with an additional section in which the participants of the Topology Conference were joined. This was held on Friday evening, September 1. The sessions were devoted to:

1. Analysis in the large. The speakers were L. Bers, S. Bergman, L. Cesari and T. Radó, and C. B. Morrey.

2. Extremal methods and geometric theory of functions of a complex variable. The speakers were L. V. Ahlfors, A. C. Schaeffer and D. C. Spencer, M. M. Schiffer, H. Grunsky, R. Nevanlinna, and G. Szegö.

3. Algebraic tendencies in analysis. This subconference consisted of reports of various aspects of the main topic of algebraic tendencies in analysis. A panel of experts surveyed the field and then prepared a report. This report of the panel was presented to the Congress by the spokesman of the group. Two sessions, one on Monday afternoon and the other on Tuesday, were devoted to this subconference. The topics treated were:

a. Group representations. Spokesman, R. Godement.
b. Topologic Algebra. Spokesman, I. Kaplansky.
c. Measure Theory. Spokesman, P. R. Halmos.
d. Spectral Theory. Spokesman, N. Dunford.
e. Applied Functional Analysis. Spokesman, N. Aronszajn.
f. Ergodic Theory. Spokesman, S. Kakutani.

4. The Joint Session with the Topology Conference was devoted to analysis and geometry in the large. The speakers were J. Leray, G. de Rham, and A. Lichnerowicz.

C. *Conference in Applied Mathematics*. This conference was under the Chairmanship of J. von Neumann. There were three sessions devoted to:

1. Random processes in physics and communications. The speakers were C. E. Shannon and S. M. Ulam.

2. Partial differential equations. The speakers were J. von Neumann, R. Courant, S. Goldstein, W. Heisenberg, W. Prager, J. J. Stoker. Professor von Neumann's address was the one which he presented by invitation of the Organizing Committee.

3. Statistical Mechanics. In this session there was the Stated Address of N. Wiener. In addition W. Feller gave a 45 minute address.

D. *Topology Conference*. This conference was under the chairmanship of Hassler Whitney and was divided into four sessions.

1. Homology and homotopy theory. The speakers were W. Hurewicz, S. Eilenberg, J. H. C. Whitehead, and G. W. Whitehead. Professor Hurewicz's address was given by invitation of the Organizing Committee.

2. Fiber bundles and obstructions. The speakers were P. Olum, W. S. Massey, G. Hirsch, and E. Spanier.

3. Differentiable manifolds. The speakers were S. Chern, E. Ehresmann, B. Eckmann, and C. B. Allendoerfer. Professor Chern's address was the Stated Address which he was invited to give by the Organizing Committee.

4. Topological groups. The speakers were P. A. Smith, Deane Montgomery, K. Iwasawa, A. Gleason, and R. H. Fox.

CONTRIBUTED PAPERS

There were the usual sessions for the presentation of contributed papers. Each member of the Congress was given the privilege of submitting one paper for presentation in person. Presentation by title was not permitted. The contributed papers were distributed among seven sections, listed below. In addition, the Chairman of each section was permitted to invite not more than three persons to deliver thirty minute addresses in his section.[3] In all 372 contributed papers were presented at the Congress.

Section I, *Algebra and theory of numbers*. Professor H. A. Rademacher was the chairman of this Section in which 58 contributed papers were presented. The following thirty minute addresses were given on invitation of the Chairman:

K. Mahler: *Farey sections in the fields of Gauss and Eisenstein.*

H. Selberg: *The general sieve method and its place in prime number theory.*

H. D. Kloosterman: *The characters of binary modulary congruence groups.*

Section II, *Analysis*. Professor G. C. Evans was the chairman of this Section in which 126 contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

H. A. Rademacher: *Remarks on the theory of partitions.*

H. Bohr: *A survey of the different proofs of the main theorems in the field of almost periodic functions.*

S. Mandelbrojt: *Théorèmes d'unicité de la théorie des fonctions.*

Section III, *Geometry and topology*. Professor S. Eilenberg was the chairman of this Section in which 56 contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

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[3] An exception was later permitted in the section on Logic and Philosophy, where four persons presented these longer papers.

L. A. Santaló: *Integral geometry in general spaces* (read by C. B. Allendoerfer).

B. Segre: *Arithmetical properties of algebraic varieties.*

Section IV, *Probability and Statistics, Actuarial Science, Economics.* Professor J. L. Doob was the chairman of this Section in which 27 contributed papers were presented. The following invited addresses were given by invitation of the Section Chairman:

C. Bose: *Mathematical theory of factorial designs.*

P. Levy: *Processus laplaciens et équations différentielles stochastiques.*

S. N. Roy: *On some aspects of statistical inference.*

Section V, *Mathematical physics and applied mathematics.* Professor Richard Courant was the chairman of this Section in which 74 contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

C. G. Darwin: *The refractive index of an ionized gas.*

F. Rellich: *Störungstheorie der Spektralzerlegung.*

H. Lewy: *Developments at the confluence of analytic boundary conditions.*

Section VI, *Logic and philosophy.* Professor Alfred Tarski was the chairman of this Section in which 16 contributed papers were presented. The following invited addresses were given by invitation of the Chairman:

T. Skolem: *Remarks on the foundation of set theory.*

A. Tarski: *Some notions and methods on the borderline of algebra and metamathematics.*

A. Robinson: *Applied symbolic logic.*

S. C. Kleene: *Recursive functions and intuitionistic mathematics.*

Section VII, *History and education.* Professor C. V. Newsom was the chairman of this Section in which 15 contributed papers were presented. The following invited address was given by invitation of the Chairman:

G. Pólya: *On plausible reasoning.*

THE SOCIAL PROGRAM

The entertainment of the members of the Congress was in the hands of an Entertainment Committee, whose chairman was L. H. Loomis. The other members of the Committee were C. R. Adams, Mrs. L. V. Ahlfors, Mrs. G. D. Birkhoff, J. A. Clarkson, Mrs. W. C. Graustein, F. B. Hildebrand, Evelyn M. Hull, J. R. Kline, E. R.

Lorch, Mrs. W. T. Martin, E. B. Mode, G. A. O'Donnell, Mrs. H. B. Phillips, Helen G. Russell, J. H. Van Vleck, Mrs. J. L. Walsh, Mrs. D. V. Widder, Mrs. Norbert Wiener.

The members and associate members of the Congress were housed in the dormitories of Harvard University. Excellent meals were served in the Harvard Union. The local members of the Entertainment Committee arranged a fine exhibition of mathematical models and a book exhibit at the Harvard Cooperative Society. In addition to tours of Harvard University and the Massachusetts Institute of Technology there were trips to historic and modern Boston as well as to Lexington and to Concord.

On the opening night there was a reception for all members of the Congress at the Fogg Art Museum of Harvard University. On the following evening the Busch String Quartet gave a concert in the Sanders Theatre. At the same time Professor Howard Aiken, the designer of the Harvard Computing Machine, gave an interesting lecture on computing machines.

On Friday afternoon the mathematicians and their friends were the guests of Wellesley College at a tea. At the same time a restricted number of members of the Congress were admitted to the Harvard Observatory at Oak Ridge. On Saturday evening, September 2, there was an informal dance in Lowell House, while other members of the Congress attended a Beer Party in Memorial Hall.

There were no sessions of the Congress on Sunday, September 3. In the afternoon, Boston University received the members of the Congress and an organ recital was provided in the Daniel L. Marsh Chapel of the University. In the evening Richard Dyer-Bennet gave a concert of ballads of various nations in Sanders Theatre.

On Monday evening, September 4, the committee arranged for a concert in Symphony Hall by Miss Helen Traubel. The concert was enthusiastically received by the audience and Miss Traubel received a tremendous ovation at the end of the performance.

The Congress banquet was held on Tuesday evening, September 5, in the Sever Quadrangle. Professor Marston Morse was the Toastmaster and Dr. Detlev Bronk, President of the National Academy of Sciences, gave the principal address. Other speakers were Edward Crane, the Mayor of Cambridge; Dr. James R. Killian, the President of Massachusetts Institute of Technology; Professor W. V. D. Hodge, of Cambridge University; and Professor Percy Bridgeman of Harvard University. Professor Hodge expressed the sincere appreciation of our foreign members of the Congress to their American hosts for their splendid hospitality.

On Wednesday evening, September 6, the mathematicians were the guests of the Director and Board of Trustees of Gardner Museum at a farewell party.

On Wednesday morning, September 6, there was held a final plenary session of the Congress in Sanders Theatre. Professor M. H. Stone gave a report on the conference which had been held in New York City immediately preceding the International Congress for the purpose of considering the formation of an International Mathematical Union. He reported that Statutes and By-Laws had been adopted and that these would be submitted to the proper scientific groups in the various national or geographic areas in which there was significant mathematical activity. When a specified number of groups have signified their acceptance of these Statutes and By-Laws, the Union will be declared in existence and a meeting of the General Assembly arranged.

Professor van der Corput, on behalf of the delegation from Holland, presented a cordial invitation to the International Congress to hold its next meeting in that country in 1954. The Congress unanimously voted to accept the gracious invitation of our Dutch colleagues. After addresses there was an address of appreciation by President Harald Cramér of the University of Stockholm. After Professor A. A. Albert had presented a resolution of thanks to Harvard University and the various committees of the Congress which was adopted unanimously, the Congress adjourned.

The Congress was undoubtedly the largest gathering of persons ever assembled in the history of the world for the discussion of mathematical research. However, the real measure of its success lies not in the large number of persons present, but in the excellence of its scientific program and in the contributions which it made to the cause of closer cooperation among scientists and to the cause of international good will.

J. R. KLINE,
Secretary, International Congress of Mathematicians

AN EXTENSION THEORY FOR A CERTAIN CLASS OF LOOPS

R. H. BRUCK

Introduction. If E is a group with a normal subgroup K one may form the quotient group $E/K \cong M$. Conversely, for preassigned groups K, M, there is the *extension* problem: to determine (in some sense) all groups E with K as normal subgroup such that $E/K \cong M$. Much progress has been made on this problem, particularly through the work of Baer [**1, 2, 3**][1] and the cohomology theory of Eilenberg and MacLane [**1, 2, 3**]. The latter authors make it clear that insight is gained by relinquishing part of the associative law; specifically, by requiring that E be merely a *loop* such that the associative law $(e_1e_2)e_3 = e_1(e_2e_3)$ holds if at least one of the e_i belongs to a distinguished subgroup of K. We take this to be K itself. It then becomes evident that the subclass of loops E consisting of the *groups* is not the only one of interest; one may consider, for example, the *Moufang* loops, in which case it seems natural to allow M also to be Moufang. Thus we approach the extension problem actually studied in the paper: M is a given loop, K is a group (not given, but with given centre G) and E is to be any loop with K as a normal subloop contained in the "associator" of E, such that $E/K \cong M$. This problem is more typical of group theory than of loop theory but is, nevertheless, a natural and significant special topic in the theory of loops.

For the sake of brevity no examples or applications are given and references to the bibliography are kept to a minimum. The Eilenberg-MacLane *kernels*, important for constructions, have been ignored. I may signal out as new: the inverse of a (noncentral) extension (§1), the specific results on central Moufang extensions (§6)[2] and the all-pervading functions F which generalize (even for M a group) the Eilenberg-MacLane cocycles. As indicated by Theorem 8 (§4), additional information about the functions F would probably increase our knowledge of cohomology groups.

1. **Extensions.** A *loop* M is a system with a multiplication such

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An address delivered before the Chicago Meeting of the Society on November 25, 1949, by invitation of the Committee to Select Hour Speakers for Midwestern Sectional Meetings; received by the editors February 28, 1950.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Slightly weaker results on central Moufang extensions were obtained in 1946–1947 with the support of a Guggenheim Fellowship supplemented by a grant from the Wisconsin Alumni Research Foundation. (See Bull. Amer. Math. Soc. Abstract 53-1-11.)

that: (a) in $xy=z$, any two of x, y, z uniquely determine the third; (b) M has a unit 1. The *associator* $A=A(M)$ is the subset of M such that $(xy)z=x(yz)$ if at least one of x, y, z is in A; the associator is an associative subloop (and therefore a group). A subloop H of M is *normal* in M if and only if H is the kernel of a homomorphism of M into a loop; equivalently, $xH=Hx$, $(xy)H=x(yH)$, $(xH)y=x(Hy)$, $(Hx)y=H(xy)$ for all x, y in M. The mapping $x\to xH$ of M set up by a normal subloop H is a homomorphism upon a *quotient* loop M/H. (See Bruck [1].)

If M is given, we wish to study all loops E such that (i) E has a homomorphism θ upon M; (ii) the kernel K of θ is a subgroup of $A(E)$. Let $G=Z(K)$ be the centre of K. For each e in E define the mapping $T(e)$ of K by

$$(1)\qquad ke = e(kT(e)), \qquad k\in K.$$

Applying θ to both sides of (1) we see that $kT(e)$ is in K. And to each k' in K corresponds a unique k in K such that $kT(e)=k'$. Furthermore, $e((k_1k_2)T(e))=(k_1k_2)e=k_1(k_2e)=k_1(e\cdot k_2T(e))=k_1e\cdot k_2T(e)$ $=(e\cdot k_1T(e))\cdot k_2T(e)=e(k_1T(e)\cdot k_2T(e))$. Thus $T(e)$ is an automorphism of K: $(k_1k_2)T(e)=k_1T(e)\cdot k_2T(e)$. In particular, $T(1)$ is the identity automorphism. Moreover, $(e_1e_2)\cdot kT(e_1e_2)=k(e_1e_2)=(ke_1)e_2$ $=(e_1\cdot kT(e_1))e_2=e_1(kT(e_1)\cdot e_2)=e_1(e_2\cdot kT(e_1)T(e_2))=(e_1e_2)\cdot kT(e_1)T(e_2)$, or $kT(e_1e_2)=kT(e_1)T(e_2)$. In other words, *the mapping* $e\to T(e)$ *is a homomorphism of* E *upon a group of automorphisms of* K.

For our purposes a *pair* (G, M) shall consist of an abelian group G, a loop M and a single-valued product gx from GM to G such that $g1=g$, $(gg')x=(gx)(g'x)$ and $(gx)y=g(xy)$ for all g, g' in G and x, y in M, where 1 is the unit of M. From (1), $T(e)$ is an inner automorphism if e is in K. Thus, for arbitrary g in $G=Z(K)$, k in K, e in E, we have $gT(ke)=gT(k)T(e)=gT(e)$. However, $e'\theta=e\theta$ if and only if $e'=ke$ for k in K; thus $gT(e)$ *depends only on* g *and* $x=e\theta$. Hence if we set $gx=gT(e)$, G and M become a pair (G, M). It is a mere matter of bookkeeping (which turns out to be useful) to pursue the study in terms of a fixed pair (G, M). This leads to the basic definition:

DEFINITION 1. Let (G, M) be a pair. A (G, M) *extension* (E, θ) consists of a loop E and a homomorphism θ of E upon M such that (i) $K=1\theta^{-1}$ is in $A(E)$; (ii) $Z(K)=G$; (iii) $ge=e(gx)$ for g in G, e in E, $x=e\theta$.

It will be convenient to list here other fundamental definitions concerning extensions.

DEFINITION 2. (E, θ) is *central* if $1\theta^{-1}=G$.

DEFINITION 3. (E_1, θ_1) is *equivalent* to (E_2, θ_2) if there exists an isomorphism π of E_1 upon E_2 such that (i) $\theta_1=\pi\theta_2$; (ii) $g\pi=g$ for g in G. (Notation: $E_1\sim E_2$.)

Equivalence is reflexive, symmetric, transitive; it will serve as equality. Equivalence should be contrasted with inverse equivalence:

DEFINITION 4. (E_1, θ_1) is *inverse equivalent* to (E_2, θ_2) if there exists an isomorphism π of E_1 upon E_2 such that (i) $\theta_1=\pi\theta_2$; (ii) $g\pi=g^{-1}$ for g in G. (Notation: $E_1\sim^{-1}E_2$.)

Inverse equivalence is symmetric, not always reflexive. Transitivity has three substitutes, one being: $E\sim^{-1}E_1$, $E_1\sim E_2$ imply $E\sim^{-1}E_2$. Therefore, since equivalence is to serve as equality, we may define *the* inverse $(E, \theta)^{-1}$ as any extension inverse equivalent to (E, θ). The inverse of (E, θ) may be constructed as follows. Let $u(x)$ be any normalized system of representatives of M in E; thus $u(x)\theta=x$, $u(1)=1$. If $K=1\theta^{-1}$, every e in E has a unique representation $e=u(x)k$ with $x=e\theta$, k in K; define π by $e\pi=u(x)k^{-1}$. Define a new operation (o) on the elements of E by $eoe'=(e\pi\cdot e'\pi)\pi$; it is easy to see that this turns E into a loop E^{-1}. I claim that (E^{-1}, θ) is the desired inverse. Indeed, π is an isomorphism of E upon E^{-1}, and $g\pi=g^{-1}$ for g in G. Also $\theta=\pi\theta$. Certainly θ is a homomorphism of E^{-1} upon M, the kernel being the group $K\pi$ anti-isomorphic to K, with centre $G\pi=G$. If at least one of e_1, e_2, e_3 is in $K\pi$, $(e_1oe_2)oe_3=((e_1\pi\ \ e_2\pi)\cdot e_3\pi)\pi=(e_1\pi\cdot(e_2\pi\ \ e_3\pi))\pi=e_1o(e_2oe_3)$; thus $K\pi$ is in $A(E^{-1})$. For g in G, e in E^{-1}, $x=e\theta$, we have $goe=(g^{-1}\cdot e\pi)\pi=(e\pi\ \ (g^{-1}x))\pi=eo(gx)$. This completes the proof.

DEFINITION 5. The *product* $(E_1, \theta_1)\otimes(E_2, \theta_2)=(E, \theta)$ of two extensions (E_j, θ_j) is defined as follows: (i) The elements of E are the pairs (e_1, e_2) with e_j in E_j and $e_1\theta_1=e_2\theta_2$. (ii) $(e_1, e_2)=(e_1', e_2')$ if and only if $e_1'=e_1g$, $e_2'=e_2g^{-1}$ for some g in G. (iii) $(e_1, e_2)(e_1', e_2')=(e_1e_1', e_2e_2')$. (iv) $(e_1, e_2)\theta=e_1\theta_1=e_2\theta_2$. (v) $(g, 1)=g$ for g in G. (Notation: $E_1\otimes E_2=E$.)

For a more detailed discussion of the product see Eilenberg and MacLane [2, 3]. Straightforward but tedious calculation shows that $E_1\otimes E_2$ is a (G, M) extension such that

$$\text{If } E_j\sim E_j' \quad (j=1, 2), \quad E_1\otimes E_2\sim E_1'\otimes E_2', \tag{2}$$

$$E_1\otimes E_2\sim E_2\otimes E_1, \tag{3}$$

$$(E_1\otimes E_2)\otimes E_3\sim E_1\otimes(E_2\otimes E_3). \tag{4}$$

Therefore *the set S of all (G, M) extensions, with equivalence as equality, and with multiplication as in Definition* 5, *is a commutative semigroup*. It may also be shown that *S has a unit* (E_o, θ_o):

DEFINITION 6. The *unit* extension (E_o, θ_o) is defined as follows: E_o is the set of all pairs (x, g), x in M, g in G, such that (i) (x, g)

$=(y, g')$ if and only if $x=y$, $g=g'$; (ii) $(x, g)(y, g')=(xy, (gy)g')$; (iii) $(1, g)=g$. And θ_o is given by (iv) $(x, g)\theta_o=x$.

It is essentially known (Baer [1], Eilenberg-MacLane [1]) that *the subset S' of S, consisting of the central extensions, is an abelian group with unit (E_o, θ_o)*. For (E, θ) central, our inverse $(E, \theta)^{-1}$ is the inverse of (E, θ) in S'. Details are deferred until §6 (see Theorem 10) but the facts are assumed in §4.

2. **The functions F.** For any positive integer n let L_n be the *free loop* (Bates [1]) with (free) generators $X_1, \cdots, X_n$. Thus L_n is a loop containing the X_j, such that any mapping $X_1\rightarrow e_1, \cdots, X_n\rightarrow e_n$ into elements e_j of a loop E may be extended uniquely to a homomorphism ρ of L_n into E. By a (nonassociative) *word* W_n we mean any element of L_n. The image $W_n\rho$ is denoted by $W_n(e_1, \cdots, e_n)$; this turns W_n into a function defined on *every* loop E (with values in E). The following fact is worth noting: if also σ is a homomorphism of E into a loop L, $W_n(e_1, \cdots, e_n)\sigma=W_n(e_1\sigma, \cdots, e_n\sigma)$, since the homomorphism $\rho\sigma$ of L_n maps X_j upon $e_j\sigma$.

DEFINITION 7. A word W_n is *purely nonassociative* (p.n.a.) if it "vanishes" on every group: If $e_1, \cdots, e_n$ are group elements,

$$W_n(e_1, \cdots, e_n) = 1.$$

As an important example of a p.n.a. word, consider A_3, defined by $(X_1X_2)X_3=(X_1(X_2X_3))A_3(X_1, X_2, X_3)$. If E is a loop, the set of all elements $W_n(e_1, \cdots, e_n)$ (n arbitrary, W_n p.n.a., the e_j in E) generates a normal subloop E_{pna} which may be characterized as follows: a necessary and sufficient condition that E/F be associative (for a normal subloop F of E) is that F contain E_{pna}.

THEOREM 1. *Let (E, θ) be a (G, M) extension, W_n, a p.n.a. word, $e_1, \cdots, e_n$, elements of E. Write $e_j\theta=x_j$, $e_o=W_n(e_1, \cdots, e_n)$. Then* (i) *$e_ok=ke_o$ for k in the kernel K*; (ii) *$W_n(x_1, \cdots, x_n)=1$ if and only if e_o is in G*; (iii) *e_o depends only on the x_j*:

$$e_o = W_n(e_1, \cdots, e_n) = F(W_n, E; x_1, \cdots, x_n). \tag{5}$$

PROOF. (i) If T is defined by (1), the mapping $e\rightarrow T(e)$ is a homomorphism of E upon a group of automorphisms of K. Thus $T(e_o)=W_n(T(e_1), \cdots, T(e_n))=1$, the identity automorphism.

(ii) $e_o\theta=W_n(x_1, \cdots, x_n)$, so (i) implies (ii).

(iii) For fixed n, and for every word A_n (not necessarily p.n.a.), define a function $H(A_n; e, k)=H(A_n; e_1, \cdots, e_n; k_1, \cdots, k_n)$ by

$$A_n(e_1k_1, \cdots, e_nk_n) = A_n(e_1, \cdots, e_n)H(A_n; e, k). \tag{6}$$

Here the e_j are assigned fixed values in E and the k_j vary over K. Applying θ to (6) we find that H takes values in K. Also from (6), direct computation, along with the fact that $(A_nB_n)(e_1, \cdots, e_n) = A_n(e_1, \cdots, e_n)B_n(e_1, \cdots, e_n)$, gives

$$(7) \quad H(A_nB_n; e, k) = H(A_n; e, k)T(B_n(e_1, \cdots, e_n)\cdot H(B_n; e, k).$$

Moreover, by specializing A_n in (6) to the "unit" word 1 and the words X_j,

$$(8) \quad H(1; e, k) = 1; \quad H(X_j; e, k) = k_j \qquad (j = 1, 2, \cdots, n).$$

In addition, if $B_nC_n = A_n = D_nB_n$, we may derive from (7) formulas involving only A_n and C_n or A_n and D_n. Hence, since L_n is free, *the recurrence formula* (7) *and the initial conditions* (8) *define a unique function* H.

Next construct the holomorph $\mathfrak{K}$ of K. This group is the set of all pairs (S, k), k in K, S an automorphism of K, under the product $(S, k)(U, k') = (SU, kU\cdot k')$. The n elements $f_j = (T(e_j), k_j)$ yield $A_n(f_1, \cdots, f_n) = (T(A_n(e_1, \cdots, e_n)), H'(A_n; e, k))$ where H' satisfies both (7) and (8). Therefore $H = H'$. Since $\mathfrak{K}$ is a group, $H'(W_n; e, k) = 1$ for every p.n.a. word W_n. Thus, by (6), $W_n(e_1k_1, \cdots, e_nk_n) = W_n(e_1, \cdots, e_n) = e_0$, showing that e_0 depends only on the images $x_j = e_j\theta = (e_jk_j)\theta$. This completes the proof of Theorem 1.

DEFINITION 8. An ordered set $x_1, \cdots, x_n$ of elements of M is called a *spot* for a p.n.a. word W_n if $W_n(x_1, \cdots, x_n) = 1$.

THEOREM 2. *At each spot for a* p.n.a. *word* W_n, *the functions* F (*of Theorem* 1) *form a multiplicative abelian group*: (i) $E_1 \sim E_2$ *implies* $F(W_n, E_1) = F(W_n, E_2)$; (ii) $E_1 \sim^{-1} E_2$ *implies* $F(W_n, E_1) = F(W_n, E_2)^{-1}$; (iii) $F(W_n, E_1)F(W_n, E_2) = F(W_n, E_1 \otimes E_2)$.

PROOF. Let $x_1, \cdots, x_n$ be a spot for W_n, and write $F(W_n, E) = F(W_n, E; x_1, \cdots, x_n)$ for any extension (E, θ). By Theorem 1 (ii), $F(W_n, E)$ is in G. Let π be an isomorphism of (E_1, θ_1) upon (E_2, θ_2) satisfying (i) of Definitions 3, 4, and let e_j in E_1 satisfy $e_j\theta_1 = x_j$ $(j = 1, 2, \cdots, n)$. Then $e_j\pi$ is in E_2, and $e_j\pi\theta_2 = e_j\theta_1 = x_j$. Hence $F(W_n, E_1)\pi = W_n(e_1, \cdots, e_n)\pi = W_n(e_1\pi, \cdots, e_n\pi) = F(W_n, E_2)$. According as π satisfies (ii) of Definition 3 or 4, we get (i) or (ii) of Theorem 2. To prove (iii), choose e_{1j} in E_1, e_{2j} in E_2 such that $e_{1j}\theta_1 = e_{2j}\theta_2 = x_j$, and set $e_j = (e_{1j}, e_{2j})$, $(j = 1, 2, \cdots, n)$. If $g_i = F(W_n, E_i)$, Definition 5 gives $F(W_n, E_1 \otimes E_2) = W_n(e_1, \cdots, e_n) = (g_1, g_2) = (g_1g_2, 1) = g_1g_2 = F(W_n, E_1)F(W_n, E_2)$.

3. **Strongly grouplike and C extensions.** An extension (E, θ) is *strongly grouplike* (s.g.) if E inherits all relations between elements

(implied by the associative law) which hold for the images in M. This means: if W_n is p.n.a., and if $W_n(e_1, \cdots, e_n)\theta = 1$, then $W_n(e_1, \cdots, e_n) = 1$. In particular, if M is a group, the s.g. extensions are precisely the associative extensions. The following theorem is an immediate consequence of Theorem 2.

THEOREM 3. (i) *For any (G, M) extension E, $E \otimes E^{-1}$ is s.g.* (ii) *If E is s.g., and if $E_1 \sim E$ or $E_1 \sim^{-1} E$, then E_1 is s.g.* (iii) *If $E_1 \otimes E_2 = E_3$, and if two of the E_j are s.g., so is the third.*

Next let C be any set of p.n.a. words. *Assume* that if W_n is in C then $W_n(x_1, \cdots, x_n) = 1$ for all x_j in M. Then a (G, M) extension (E, θ) is "C" if $W_n(e_1, \cdots, e_n) = 1$ for each W_n in C and all e_j in E. We get at once the following theorem.

THEOREM 4. *Every s.g. extension is C, and Theorem 3 remains true with "s.g." replaced by "C".*

The following examples are of interest: (1) C consists of A_3, introduced after Definition 7. M is a group and the C-extensions are the associative ones. (2) C consists of B_3, defined by $X_1X_2 \cdot X_3X_1 = (X_1(X_2X_3 \cdot X_1))B_3(X_1, X_2, X_3)$. M is a Moufang loop (Bruck [1]), characterized by the identity

$$xy \cdot zx = x(yz \cdot x), \tag{9}$$

and the C-extensions are the Moufang ones.

4. **Groups of extensions.** First let S be any commutative semigroup. A subset N is a *nucleus* of S if there exists a homomorphism ρ of S, with kernel N, upon a group. Equivalently: (i) if $a_1a_2 = a_3$ for a_j in S, and if two of the a_j are in N, so is the third; (ii) to each a in S corresponds an a^{-1} in S such that $aa^{-1} \in N$. The necessity of (i), (ii) is obvious. As for sufficiency, define $a \equiv b \bmod N$ if $an_1 = bn_2$ for n_j in N, and let $a\rho$ be the equivalence class of a mod N, then ρ is a homomorphism, with kernel N, of S upon the *quotient group* $S\rho = S/N$. If the nucleus N' contains the nucleus N, one may establish the isomorphism $S/N' \cong (S/N)/(N'/N)$. Furthermore, if S has a unit contained in a *subgroup* S' of S, then NS' is a nucleus and one may establish the isomorphism $(NS')/N \cong S'/(S' \cap N)$. These remarks lead to the following (restricted) definition.

DEFINITION 9. A subset N of the semigroup S of (G, M) extensions (or of the group S' of central extensions) is a *nucleus* of S (or S') provided (i) if $E_1 \otimes E_2 = E_3$ for (central) extensions E_j, and if two of the E_j are in N, so is the third; (ii) for every (central) extension E, $E \otimes E^{-1}$ is in N, where E^{-1} denotes the inverse extension.

The following are nuclei of S: (i) the set N_{sg} of s.g. extensions (Theorem 3); (ii) the set N_C of C-extensions (Theorem 4); (iii) $S'\otimes N_{sg}$; (iv) $S'\otimes N_C$. As nuclei of S' we have the subgroups $N'_{sg} = S'\cap N_{sg}$, $N'_C = S'\cap N_C$. We define abelian groups $\mathfrak{Z}$, $\mathfrak{B}$, $\mathfrak{H}$ by

$$\mathfrak{Z} = S/N_{sg}, \qquad \mathfrak{B} = (S'\otimes N_{sg})/N_{sg} \cong S'/N'_{sg}, \qquad \mathfrak{H} = \mathfrak{Z}/\mathfrak{B}. \tag{10}$$

Similar definitions hold for $\mathfrak{Z}_C$, $\mathfrak{B}_C$, $\mathfrak{H}_C$. In view of Theorem 2, these groups are isomorphic to certain groups of functions F. A characterization of the latter would be highly enlightening. So far, however, not much is known. At the one end of the scale we have the following theorem.

THEOREM 5. *If the loop M is free, $\mathfrak{Z}$, $\mathfrak{B}$, and $\mathfrak{H}$ are groups of order* 1.

PROOF. Let (E, θ) be a (G, M) extension. In particular, θ is a homomorphism of E upon M. Since M is free, there exists (Bates [1, Theorem 3.5]) an isomorphism ρ of M into E such that $x\rho\theta = x$ for each x in M. Let W_n be any p.n.a. word, $x_1, \cdots, x_n$ any spot for W_n. Then $F(W_n, E; x_1, \cdots, x_n) = W_n(x_1\rho, \cdots, x_n\rho) = W_n(x_1, \cdots, x_n)\rho = 1\rho = 1$. Therefore $S = N_{sg}$, which implies Theorem 5.

A similar result holds for C-extensions. Define a loop L to be a C-loop if $W_n(y_1, \cdots, y_n) = 1$ for every W_n in C and all $y_1, \cdots, y_n$ in L. By previous agreement, M is a C-loop, and E is a C-loop for every C-extension (E, θ). The notion of a free C-loop may be defined as in Bates [1, Appendix]. Restricting attention to C-extensions, the proof of Theorem 5 may be paralleled exactly to give the following theorem.

THEOREM 6. *If M is a free C-loop, $N_C = N_{sg}$. In words: the C-extensions coincide with the strongly grouplike extensions.*

At the other end of the scale, take M to be a group. For $n \geqq 0$, a *(normalized) n-cochain* f_n is (Eilenberg and MacLane [1, 2, 3]) a single-valued function from M to G, with values $f_n(x_1, \cdots, x_n)$, taking the value 1 if at least one of the x_i is 1. These n-cochains form the n-cochain group $\mathfrak{C}_n$ under the product $(f_nh_n)(x_1, \cdots, x_n) = f_n(x_1, \cdots, x_n)h_n(x_1, \cdots, x_n)$. We define the *$(n+1)$-coboundary* δf_n of f_n as the normalized cochain

$$\begin{aligned}(\delta f_n)(x_1, \cdots, x_{n+1}) = {} & (f_n(x_1, \cdots, x_n)x_{n+1}) \cdot f_n(x_2, \cdots, x_{n+1})^{c(0)} \\ & \cdot \prod_{i=1}^{n} f_n(x_1, \cdots, x_{i-1}, x_ix_{i+1}, x_{i+2}, \cdots, x_{n+1})^{c(i)},\end{aligned} \tag{11}$$

where $c(j) = (-1)^{n+1+j}$ for $j = 0, 1, \cdots, n$. For $n > 0$, $\mathfrak{B}_n$ is the group of the n-coboundaries; $\mathfrak{B}_0$ consists of the 0-cochain $1_0=1$. An *n-cocycle* is an n-cochain f_n such that $\delta f_n=1_{n+1}$ (the identity of $\mathfrak{C}_{n+1}$) and $\mathfrak{Z}_n$ is the group of the n-cocycles. As a consequence of the associativity of M, one may verify that $\delta^2=0$, in the sense that $\delta(\delta f_n)=1_{n+2}$; hence $\mathfrak{B}_n$ is a subgroup of $\mathfrak{Z}_n$. The nth cohomology group $\mathfrak{H}_n$ is defined by $\mathfrak{H}_n=\mathfrak{Z}_n/\mathfrak{B}_n$. The next theorem is due to Eilenberg and MacLane [**2, 3**]:

THEOREM 7. *If M is a group, the homomorphism $(E, \theta)\to F(A_3, E)$ induces the isomorphism $\mathfrak{H}\cong\mathfrak{H}_3$.*

A partial sketch of the proof will be useful. For any (G, M) extension (E, θ), define (see §3) f_a and f_m by

$$(12)\qquad f_a(x, y, z) = F(A_3, E;\ x, y, z);\ f_m(x, y, z) = F(B_3, E;\ x, y, z).$$

Choose e_j in E such that $e_j\theta=x_j$ for $j=1, 2, 3, 4$. Then

$$(13)\quad (e_1e_2)e_3 = e_1(e_2e_3)f_a(x_1, x_2, x_3);\ e_1e_2\cdot e_3e_1 = e_1(e_2e_3\cdot e_1)f_m(x_1, x_2, x_3),$$

showing that f_a, f_m are normalized 3-cochains. If (E, θ) is central and if $u(x)$ is a normalized system of representatives of M in E, then $u(x)u(y)=u(xy)h(x, y)$ for a normalized 2-cochain h. Setting $e_j=u(x_j)$ in (13) we find $f_a=\delta h$. In any case, by (13), $(e_1e_2\cdot e_3)e_4=(e_1e_2\cdot e_3e_4)\cdot f_a(x_1x_2,\ x_3,\ x_4)=(e_1(e_2\cdot e_3e_4))f_a(x_1,\ x_2,\ x_3x_4)f_a(x_1x_2,\ x_3,\ x_4)$ and also

$$\begin{aligned}(e_1e_2\cdot e_3)e_4 &= ((e_1\cdot e_2e_3)f_a(x_1, x_2, x_3))e_4 = (e_1\cdot e_2e_3)e_4(f_a(x_1, x_2, x_3)x_4)\\ &= e_1(e_2e_3\cdot e_4)f_a(x_1, x_2x_3, x_4)(f_a(x_1, x_2, x_3)x_4)\\ &= (e_1(e_2\cdot e_3e_4))f_a(x_2, x_3, x_4)f_a(x_1, x_2x_3, x_4)(f_a(x_1, x_2, x_3)x_4),\end{aligned}$$

whence comparison gives $\delta f_a=1_4$. Thus f_a is a 3-cocycle. It can be shown conversely that every 3-cocycle (3-coboundary) is an $F(A_3, E)$ (an $F(A_3, E)$ for E central).

Again, $e_1e_2\cdot e_3e_1=e_1(e_2\cdot e_3e_1)f_a(x_1,\ x_2,\ x_3x_1)$ and $e_1(e_2e_3\cdot e_1)=e_1(e_2\cdot e_3e_1)$ $f_a(x_1, x_2, x_3)$, whence, by (13),

$$(14)\qquad f_m(x, y, z) = f_a(x, y, zx)f_a(y, z, x)^{-1}.$$

The homomorphism ρ of $\mathfrak{Z}_3$ into $\mathfrak{C}_3$, defined by $(f_3\rho)(x, y, z)=f_3(x, y, zx)$ $\cdot f_3(y,\ z,\ x)^{-1}$, induces a homomorphism of $\mathfrak{H}_3$ upon a group $\mathfrak{H}_3\rho$ $=\mathfrak{Z}_3\rho/\mathfrak{B}_3\rho$. In view of (14) we may state the following theorem.

THEOREM 8. *If M is a group, let C-extensions be Moufang extensions. Then the homomorphism $(E, \theta)\to F(B_3, E)$ induces an isomorphism $\mathfrak{H}_C\cong\mathfrak{H}_3\rho$.*

Theorems 5–8 have analogues for central extensions, for example (Baer [1], Eilenberg and MacLane [1]): *if M is a group, the group of* *central group extensions is isomorphic to the second cohomology group* $\mathfrak{H}_2$.

5. **Grouplike extensions. Conjugate extensions.** A (G, M) extension (E, θ) is *grouplike* if, for every subgroup (=associative subloop) H of M, $H\theta^{-1}$ is a subgroup of E. Thus (E, θ) is grouplike if and only if $F(A_3, E; x, y, z)=1$ for all triples x, y, z which generate a subgroup of M. Note that s.g. extensions are grouplike.

If E is any loop, define for each p in E permutations R_p, L_p by $eR_p=ep$, $eL_p=pe$, all e in E. Choosing fixed p, q in E, we may define a new operation (o) on E by

$$e_1oe_2 = (e_1R_q^{-1})(e_2L_p^{-1}). \tag{15}$$

The elements of E form a loop E_o under (o); the unit is pq. E_o is (Albert [1]) a *(principal) isotope* of E. If, further, (E, θ) is a (G, M) extension, write $p\theta=u$, $q\theta=v$. Then, if M_o is the principal isotope of M defined by

$$xoy = (xR_v^{-1})(yL_u^{-1}), \tag{16}$$

we find from (15), (16), with $e_i\theta=x_i$, that $(e_1oe_2)\theta=x_1ox_2$.

For each a in the associator $A(M)$, and for each (G, M) extension (E, θ), define a loop E^a as follows: Choose p in E so that $p\theta=a^{-1}$, and q in E so that $pq=1$. Then E^a is the loop E_o given by (15). We define $(E, \theta)^a=(E^a, \theta)$ to be a *conjugate* of (E, θ).

THEOREM 9. *Let* (E, θ) *be a* (G, M) *extension, and let* a, b *be in* $A(M)$. *Then*: (i) E^a *is independent of the choice of* p *in its definition*; (ii) (E^a, θ) *is a* (G, M) *extension*; (iii) $E_1\sim E_2$ *implies* $E_1^a\sim E_2^a$; (iv) $E_1\sim^{-1}E_2$ *implies* $E_1^a\sim^{-1}E_2^a$; (v) $(E^a)^b\sim E^{ab}$; (vi) $(E_1\otimes E_2)^a\sim E_1^a\otimes E_2^a$.

PROOF. (i) In (15), $pq=1$. Clearly we can construct a word W_3, independent of the loop E, so that (15) becomes $e_1oe_2=e_1e_2\cdot W_3(e_1, e_2, p)$. If E is a group, (15) yields $e_1oe_2=(e_1q^{-1})(p^{-1}e_2)=e_1pp^{-1}e_2=e_1e_2$; thus W_3 is p.n.a. Since, in (16), $u=p\theta=a^{-1}$, $v=q\theta=a$, with a in $A(M)$, we have $xoy=xa^{-1}\cdot(a^{-1})^{-1}y=x(a^{-1}\cdot ay)=xy$. Hence $W_3(e_1, e_2, p)\theta = W_3(x_1, x_2, a^{-1})=1$. By Theorem 1, $W_3(e_1, e_2, p)$ lies in G and depends only on x_1, x_2, a.

(ii) Since $xoy=xy$, θ is a homomorphism of E^a upon M. The kernel of θ (in E^a) is the subloop K_o consisting of K under (o). Since $pq=1$ is the unit of E^a, $W_3(1, e, p)=1=W_3(e, 1, p)$ for all e in E^a. Hence, for k in K, $eok=ekW_3(e, k, p)=ekW_3(e, 1, p)=ek$ and $(e_1oe_2)ok=(e_1oe_2)k=e_1e_2W_3(e_1, e_2, p)k=e_1e_2kW_3(e_1, e_2, p)=e_1o(e_2k)=e_1o(e_2ok)$. Similarly $(e_1ok)oe_2=e_1(koe_2)$, $(koe_1)oe_2=ko(e_1oe_2)$, so that K_o is in

$A(E^a)$. The element c of K_o is in $Z(K_o)$ if and only if $cok=koc$, $ck=kc$, c is in $G=Z(K)$. If g_1, g_2 are in G, $g_1og_2=g_1g_2$; thus $G=Z(K_o)$. Finally, for g in G, e in E, $x=e\theta$, $goe=ge=e(gx)=eo(gx)$. This completes the proof that (E^a, θ) is a (G, M) extension.

(v) Assume $E^a=E_o$ is defined by (15), with $p\theta=a^{-1}$, $pq=1$. Then $(E^a)^b=E_o^b$ must be defined, with operation $(*)$, by $e_1*e_2=(e_1T)o(e_2S)$ $=(e_1TR_q^{-1})(e_2SL_p^{-1})$, where $eS^{-1}=soe=(sR_q^{-1})(eL_p^{-1})$, $eT^{-1}=eot$ $=(eR_q^{-1})(tL_p^{-1})$ for s, t in E such that $s\theta=b^{-1}$, $1=sot=(sR_q^{-1})(tL_p^{-1})$. The elements $f=sR_q^{-1}$, $h=tL_p^{-1}$ satisfy $f\theta=b^{-1}a^{-1}$, $fh=1$. Moreover, $SL_p^{-1}=L_f^{-1}$ and $TR_q^{-1}=R_h^{-1}$. Therefore $e_1*e_2=(e_1R_h^{-1})(e_2L_f^{-1})$, showing that $(E^a)^b=E^{ab}$. The proofs of (iii), (iv), (vi) offer no difficulty, hence are omitted.

6. **Central and central Moufang extensions.** For any pair (G, M) we may define the groups $\mathfrak{C}_n$, $\mathfrak{B}_n$ of (normalized) n-cochains and n-coboundaries. By (11), the n-coboundaries for $n=2$, 3 are given by

$$(\delta f_1)(x, y) = (f_1(x)y)f_1(y)f_1(xy)^{-1}, \tag{17}$$

$$(\delta f_2)(x, y, z) = (f_2(x, y)z)f_2(y, z)^{-1}f_2(xy, z)f_2(x, yz)^{-1}. \tag{18}$$

If M is not associative we lose the important property $\delta^2=0$; in particular,

$$(\delta^2 f_1)(x, y, z) = f_1(x\cdot yz)f_1(xy\cdot z)^{-1}. \tag{19}$$

Definition 10. Let f, h be normalized 2-cochains of (G, M). Then f is equivalent to h if $f=h\cdot\delta c$ for some (normalized) 1-cochain c. (Notation: $f\sim h$.)

Definition 11. If f is a normalized 2-cochain of (G, M), then (G, M, f) is the central (G, M) extension (E, θ) defined as follows: (i) The elements of E are the pairs (x, g), x in M, g in G. (ii) (x, g) $=(y, g')$ if and only if $x=y$, $g=g'$. (iii) $(x, g)(y, g')=(xy, f(x, y)$ $\cdot(gy)g')$. (iv) $(x, g)\theta=x$. (v) $(1, g)=g$.

By Definition 6, the unit extension (E_o, θ_o) may be identified with $(G, M, 1)$ where 1 is the identity 2-cochain 1_2.

Theorem 10. (i) *Each central (G, M) extension is equivalent to at least one extension (G, M, f).* (ii) *$(G, M, f)\sim(G, M, h)$ if and only if $f\sim h$.* (iii) *$(G, M, f)\sim^{-1}(G, M, h)$ if and only if $f\sim h^{-1}$.* (iv) *(G, M, f) $\otimes(G, M, h)\sim(G, M, fh)$.* (v) *$(G, M, f)$ is grouplike if and only if $(\delta f)(x, y, z)=1$ for all x, y, z which generate a subgroup of M.* (vi) *For a in $A(M)$, $(G, M, f)^a\sim(G, M, f^a)$ where*

$$f^a(x, y) = f(x, y)\cdot(\delta f)(a^{-1}, a, y)\cdot((\delta f)(xa^{-1}, a, y))^{-1}. \tag{20}$$

Corollary. *The set S' of central (G, M) extensions is an abelian* *group with unit* (E_o, θ_o) *and inverse* $(E, \theta)^{-1}$.

PROOF. (i) Let (E, θ) be a central extension, $u(x)$ a normalized system of representatives of M in G. Since $(u(x)u(y))\theta = xy = u(xy)\theta$, $u(x)u(y) = u(xy)f(x, y)$ for $f(x, y)$ in G. Since $u(1) = 1$, f is a normalized 2-cochain. Every e in E has a unique representation $e = u(x)g$ with g in G, $x = e\theta$. Moreover $u(x)g \cdot u(y)g' = u(x)u(y)(gy)g' = u(xy)f(x, y) \cdot (gy)g'$. Hence the mapping $u(x)g \rightarrow (x, g)$ gives the equivalence of (E, θ) and (G, M, f).

(v) In the notation of (i), consider the equality $u(x)u(y) \cdot u(z) = u(x) \cdot u(y)u(z)$.

(vi) In view of Theorem 9, E^a may be defined by $e_1oe_2 = (e_1R_q^{-1}) \cdot (e_2L_p^{-1})$ where $p = u(a^{-1})$ and $q = u(a)f(a^{-1}, a)^{-1}$. Write $u(x)ou(y) = u(xy)h(x, y)$, so that $h = f^a$. Let $P = (u(xa^{-1})q)o(pu(ay))$. On the one hand, $P = (u(xa^{-1})R_qR_q^{-1})(u(ay)L_pL_p^{-1}) = u(xa^{-1})u(ay) = u(xa^{-1} \cdot ay)f(xa^{-1}, ay) = u(xy)f(xa^{-1}, ay)$. On the other hand, since $u(xa^{-1})q = u(xa^{-1}) \cdot u(a)f(a^{-1}, a)^{-1} = u(x)f(xa^{-1}, a)f(a^{-1}, a)^{-1}$, $pu(ay) = u(a^{-1}) u(ay) = u(y)f(a^{-1}, ay)$ and $(u(x)g)o(u(y)g') = u(xy)h(x, y)(gy)g'$, $P = u(xy)h(x, y)(f(xa^{-1}, a)y)(f(a^{-1}, a)y)^{-1}f(a^{-1}, ay)$. Comparison of the two expressions for P gives $h(x, y) = f(xa^{-1}, ay)(f(a^{-1}, a)y) \cdot (f(xa^{-1}, a)y)^{-1}f(a^{-1}, ay)^{-1}$. However, substitution from (18) in the right-hand side of (20) yields precisely this expression for $h = f^a$.

(ii), (iii), (iv). For $j = 1, 2$, denote the elements of $(E_j, \theta_j) = (G, M, f_j)$ by $(x, g)_j$, where $(x, g)_j\theta_j = x$. Set $u_j(x) = (x, 1)_j$. If π is an isomorphism of E_1 upon E_2 such that $\pi\theta_2 = \theta_1$, then necessarily $u_1(x)\pi = u_2(x)c(x) = (x, c(x))_2$ for a normalized 1-cochain c; and $(x, g)_1\pi = (x, (g\pi)c(x))_2$. Also $g\pi x = gx\pi$. Conversely, if π is any automorphism of G (such that $g\pi x = gx\pi$) and c any normalized 1-cochain, the definition $(x, g)_1\pi = (x, (g\pi)c(x))_2$ extends π to an isomorphism of E_1 upon E_2 such that $\pi\theta_2 = \theta_1$. Direct calculation gives $f_1(x, y)\pi = f_2(x, y) \cdot (\delta c)(x, y)$; (ii), (iii) come by assuming $g\pi = g$, $g\pi = g^{-1}$ respectively. For $E_1 \otimes E_2$ take the representatives $u(x) = (u_1(x), u_2(x))$; Definition 5 gives $u(x)u(y) = (u_1(xy)f_1(x, y), u_2(xy)f_2(x, y)) = u(xy)f_1(x, y)f_2(x, y)$, proving (iv). The corollary should be obvious.

Note that if c is a 1-cochain and if a is in $A(M)$, (19) gives $(\delta^2 c)(xa^{-1}, a, y) = c(xa^{-1} \cdot ay)c(xy)^{-1} = 1$. Thus it is evident from (20) that *the cochain* f^af^{-1} *is invariant under replacement of* f *by an equivalent cochain.* We now turn to Moufang loops.

THEOREM 11. *Let* M *be a Moufang loop. Then*: (i) $xy \cdot zx = x(yz \cdot x)$ *for all* x, y, z *in* M. (ii) $x(y \cdot xz) = (xy \cdot x)z$ *for all* x, y, z *in* M. (iii) *Every loop* M_o *isotopic to* M *is Moufang.* (iv) *The subloop generated by any two elements* x, y *of* M *is a group.* (v) *If the three elements* x, y, z *satisfy*

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$xy\cdot z = x\cdot yz$, they generate an associative subloop. (vi) *The central extension (G, M, f) is Moufang if and only if f satisfies one of the (equivalent) conditions for a Moufang cochain:*

$$f(xy, zx)(f(x, y)zx)f(z, x) = f(x, yz\cdot x)f(yz, x)(f(y, z)x); \tag{21a}$$

$$f(x, y\cdot zx)\cdot(\delta f)(x, y, zx) = f(x, yz\cdot x)\cdot(\delta f)(y, z, x). \tag{21b}$$

(vii) *The central Moufang (G, M) extensions form a subgroup of the group of central extensions.* (viii) *If f is a Moufang cochain,* (20) *simplifies to*

$$f^a(x, y)f(x, y)^{-1} = (\delta f)(xa^{-1}, a, y)^{-1}; \tag{22}$$

in particular, for each a of $A(M)$, the 2-cochain defined by the right side of (22) *is Moufang.*

Proof. Items (i)–(v) are included for reference. For a proof that (i) and (ii) are equivalent, and for (iii), see Bruck [1, Chapter II]. Items (iv), (v) are due to Moufang [1]; see also Bruck [2]. As for (vi), the extension $E=(G, M, f)$ is Moufang if and only if the word B_3 of §3 vanishes on E. Assuming $u(x)u(y) = u(xy)f(x, y)$, the condition $B_3(u(x), u(y), u(z)) = 1$ gives precisely (21a), which, by (18), is equivalent to (21b). (vii) follows from (21) and Theorem 10. As for (viii), the elements $u(x)$, $u(y)$ of the Moufang loop E generate a group, by (iv). Since $u(x)^{-1} = u(x^{-1})g$ for some g in G, the condition $u(x)^{-1}u(x)\cdot u(y) = u(x)^{-1}\cdot u(x)u(y)$ reduces to $(\delta f)(x^{-1}, x, y) = 1$. In particular, (20) becomes (22). Since $E^a\otimes E^{-1}\sim(G, M, f^af^{-1})$, (iii), (vii) imply the concluding statement of (viii).

Theorem 12. *Let M be a finite Moufang loop of order m. Let the least common multiple of the orders of the elements of M be n. For any a in $A(M)$, and for any central Moufang (G, M) extension (E, θ):* (i) *E^a is Moufang.* (ii) *$E^{mn}\sim E_o$.* (iii) *$(E^a\otimes E^{-1})^{2m}$ is grouplike.* (iv) *If M is commutative, E^{2m} is grouplike.* (v) *If n is odd, the exponent $2m$ in* (iii), (iv) *may be replaced by m.* (vi) *If $gx = g$ for all g in G, x in M, $E^m\sim E_o$.*

Proof. (i) reflects Theorem 11 (iii) and was used for (viii). For the proof of (ii)–(vi), take $(E, \theta) = (G, M, f)$ where f satisfies (21). Define the following (normalized) cochains:

$$c(x) = \prod_y f(x, y), \qquad d(x) = \prod_y f(y, x), \tag{23}$$

where the products are taken over the m elements y of M, and

$$h(x, y) = (c(x)y)c(x)^{-1}. \tag{24}$$

From (24),

$$h(x,\ yz) = (h(x,\ y)z)h(x,\ z). \tag{25}$$

This implies

$$h(w,\ xy\cdot z) = h(w,\ x\ yz), \tag{26}$$

since both sides reduce to $(h(w,\ x)yz)(h(w,\ y)z)h(w,\ z)$. If $f_1(x,\ y) = h(y,\ xy)^{-1}$, we take products in (21a) over all y, use (23), (24) and find $f(z,\ x)^m = (\delta d)(z,\ x)\cdot f_1(z,\ x)$, or

$$f^m \sim f_1, \qquad f_1(x,\ y) = h(y,\ xy)^{-1}. \tag{27}$$

If $gx = g$ for all g, x, $h = 1$ by (24) and $f^m \sim 1$ by (27), proving Theorem 12 (vi).

Since $1 = f_1(1,\ y) = h(y,\ y)^{-1}$, or $h(y,\ y) = 1$, (25) implies

$$h(x,\ x) = 1, \qquad h(x,\ xy) = h(x,\ y), \qquad h(x,\ yx) = h(x,\ y)x. \tag{28}$$

Since (21a) applies to f_1, set $z = 1$ and get $f_1(xy,\ x)(f_1(x,\ y)x) = f_1(x,\ yx)\cdot f_1(y,\ x)$. By (27), (28), $f_1(xy,\ x) = h(x,\ xyx)^{-1} = h(x,\ yx)^{-1} = f_1(y,\ x)$, leaving $f_1(x,\ y)x = f_1(x,\ yx) = h(yx,\ xyx)^{-1} = (h(yx,\ x)yx)^{-1}$, or $f_1(x,\ y) = (h(yx,\ x)y)^{-1}$. Thus $h(yx,\ x)y = f_1(x,\ y)^{-1} = h(y,\ xy) = h(y,\ x)y$, $h(yx,\ x) = h(y,\ x)$, or

$$h(xy,\ y) = h(x,\ y). \tag{29}$$

Returning to (21a), take products over all z, getting

$$\prod_z (f(x,\ y)z) = (c(y)x)c(x)c(xy)^{-1} = h(y,\ x)c(x)c(y)c(xy)^{-1}. \tag{30}$$

The left-hand element of (30) remains fixed when we operate with w. Thus, by (24), $(h(y,\ x)w)h(y,\ x)^{-1}h(x,\ w)h(y,\ w)h(xy,\ w)^{-1} = 1$; whence, by (25),

$$h(y,\ xw)h(x,\ w) = h(y,\ x)h(xy,\ w). \tag{31}$$

Set $w = y$ in (31) and use (29). Thus $h(y,\ xy)h(x,\ y) = h(y,\ x)h(xy,\ y) = h(y,\ x)h(x,\ y)$, $h(y,\ xy) = h(y,\ x)$, and

$$h(x,\ yx) = h(x,\ y). \tag{32}$$

In view of (28.3), (32), $h(x,\ y)x = h(x,\ y)$. Hence, by (29), $h(x,\ y)y = h(x,\ y)xy = h(xy,\ y)xy = h(xy,\ y) = h(x,\ y)$. Therefore

$$h(x,\ y)x = h(x,\ y)y = h(x,\ y). \tag{33}$$

From (29) with y replaced by $x^{-1}y$, $h(y,\ x^{-1}y) = h(x,\ x^{-1}y)$. By (32) and (28.2), this implies $h(y,\ x^{-1}) = h(x,\ y)$. Then, by (33), (25),

$h(x, y)h(y, x) = h(y, x^{-1})h(y, x) = (h(y, x^{-1})x)h(y, x) = h(y, x^{-1}x) = h(y, 1) = 1$, or

$$h(y, x)^{-1} = h(x, y). \tag{34}$$

Hence (32), (34) give $f_1(x, y) = h(y, xy)^{-1} = h(y, x)^{-1} = h(x, y)$, so

$$f_1 = h. \tag{35}$$

Since $h(x, y)y = h(x, y)$, a simple induction using (25) gives $h(x, y^j) = h(x, y)^j$. Combining this with (34),

$$h(x^i, y^j) = h(x, y)^{ij} \tag{36}$$

for all integers i, j. In particular, $f_1(x, y)^n = h(x, y)^n = h(x, y^n) = h(x, 1) = 1$, and so $f^{mn} \sim f_1^n = 1$. This proves Theorem 12 (ii).

If $p = \delta f_1 = \delta h$, (18) and (25) combine to give

$$h(xy, z) = h(x, z)h(y, z)p(x, y, z), \qquad p = \delta h. \tag{37}$$

Since h satisfies (21b), (26),

$$p(x, y, zx) = p(y, z, x). \tag{38}$$

Operating on (37) by w, and using (25), we find

$$p(x, y, zw) = (p(x, y, z)w)p(x, y, w). \tag{39}$$

Again, since $h(x, z)z = h(x, z)$, (37) gives $p(x, y, z)z = p(x, y, z)$. Hence, by (38), $p(x, y, zx)x = p(y, z, x) = p(x, y, zx)$, or $p(x, y, z)x = p(x, y, z)$. Thus, finally, $p(x, y, zx)y = p(y, z, x)y = p(y, z, x) = p(x, y, zx)$, and

$$p(x, y, z)w = p(x, y, z), \qquad w = x, y, z. \tag{40}$$

Since $h(xy, x) = h(x, xy)^{-1} = h(x, y)^{-1} = h(y, x)$ and $h(x, x) = 1$, (37) with $z = x$ gives $p(x, y, x) = 1$. Therefore, by (38), (39), (40), $p(x, y, zx) = (p(x, y, z)x)p(x, y, x) = p(x, y, z)$, so that (38) becomes

$$p(x, y, z) = p(y, z, x). \tag{41}$$

By (37), (24), and (25), $p(x, y, z) = h(z, x)h(z, y)h(z, xy)^{-1} = h(z, x) \cdot (h(z, x)y)^{-1}$. Therefore, by (41), (34),

$$p(x, y, z) = h(x, y)(h(x, y)z)^{-1} = p(y, x, z)^{-1}. \tag{42}$$

By this and (37),

$$h(xy, z)h(yx, z)^{-1} = p(x, y, z)^2. \tag{43}$$

Hence, *if M is commutative*, (43) gives $((\delta h)(x, y, z))^2 = 1$ for all x, y, z. In view of (19). the best we can say for $k = f^{2n}$ is that $(\delta k)(x, y, z) = 1$ *for all x, y, z such that $xy \cdot z = x \cdot yz$*. By Theorems 11(v), 10(v), this is precisely the condition that E^{2m} be grouplike. We have proved Theorem 12(iv).

Since $f^m \sim h$ and $p = \delta h$, we see from Theorem 11(viii) that $(E^a \otimes E^{-1})^{2m} \sim (G, M, q)$ where

$$q(x, y) = p(xa^{-1}, a, y)^{-2}. \tag{44}$$

Define the (normalized) 4-cochain r by

$$r(w, x, y, z) = (p(w, x, y)z)p(w, x, y)^{-1}. \tag{45}$$

By (45), r has the skew-symmetry (41), (42) of p on its first three arguments. By (39),

$$p(w, x, yz) = p(w, x, y)p(w, x, z)r(w, x, y, z). \tag{46}$$

By (34), (26), $h(wx \cdot y, z) = h(w \cdot xy, z)$. Expand each side of this last equation by (38), in the form $h(w, z)h(x, z)h(y, z)$. Equate, and use (46) to get $r(y, z, w, x) = r(z, w, x, y)$, whence $r(z, w, y, x) = r(z, w, x, y)$ or

$$r(w, x, y, z) = r(w, x, z, y). \tag{47}$$

By (47) and skew-symmetry, $r(w, x, y, z) = r(w, x, z, y) = r(x, z, w, y)$ $= r(x, z, y, w) = r(y, x, z, w) = r(y, x, w, z) = r(w, x, y, z)^{-1}$, or

$$r(w, x, y, z)^2 = 1. \tag{48}$$

From (44), (46), (48), $q(x, y)^{-1} = p(a, y, xa^{-1})^2 = p(a, y, x)^2 p(a, y, a^{-1})^2$. Since $q(1, y) = 1$, the second factor is 1, and, by (42),

$$q(x, y)^{-1} = p(x, y, a)^2 = h(x, y)^2(h(x, y)a)^{-2}. \tag{49}$$

Therefore, since $p = \delta h$, $(\delta q)(x, y, z)^{-1} = p(x, y, z)^2(p(x, y, z)a)^{-2}$. Hence, by (45), (48), $(\delta q)(x, y, z) = 1$ for all x, y, z. This proves Theorem 12 (iii).

As for (v), since $h^n = 1$, (37) gives $p^n = 1$ and then (45) gives $r^n = 1$. However, $r^2 = 1$, by (48). Hence, if n is odd, $r = 1$ and (iii) holds with $2m$ replaced by m. A similar remark is true of (iv). This completes the proof of Theorem 12.

Theorem 12 should be compared with the simpler result for groups (Marshall Hall [1]): *If M is a group of order m and if (E, θ) is a central associative (G, M) extension, $E^m \sim E_0$.*

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PROBLEMS OF SENSORY PROSTHESIS

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This is an age in which a great deal is being done to help the handicapped. There is an improvement in artificial limbs for the maimed, in new sorts of spectacles, and in hearing aids. All this is returning to useful life a considerable part of those who would in other generations have been hopelessly handicapped. Even the two greatest handicaps from which humanity suffers—blindness and deafness—are the object of powerful and determined attacks with the object that where these defects cannot be removed or corrected by medical means, these victims may still be allowed to take, as nearly as possible, a normal part in life.

Notice that these two afflictions are sensory defects, that is, defects in the reception of impressions from the outer universe and from other human beings. From the standpoint of the outer universe, blindness is overwhelmingly the greater of the two losses. The sense of hearing, as brought out by such investigations as have been made by the Bell Telephone Company in their study of speech, and as further evidenced by the size of the speech areas in the brain, is a sense with vastly less variety than the sense of sight. On the other hand, normal communication between man and man goes by mouth and ear much more than by all other channels, and the social and emotional damage done by deafness is disproportionately great when one compares it with the social loss of the blind. The typical emotional picture of the deaf man is that of a reserved, self-contained, neurotic personality, whereas the typical picture of the blind man who has achieved any degree whatever of equilibrium is that of a euphorically confident and rather cocky personality.

In order to make any effort to replace the lost sense, either of the blind or of the deaf, it is necessary for us to make a rather accurate measure of what they have lost. It is then necessary for us to see if there are any relatively unused channels into the human nervous system which are capable of supplying the whole or any considerable part of what is lost. These are problems of physiology and communication engineering, and have a most important mathematical side.

To bring out this mathematical side adequately, I wish to refer to

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communication mechanisms which are more accessible to us in their whole than the human nervous system. A very interesting type of machine which is engaged in continual conversation with its surroundings is the remotely controlled power station. It is not unusual for water power to be available under just those circumstances which make the site highly unsuitable for human habitation. Under these conditions, a power station may be built without any resident engineer or operative. The place of such a resident operative is taken in part by a visiting operative who packs in at long intervals to replace and repair vital parts, but who is much more effectively replaced by automatic operation and long distance control.

In the automatic operation of a power station, one meets much the same problems which one finds in the operation of a railroad switching and signal system. There are certain things which must not be done at all costs. No generator should be connected with the bus bars until it is brought up to speed, and is running in the proper phase relations with the others. A penstock must not be thrown suddenly open in the lower end, or it will collapse under atmospheric pressure; and, on the other hand, it must not be subjected to the shock of too sudden turning on of water, or it will burst. These contingencies must be blocked in the same way as the conjunction of a closed switch and an open signal must be blocked in railroad practice.

These are relatively simple undertakings. As stated, they seem scarcely deep enough to have a mathematical form. However, behind them is a principle which is well susceptible to mathematical treatment. The principle, namely, is that the performance of a piece of apparatus should be fed back to it as information on which it is to operate.

What is more interesting to us mathematically at present, however, is not these purely local devices which modify the nature of the terminal apparatus in the generating station, and may be considered as part of the operation itself, but rather the messages which go between the load dispatcher and the station. Such messages must exist. A little cloud passes over the city to which the power is sent, and thousands of lights go on in the offices, shops and homes. This immediately demands the sending of more power; and this in turn means the starting of new turbines, the switching in of new generators, and a thousand and one other operations. Let there be an accident in the remote generating station, and even if this accident can be compensated for temporarily by automatic machinery, a message must go out to the plant engineer to send in a troubleshooter to restore the plant to maximum efficiency. In other words, the automatic substation must listen, and it must speak. Since, however, its ears and its mouth have been made by engineers for these specific purposes, and not installed by a complicated embryological process, we know far better what to expect of them than we do of human mouths and ears.

It may be remarked in passing that it is quite common practice to transmit these messages to and from the power station either over ordinary telephone lines and similar facilities, or over a carrier sent over the power line itself. In either case, they are generally coded messages like those of teletypewriters. If we wish to go to the trouble to make them vocal, all of the resources of the telephone and phonograph are at our disposal, but there is generally no particular point in this rather precious imitation of human function.

What the casual listener may miss in this account is the fact that the operation of such a system involves several stages of linguistic translation. The message from the dispatcher to the station is or should be committed to the line in the form best suitable for line transmission. On the other hand, it must appear in the station, not in the form of a coded sequence of dots and dashes, but in the form of a series of openings of switches, turnings on of valves, measurements of the speeds of motors or generators, and so on. The language of command must be translated into the language of action.

To understand this, we must realize what information is, and how it can be measured. When I send a series of dots and dashes, the problem of the information that it carries cannot be answered in terms of the message I send, taken by itself apart from all other messages. A single click may be able to do a large number of things, such as opening a sluice, connecting a generator to the bus bars, or replacing a burnt out electronic valve; but it cannot do these things *differentially*. It is only when I know the number of separate alternatives which must be transmitted and combined that I can determine how many different messages I must be able to send. It then appears that the significance of a signal is determined not only by that signal itself, but by the whole set of signals I might have sent in place of the actual one. In other words, significance is a property belonging to ensembles.

Another property belonging to larger and smaller ensembles is that of probability, or what is its equivalent, their Lebesgue measure. I do not intend to go into the whole question of the theoretical basis of probability, but shall only state that, under certain rather wide conditions, a smaller set of signals has a probability when taken to be part of a larger set, and that this probability is a number between 0 and 1. Now, the probability that two independent events should happen simultaneously is the *product* of their individual probabilities. On the other hand, if information is to have a significance even remotely resembling its ordinary meaning, the amount of information given by two independent signals should be the *sum* of the amount of information which they convey separately. Under these conditions, since multiplication of probabilities corresponds to addition of information, the amount of information given by a signal can only be a constant multiple of the logarithm of its probability. Moreover, since the information is positive when the probability is less than one, the ratio between the logarithm of the probability and the amount of information must be negative.

The notion of the logarithms of probability is a familiar one for statistical mechanics. The logarithm of a probability is called *entropy*. In an ordinary statistical system, the entropy can increase spontaneously, but can never so decrease. In the transmission of information, we may lose information, but can never gain it, except from a new source. Thus in the translation which occurs between the message sent into the automatic station, and the operation of that station, there may be a loss of information, but there cannot be a gain. Under ideal circumstances, we attempt a coding which will correspond as nearly as possible to the actual operation of the power station, so that we are not sending in unused material, as we should for example if we were to transmit our orders by voice. Nevertheless, it is hardly to be expected that our code is so perfect that we lose no information.

Let us go from this problem to the problem of transmission of a message by voice or by eye from person to person. There is in the first instance a limitation of the rate at which information can be transmitted. It depends on the system of repeatable and recognizable signals which can be made with our vocal organs or our pen, or whatever other means we use of active communication, and which will be accepted by the air, or whatever other channel of transmission we use. I say recognizable, but I have already introduced an ambiguity thereby. Recognizable how, and by what? If I allow the recipient the full choice of any detecting and analyzing instrument which he may choose to demand, I get a certain rate of transmission of information, which is characteristic of the sender and of the means of transmission but which is not characteristic of the reception of this message by the ordinary listener or viewer. It is information of this type with which the communication engineer has been chiefly concerned in the past, as it is by all means the simplest to handle. However, when this information penetrates through the ear or the eye into those parts of the human nervous system which are specifically concerned with sound or sight, it undergoes a translation by a mechanism of which we know something, but not everything. As a matter of fact, it undergoes two or more translations. The first of these represent what is necessary to penetrate into the human sense organ, and through that sense organ to those parts of the brain which are in more or less permanent connection with it. The later translations interpret this superficial sensory representation into meaning, and give to what is received a more or less permanent place in memory and understanding. In the case of sound, the first translation is called *phonetic* and the later translations are termed *semantic.*

The problems of the loss of information in the two human translations, phonetic and semantic, are not at present susceptible to a direct investigation through the structure of the nervous system. They are however susceptible to an investigation which is empirical and experimental. Work of this sort has been done by the Bell Telephone Company, with special reference to the recognizability of sound as speech. The original measures of information, which were given more or less crudely, referred to the bands of frequency necessary for the formation of intelligible speech; and in other words were almost more measures of what the outside line and the air could carry, than of what the ear could translate. The reconstruction of intelligible speech from clicks and buzzes, by way of wave filters and resonators and other apparatus of the sort, shows that the loss of information between the ear and the part of the nervous system belonging to the ear is overwhelmingly great, and that a crude imitation of speech involving less than 1/10 and perhaps even 1/100 of the information carried over a telephone line, is enough for adequate conversation.

It was in view of this work that Dr. Wiesner of the Electronics Laboratory of MIT discussed with me the question as to whether anything could be done for a totally deaf man, to replace the missing sense of hearing in its most important employment, that of the recognition of conversation. The question was put to me by Dr. Wiesner, and the two of us were independently led to the same answer: namely, that the recognizable part of speech was so scanty, that it was not beyond our hope to replace it by the sense of touch backed up with adequate equipment. In other words, it appeared to us that it is perfectly possible to make the transition between sound in the outside air, and the semantic recognition of speech, by an artificial phonetic stage, making use of touch, and supplemented by adequate electrical tools. Having come to this conclusion, we started the detailed design of such apparatus. We took speech through an ordinary microphone.

not dissimilar to that used as a receiver for the usual hearing aids, and quite adequate to take in all that is intelligible in speech. We then split the band received into a number of frequency bands, making use of wave filters. As we intended to use the hand as a receiver, five seemed the natural number of bands to take, because we could then carry them to the five fingers. However, we are not yet finally decided that five is the right number and may experiment on supplementing it by additional stations in the palm of the hand, so as to bring the number up to six or seven. The different bands we have so far taken to be of equal logarithmic length, or length when measured in terms of the number of keys of the piano corresponding to them. Whether this is the best mode of speech, we are not yet sure. It is a subject for further experiment.

At any rate, after receiving and separating these bands we rectify them, which means, nearly, that we replace the time-patterns of that band by its envelope. These rectified messages are then used to modulate an equal number of vibrators at frequencies of something like 100 cycles each. This is because the skin is very insensitive, both to nearby constant messages and to messages of excessively high frequency. These modulated messages are then carried either to electrical mechanical vibrators placed on the five fingers, or to electrodes designed for the direct stimulation of the nerve endings in the skin.

From the standpoint of lightness of weight, it would seem that a direct electrical stimulation is desirable. However, there are several difficulties to be overcome. The variable resistances of the skin, and the fact that current rather than voltage is what stimulates touch, both indicate that it is desirable to use constant current electrodes, which can be properly designed. However, even with these, the margin between the lower threshold of feeling and the lower threshold of pain is so small that our apparatus must work over a very narrow range, and our present models need considerable modification.

The mechanical vibrator, on the other hand, tends to be undesirable, because its current demand is at present so great as to require an apparatus which is still too heavy to be comfortably portable. We believe that whether we finally arrive at mechanical or electrical stimulators, the problem of making the apparatus portable, and indeed of comparable weight with that actually used in existing hearing aids, is not too difficult to be solved, and is more one of development than anything else.

Let me first state how our results stood when we left off work early in the summer. This work was undertaken by Dr. Leon Levine, with the cooperation of Dr. Alex Bavelas of our Psychology Department in matters which concerned techniques of teaching. It was desirable to see if there was in fact a sufficient basis of recognizable and distinguishable syllables in speech for us to go ahead. In the beginning, we did not experiment with the deaf, but with hearing people whose ears were filled with such a jumble of artificial noise that they were quite completely deaf for the purposes of the experiment. We then gave them a sequence of syllables by the machine accompanied with a visual pattern display of these syllables. After they had learned these, we subjected them to a series of trial syllables selected from the sequence; and we recorded the successes and failures. The best run we had was a series of 80 selections among 12 syllables with only six errors.

It has been our idea from the beginning that the learning process necessary for achieving proficiency with the new method of receiving speech must be something quite long and arduous, and we have not wished to involve any deaf mutes in an experiment which might give them a false hope, or at any rate a bad learning technique. However, we did make a certain number of experiments with a blind deaf mute, who came to us with his brother. The two had learned to communicate together by speech, which was recognized by the deaf mute when he placed his hands on his brother's larynx. The deaf mute could himself enunciate a few words, but the enunciation was very breathy and bad. After he had been put in the apparatus, he was for the first time in his life able to compare his brother's speech with his own, particularly in the enunciation of his own name and that of his brother. Within a matter of minutes, and certainly of hours, the improvement in his enunciation was not merely noticeable, but overwhelming. It was perfectly clear to us that our apparatus is an adequate way for a deaf mute, even for a blind deaf mute, to monitor his own speech.

I want to say at this point that soon after we had begun our work, we recognized very definitely that it was closely related to the visual speech developed by the Bell Telephone Company. I also wish to say that the patents which the Bell Telephone Company has based on this work were so phrased as to cover the work we had done, although our methods had not been actually employed experimentally by the Bell workers. The advantage which we believe our methods possess over visual speech, in addition to the obvious one that they can be used by blind deaf mutes, are that we have been working for portability in the apparatus. It was our conviction before we started, and it has been very greatly confirmed by our results, that the apparatus which we finally achieve should be completely portable, and should not deprive one of the concurrent use of such an important sense as vision. It is only by this portability that the speaker's words can be continually monitored, so that he is never allowed to fall into the faulty speech of the congenital deaf mute. We hope that, with the use of this apparatus, the speech-training of the deaf mute can be begun in early childhood, and that although it will be arduous training, it may not at that period be too different from that training which hearing children receive in learning to speak.

It will be a curious study to see what these hearing aids will do for the musical sense. They were not designed for this purpose, and I do not expect any very great results. Still, people like Helen Keller have reached a surprising degree of proficiency in recognizing musical vibrations with their fingers, and it is scarcely to be expected that the results with these machines will be worse than with the bare fingers. All in all, however, the social function of hearing is so nearly exclusively a matter of the reception of speech, that a person who can follow speech in the basis of sound carried by the air, and can do this with reasonable proficiency, can scarcely be considered socially deaf.

It is a little unfortunate that we have to use so valuable a part of the body as the hand for a receiver. We intend to use the left hand alone, and even as far as this is concerned, we intend to hold the receivers in a flexible glove, so that a gross use of the grasp by the hand is not incompatible with the simultaneous reception of speech. The trouble with the other sensitive receiving parts of the body is that the best of them lie around the mouth which must be left without impediment to participate in conversation.

It is thus conceivable at least to us that deafness as a considerable social problem might even be eliminated, except for those not mentally able to face the stringent training needed, or for those with serious concomitant physical defects. We have no such expectation for the replacement of vision.

There is reason to believe that the total amount of information received at a given time by a normal eye is of the order of 50 to 100 times what is received in the same time by a normal ear. In this, the type of measurement of vision as given by the oculist's ratio is deceptive. Vision which is 50/100 means ability to recognize details at 50 feet which would ordinarily be recognized at 100. However, the number of details varies as the *square* of the distance, so that from this point of view 30/100 vision should give about a quarter of the amount of possible information which normal vision gives. On this basis, if the whole auditory region of the cortex were turned over to the purposes of vision, we should expect about 10/100 vision. This is not blindness, but very poor vision indeed. Touch is an even worse surrogate for vision. Under these circumstances, it scarcely seems worth while to try to make a universal apparatus to replace vision for all purposes.

There are two directions in which progress should be expected. One of these is a reading apparatus, which will translate the pages of an ordinary book into a sound or touch pattern recognizable by the blind. The other is an aid whereby the blind man may go around more freely out of doors and in unfamiliar rooms. In the construction of an apparatus of this latter sort, there is a limitation below as well as a limitation above. Any apparatus must be better than what the blind man can do for himself by using such odds and ends of sensation as auditory echo, air temperature, air pressure of the forehead, and so forth.

Finally, besides the problem of the blind and the deaf, the problem of the maimed is also one with a profound sensory aspect which is often neglected. The better an artificial limb is from the standpoint of variability of performance and flexibility, the more information as to its position and as to the strains on it which the cripple needs if he is to walk securely. Similar remarks apply to artificial hands. There is a very considerable future in the art of applying strain and pressure gauges to artificial limbs, in such a way as to furnish the cripple with better sensory monitoring than he can obtain with the aid of anything but the skin of his stump. What the cripple has lost is not only motion, but sensation as well. He is an ataxic, as well as a paralytic, and anything that can be done to furnish him with sensory feedback channels other than those which he normally possesses is likely to make a great contribution to his effectiveness and to his well-being.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SOME QUESTIONS CONCERNING ALTERNATIVE RINGS

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1. **Introduction.** Our purpose is to summarize our present knowledge of alternative rings in the case in which no a priori finiteness assumptions are made and to indicate a number of problems in this field. One defines an alternative ring by replacing the law $a(bc)=(ab)c$ in the definition of an associative ring by the laws $a(ab)=a^2b$ and $(ab)b=ab^2$. The name is derived from the fact that the *associator* $(a, b, c)=(ab)c-a(bc)$ is an alternating function of its arguments. The name as well as much of our knowledge of the finite-dimensional case is due to M. Zorn [52–55],[1] although N. Jacobson [25], A. A. Albert [1], R. D. Schafer [41–43], and Dubisch and Perlis [19] have also contributed.

We shall use the terms *ring* and *algebra* in place of *nonassociative ring* and *nonassociative algebra*. If the nonzero elements of a ring form a *loop* [4] under multiplication (that is, if each pair of elements in the equation $ab=c$ uniquely determines the remaining element and a unit element 1 is present), we call the ring a *division ring* [cf. **2**; **16**]. The *center* [**2**; **26**] of a ring A consists of those elements c in A for which $cx=xc$ and $(cx)y=c(xy)=x(cy)$ for every x and y in A. When the center of A is a division ring, then A is a vector space over its center and we call the dimension of this vector space the *dimension* of A.

We divide our discussion into three parts wherein the primary interest is geometric, algebraic, and topological, respectively.

2. **Geometry.** Ruth Moufang [33–38] was the first to derive the geometric meaning of the alternative law as a weak form of Desargues' Theorem in plane projective geometry. Marshall Hall, Jr. [22–23] has given a new proof which is mainly algebraic and which avoids assumptions concerning order or characteristic in the plane. In its affine form, the theorem used by Hall merely asserts that *if corresponding vertices of two triangles are on parallel lines, while two pairs of corresponding sides are parallel, then the remaining sides are also parallel.* The works of Moufang and of Hall suggest the following questions.

1. Is there an ordered alternative division ring which is not associative? (Moufang [38])

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[1] Numbers in brackets denote references given at the end of the paper.

2. Is every isotope [2] with unit of an alternative division ring really an isomorph? (Hall [22])

3. Is there a division ring for which the identity $a(ab)=a^2b$ holds but $(ab)b=ab^2$ fails?

These questions have been answered by Zorn [53], Schafer [41], and Albert [9], respectively no, yes, and no,[2] in the finite-dimensional case. R. H. Bruck has shown that a natural generalization of Hilbert's procedure fails to answer the first question [16]. Albert[3] has remarked that the analogous question for Jordan rings [6] has a very easy answer for we may set $2(x\cdot y)=xy+yx$ in any ordered associative but not commutative ring A to obtain an ordered Jordan ring A_+, provided that the positive elements of A_+ coincide with those of A.

Since configuration theorems in plane projective geometry are reflected in algebraic identities in the coördinate ring, we are led to consider these next. Kaplansky [29] has given a sweeping generalization of a result of Hall [22] by showing that a primitive ring [27] which satisfies a polynomial identity is finite-dimensional. An important question (no. 4) is how this may be generalized to alternative rings. For the case of an alternative algebraic division algebra of degree two, Albert [8] has given an affirmative answer. (As noted by Jacobson [29], every alternative algebraic algebra of bounded degree satisfies a polynomial identity.) We have given [49] an independent proof of Albert's result (including the case of characteristic two, avoided by Albert), starting from the appropriate identity $[[x, y]^2, z]=0$ of Hall [22]. The identity $xy=yx$ in an alternative division ring implies the associative law,[4] as Bruck has proved by a simple algebraic argument [46], and whose geometric counterpart in the form of the Theorem of Pappus and the weakened Desargues Theorem seems to be well known [40]. In fact, a commutative alternative ring with zero radical (see §3) is associative, but commutative alternative but not associative rings exist [47]. Nothing further seems to be known about this question. The identity attributed by Kaplansky to Kolchin [29; cf. 51, p. 98] for associative algebras of finite dimension applies to algebras if we use a fixed association in each of its terms. This indicates that an algebra of finite dimension is not *completely* nonassociative since, for example, algebras of dimension two satisfy the identity $x(yz)+y(zx)+z(xy)-x(zy)-y(xz)-z(yx)=0$. A similar remark applies to an arbitrary binary operation defined on a *finite* set.

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[2] At least when the characteristic is zero.

[3] This remark of Albert was mentioned by the discussant, Saunders MacLane.

[4] It is interesting to note that M. H. A. Newman (*A characterization of Boolean algebras and rings*, J. London Math. Soc. vol. 16 (1941) pp. 256–272, and *Relatively complemented algebras*, J. London Math. Soc. vol. 17 (1942) pp. 34–47) observed that the rule $(ab)b=a(bb)$ was sufficient to ensure the associativity of his generalized Boolean rings.

Let us finally mention the following extension of a theorem of H. Cartan [18] which is due to R. Brauer [12]. *If B is a normal division subring of a division ring A, then $B=A$ or B is contained in the center of A.* We use the term *normal* as it applies to the *subloop* of nonzero elements of B [4; 5; 11; 17; 21; 45] and *center* as defined in §1. The additional[5] requirements on the elements of B, namely, that $b(xy)=(bx)y=y(by)$ for every $x, y\in A$ and $b\in B$ when $B\neq A$, are easily proved using Brauer's method. L. K. Hua [24] has also given an elementary proof of Cartan's theorem which is not greatly different from Brauer's. However, it is interesting to observe that several of the related results stated by Hua in [24] are also valid in alternative division rings. In particular, *the conjugates $x^{-1}ax$ of an element a of an alternative division ring A are all equal to a or they generate A.* This follows from Hua's proof and the fact that $3(a, x, y)=(a, x, y)-(x, a, y)+(x, y, a)=0$ for every $x, y\in A$, provided that $az=za$ for every $z\in A$. When $3\neq 0$, the center of A coincides with the set of all a satisfying $az=za$ for every $z\in A$. But when $3=0$, this question is unanswered, and we may assert Theorems 1–8 of Hua [24] for alternative division rings only if we replace the word "center" by the words "the set of elements $a\in A$ for which $az=za$ for every $z\in A$."

3. **Algebra.** The structure theory of associative rings presented[6] by Jacobson [26–27; 10] is so elegant that its generalization to rings has attracted the interest of several writers. In fact, Dubisch and Perlis were aware in 1943 (prior to the publication of this theory) that the set of all properly nilpotent elements of an alternative algebra (that is, the *radical* in the sense of Zorn [55]) could be characterized as what is now called the Perlis-Jacobson radical:[7]

$$R = [a;\ b \in (a)_r \text{ implies that } (1 - b)A = A].$$

Our observation that the proof of Forsythe and McCoy [20] that an associative regular ring without nonzero nilpotent elements is a subdirect sum [32] of associative division rings is easily extendable to alternative rings [46] led us to study this problem and we were able to show that the Perlis-Jacobson radical of an alternative ring is an ideal [47]. Brown and McCoy offered a variant of the Perlis-Jacobson radical [14] for associative rings which we have shown to be applicable to arbitrary rings [48] and which reduces to Zorn's radical of an alternative ring when his chain conditions hold [55].

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[5] That is, those not already proved in [12].

[6] Cf. J. Dieudonné, *Sur le socle d'un anneau et les anneaux simple infinis*, Bull. Soc. Math. France vol. 70 (1942) pp. 46–75.

[7] Dubisch and Perlis [19]. In this section we shall use $(a)_r$ and (a) to denote the right ideal and the ideal, respectively, generated by a. We shall also use the abbreviation $(1-b)A$ for the set of elements $[x-bx; x\in A]$.

A connection of the Perlis-Jacobson radical of an alternative ring with a subdirect decomposition [32] of the alternative ring is still lacking in the general case, although we have succeeded in exhibiting such a connection for an alternative ring A (which we call *special*) in which the set $(1-a)A$ is a right ideal of A for every a in A [50]. Under this assumption, we may prove as in the associative case that $R\geqq\bigcap(M:A)$, where M ranges over the modular[8] maximal right ideals of A and $(M:A)=[a; xa\in I$ for every $x\in A]$ is the *quotient ideal* of M. To secure $R\leqq\bigcap(M:A)$, we consider an element a not in $\bigcap(M:A)$ and hence not in some modular maximal right ideal M. Then A is the sum of M and $(a)_r$. In particular, the left unit e of A modulo M has the form $e=m+b$, with m in M and b in $(a)_r$. But then $(1-b)x=x-(e-m)x=x-ex+mx$ is in M for every x in A and a cannot be in R. Thus $R=0$ entails $\bigcap(M:A)=0$ and a consequent representation of A as a subdirect sum of special alternative rings which are *primitive* in Jacobson's sense; each summand possesses a modular maximal right ideal whose quotient ideal is zero. One may easily prove the converse statement that a subdirect sum of primitive special alternative rings has zero Perlis-Jacobson radical.

Brown and McCoy have discovered a generalization of the Perlis-Jacobson radical which applies to arbitrary rings [15] and which Brown has associated with a subdirect decomposition of the ring[9] [13]. They define N_r as the set of all elements a of a ring A for which every element b in (a) is such that the right ideal generated by the set $(1-b)A$ is A. They then show that N_r is an ideal of A. Brown [13] replaces Jacobon's quotient $(I:A)$ of a right ideal I of A by the largest ideal I' contained in A. It should be noted that for a modular right ideal I, we have $I'=(I:A)$ provided that $(I:A)$ is an *ideal* of A. Brown then proves that $N_r=\bigcap M'$ as M ranges over the modular maximal right ideals of A by a slight modification of the argument of the preceding paragraph. This argument yields another proof that N_r is an ideal. The result also shows that $R=N_r$ for special alternative rings, but the question of the validity of this equation for arbitrary alternative rings is open. When $N_r=0$, we see that A is a subdirect sum of rings each of which is (right-) primitive in the sense that it contains a modular maximal right ideal M with $M'=0$. The duality of the Perlis-Jacobson radical of an alternative ring with respect to interchange of right and left multiplications is lost in this process of generalization as a simple example shows. By use of the theory of normal subloops of a loop [**5**; **11**; **17**; **45**], we may dispense with the associativity (as well as the commutativity) of addition[10] provided that we interpret $(1-b)A$ as the set $[x+by; x\in A, x+y=0]$. The distributive laws and associativity of addition enforce, however, the commutativity of addition when $N_r=0$. It is possible to phrase this theory of Brown and McCoy in the setting of groups with operators [**15**]. A consequence of this more general formulation is the fact, proved by Brown and McCoy in [**15**], that every alternative ring has a greatest regular ideal.[11]

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[8] A right ideal I is *modular* in case there is an element $e\in A$ such that $(1-e)A\leqq I$, that is, e is a left unit for A modulo I. Segal [44] uses the term *regular*, but we prefer the present terminology of Jacobson.

[9] We are indebted to Brown and to McCoy for a stimulating correspondence in the source of which we were informed of some of their most recent results.

The results of Brown and McCoy and of Brown suggest a host of questions which will readily occur to the reader. One should note especially that nil rings are automatically placed in the limbo of radical rings. The vital question (no. 5) seems to be: "What is the nature of primitive alternative rings?" For associative rings, Jacobson shows that a primitive ring is a dense ring of linear transformations in a vector space over a division ring. All that seems to be known in the alternative case is that if the modular maximal right ideal M contains every associator, then A is associative.

One of the main difficulties in the study of alternative rings is the lack of some sort of representation closely connected with an associative ring. Saunders MacLane has suggested (in conversation) that there is a cohomology theory of alternative rings (question no. 6) and perhaps such a theory might be helpful in the study of primitive alternative rings.

In concluding this section, we mention the so-called Theorem of Artin which states that every two elements of an alternative ring generate an associative subring [**52**]. R. Moufang proved a generalization for alternative division rings: if $(a, b, c)=0$, then a, b, c generate an associative division subring [**39**]. The analogous statement is true in an arbitrary alternative ring [**47**], and no doubt still more general formulations are possible.[12]

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[10] It is possible to have a finite system of this kind in which the nonzero elements form a commutative group under multiplication but in which addition is not associative. (See *Neofields* by L. J. Paige, Duke Math. J. vol. 16 (1949) pp. 39–60.)

[11] A subset S of an alternative ring A is *regular* (in von Neumann's sense) in case every $a\in S$ has the form axa with $x\in A$.

[12] (Added July 12, 1950.) Since this was written, R. H. Bruck has shown by a remarkably simple argument that a subset S of an alternative ring A which satisfies

4. **Topology.** We turn finally to topology as a method of studying alternative rings. Here the main results are those of Albert [7–8] and of Jacobson and Taussky [28]. Albert assumes that his ring A is algebraic relative to the real field and has a homogeneous valuation. He then shows that A is alternative and applies his result, mentioned in §2, to show that A is associative (and hence already characterized) or the Cayley numbers. In his discussion of the results of Jacobson and Taussky, Kaplansky [31] also obtains the Cayley numbers as the only not associative alternative division ring which is both connected and locally connected, and he conjectures that a similar result holds in the totally disconnected, locally compact case. It seems likely that topological methods will yield further results. In fact, many of the preliminary results of Kaplansky in [30] are valid at least for special alternative rings. A difficulty appears in the proof that group neighborhoods contain ideal neighborhoods in the bounded case. We may obtain this result if we strengthen the definition of (right-) bounded ring A to: "For every neighborhood U of 0, there is a neighborhood V such that $V\rho \leqq U$ for every product ρ of finitely many right multiplications of A." (This is automatically satisfied in a right bounded associative ring.) But then it seems difficult to prove that a compact ring is bounded. However, we can show that *a compact and bounded alternative ring with zero Perlis-Jacobson radical is isomorphic and homeomorphic to a Cartesian direct sum of finite simple alternative rings.* This subsumes the corresponding result of Kaplansky in [30]. By a result of Zorn [52], we see that the only not associative summands permitted are merely finite Cayley-Dickson algebras with divisors of zero.

Added in proof, (January 5, 1951). At the International Congress of Mathematicians 1950, R. H. Bruck announced that he and Erwin Kleinfeld have proved that an alternative division ring in which $2\neq 0$ is either associative or is a Cayley-Dickson algebra over its center. This outstanding result answers a number of questions suggested in this paper. Thus, aside from the limitation on the characteristic, Questions 1 and 2, as well as the conjecture of Kaplansky mentioned in §4, are settled.

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$(x, y, z)=0$ for $x, y, z\in S$ generates an associative subring provided that A is commutative or satisfies "$3(a, b, c)=0$ for $a, b, c\in A$ implies that $(a, b, c)=0$." We have given an example of an alternative algebra A of dimension 65 over the field of three elements which is generated by a set S of *four* elements and which is not associative although $(x, y, z)=0$ for $x, y, z\in S$.

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ON THE DIFFERENTIAL GEOMETRY OF CLOSED SPACE CURVES

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1. The subject of this lecture, the study of the differential geometric properties of a space curve which depend on the assumption that the curve is closed, is a rather modest chapter of the differential geometry in the large. The results are often comparatively elementary and seem to be isolated. On the other hand, the intuitive character of the statements and the lack of a general method of approach make the field attractive, and in the latest years several authors have contributed to it. Therefore, it might be justified to give a survey of its actual state. In an attempt to gather some results and problems from a common point of view it turned out that the differential geometry of space curves becomes more satisfactory if it is developed under slightly weaker assumptions than those usually adopted. Furthermore it proves to be useful to attach more importance to the simple geometrical relations between the spherical indicatrices of a curve and to the kinematical interpretation initiated by G. Darboux. Though many of the results are or can be generalized to curves in a euclidean space of arbitrary dimension, the following exposition is restricted to the case of the ordinary space, and it will not be possible to mention all contributions to the subject.

2. Let s, $0 \leqq s \leqq l$, denote the arc length and $\boldsymbol{r}(s)$ the position vector of a variable point of a space curve K. Differentiation with respect to s is indicated by a prime. Our assumptions are the following: The coordinates of $\boldsymbol{r}(s)$ are functions of class 4. To every point $\boldsymbol{r}(s)$ is attached an osculating plane, that is, a plane containing the vectors $\boldsymbol{t}=\boldsymbol{r}'$ and $\boldsymbol{r}''$ such that its suitably oriented normal unit vector $\boldsymbol{b}(s)$, the binormal vector of K, has coordinates which are functions of class 2.[1] The vectors $\boldsymbol{t}'$ and $\boldsymbol{b}'$ do not vanish simultaneously. For the sake of simplicity we further assume that they vanish only at a finite number of points. This implies that no arc of K is contained in a plane.

By this formulation we avoid the usual assumption that $\boldsymbol{r}''$ and,

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[1] Actually we are given an osculating strip in the sense of Blaschke [1, p. 72]. (Numbers in brackets refer to the references at the end of the paper.)

hence, the curvature never vanish. Define now the principal normal vector by

$$n = b \times t \tag{1}$$

($\times$ denoting the vector product) and the curvature κ and the torsion τ by

$$t' = \kappa n, \qquad b' = -\tau n \tag{2}$$

which is possible since these derivatives are perpendicular to both t and b. From the definition of n follows

$$n' = -\kappa t + \tau b. \tag{3}$$

Hence Frenet's formulae are valid in the usual form, but here the curvature may vanish and even be negative. Of course, only a change of the sign of κ, not the sign itself, has a geometrical meaning; for, b (for all s) and, hence, n may be replaced by the opposite vectors. From the above assumptions we have

$$\omega = (\kappa^2 + \tau^2)^{1/2} > 0.$$

A point of K where either κ or τ changes sign is called a point of inflection. If necessary we distinguish between κ-inflections and τ-inflections. The number of these points is finite.

3. If the trihedral t, b, n is laid off from the origin O, the vectors t, b, and n describe three curves T, B, and N, the spherical indicatrices of the tangent, of the binormal, and of the principal normal of K. With s considered as the time, the motion of the trihedral is a rotation about O with the angular velocity $\omega = \omega(s)$ and an instantaneous axis of rotation which must lie in the plane b, t, because it is perpendicular to the velocity vectors (2). Let c be the unit vector on this axis such that its sense together with the sense of the rotation form a right-handed screw. Since c is perpendicular to the velocity vector (3) we then have

$$\omega c = n \times n' = \kappa b + \tau t. \tag{4}$$

Let $\phi = \phi(s)$ denote the angle from b to c measured in the sense of the shortest rotation which brings b into t. Then we have

$$\kappa = \omega \cos \phi, \qquad \tau = \omega \sin \phi.$$

Besides the fixed unit sphere S, consider a moving unit sphere S^* with center O rigidly attached to the trihedral. The two loci of the instantaneous center of rotation on S and on S^* respectively are called the fixed centrode and the moving centrode of the motion of S^*. It is well known that this motion is obtained by letting the moving centrode roll on the fixed centrode. In our case the fixed centrode is the curve C described by $\mathbf{c}=\mathbf{c}(s)$ on S and the moving centrode is the great circle determined by $\boldsymbol{t}$ and $\boldsymbol{b}$. Hence the motion of the trihedral about O is obtained by letting this great circle roll on C. From this it follows that the indicatrices T and B are spherical involutes of C. They are described by two points of the rolling great circle which have the spherical distance $\pi/2$ (Fig. 1).

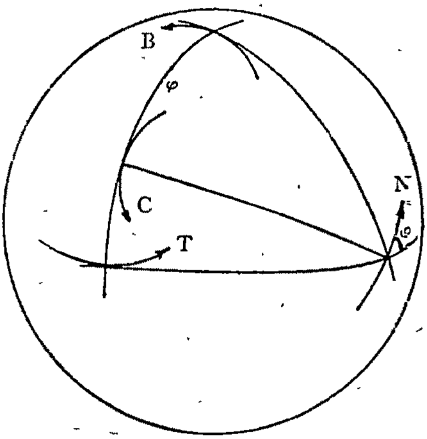

FIG. 1

4. A line element on the unit sphere is by definition a pair consisting of a point on S and an oriented great circle through this point. The spherical polarity is an involutory mapping of the set of all line elements onto itself which maps an oriented great circle on that one of its poles which lies on its left-hand side seen from the exterior of the sphere. To every spherical curve—considered as the set of its line elements, point and tangent great circle—corresponds a unique polar curve, and this correspondence is involutory. The tangent vectors at corresponding points of two polar curves are opposite. To a point of inflection of one of the curves corresponds a cusp of the first kind of the other. In order to preserve complete duality we take the arc length on a spherical curve with a sign in such a way that the differential of the arc changes sign at a cusp. The infinitesimal length of the arc between two neighboring points of a curve then equals the angle between the corresponding tangent great circles of the polar curve, this angle being taken with a sign in the usual way. From this it follows that polar curves have reciprocal geodesic curvatures.

(1) and (2) yield that T and B are mutually polar and (4) that C

and N are polar.

As mentioned in §3, T is an involute of C; it is described by a point which is fixed on a great circle rolling on C. The dual of this construction must lead from N to B. By means of the above remarks it is easily seen that B is obtained as the envelope of a family of half great circles emanating from the points of N and being parallel along N in the sense of Levi-Civita's surface parallelism. Analogously T is enveloped by the half great circles obtained by turning those enveloping B through $\pi/2$ about their initial points on N.

Introducing the notation

$$\sigma(s) = \int_0^s \omega(s)ds,$$

we have for the arc lengths of the curves T, B, N, C measured from the points corresponding to $s=0$

$$(5) \qquad s_T = \int_0^s \kappa ds = \int_0^s \omega \cos \phi ds, \qquad s_B = \int_0^s \tau ds = \int_0^s \omega \sin \phi ds,$$

$$(6) \qquad s_N = \sigma(s), \qquad s_C = \phi(s) - \phi(0),$$

and for their geodesic curvatures (with respect to the sphere)

$$(7) \qquad \gamma_T = \tan \phi, \qquad \gamma_B = \cot \phi,$$

$$(8) \qquad \gamma_N = \frac{\phi'}{\omega'} = \frac{d\phi}{d\sigma}, \qquad \gamma_C = \frac{\sigma'}{\phi'} = \frac{d\sigma}{d\phi}.$$

A κ-inflection of K [where ϕ passes one of the values $\phi \equiv \pi/2 \pmod{\pi}$] corresponds to a cusp of T and to an inflection of B. A τ-inflection of K [where ϕ passes one of the values $\phi \equiv 0 \pmod{\pi}$] corresponds to an inflection of T and to a cusp of B. The curve N has no cusps and C has no inflections ($\omega > 0$). Inflections of N and cusps of C do not correspond to singularities of K. At these points ϕ, hence τ/κ, is stationary and K behaves locally like a helix.

5. It follows from the Frenet formulae that there exists one and (up to a rigid motion) only one curve K with prescribed functions $\kappa(s)$ and $\tau(s)$ [or $\omega(s)$ and $\phi(s)$]. As a main problem of the differential geometry of closed curves I should like to formulate: To find necessary and sufficient conditions for the curvature and the torsion as periodic functions of s with period l in order that the curve be closed. Probably there is no simple answer to this problem, and so far only a few necessary conditions are known.

However, the problem splits up in a natural way. The Frenet system of differential equations determines the indicatrices T, B, N, and from T the curve K itself is found by a simple integration

$$r = \int_0^s t ds = \int_0^{s_T} (1/\kappa) t ds_T. \tag{9}$$

Instead of the arc length s of K we introduce as a new parameter the arc length $\sigma = s_N$ of N, which is an increasing function of s. Then the Frenet system becomes

$$\begin{aligned} \frac{dt}{d\sigma} &= \qquad\qquad\qquad n\cos\phi \\ \frac{dn}{d\sigma} &= -t\cos\phi \qquad\qquad\qquad + b\sin\phi \\ \frac{db}{d\sigma} &= \qquad\qquad -n\sin\phi \end{aligned} \tag{10}$$

which shows that the indicatrices are already determined if the single function $\phi = \phi(\sigma)$ is given. Since (8) yields

$$\phi(\sigma) = \phi(0) + \int_0^\sigma \gamma_N d\sigma \tag{11}$$

we see that T, B, N are determined if a constant $\phi(0)$ and the geodesic curvature γ_N of N as a function of the arc length of N are given. Elimination of t and b from (10) yields the equation

$$\frac{d^2n}{d\sigma^2} = -n + \gamma_N n \times \frac{dn}{d\sigma} \tag{12}$$

which determines the curve N (up to a rotation about O) when its geodesic curvature γ_N is given as a function of the arc length σ.

Summarizing we are led to the following problems:

I. To find conditions which a periodic function $\gamma_N(\sigma)$ must satisfy in order that the spherical curve for which σ is the arc length and γ_N the geodesic curvature be closed.

No contributions to this problem seem to exist. The analogous problem for plane curves is trivial since the differential equation corresponding to (12) can be solved explicitly in terms of simple integrals.

II. To find conditions which a closed spherical curve must satisfy in order that it be: (1) the indicatrix of the tangents; (2) the indicatrix of the binormals; (3) the indicatrix of the principal normals of a closed space curve.

We are going to discuss this last problem, which is considerably more elementary than I. For, from the geometrical relations between the indicatrices described in §4 we get

$$t = -\frac{dn}{d\sigma}\cos\phi + n \times \frac{dn}{d\sigma}\sin\phi, \quad t = b \times \frac{db}{ds_B}\cdot$$

Together with (9) this shows that, given any one of the indicatrices T, B, N, the curve K can be obtained by the simple integrations (9), (11), a fact which, of course, has been known for a long time [13].

6. A necessary condition which the tangent indicatrix T of a closed curve K must satisfy is easily derived. Consider any plane which does not intersect K. Then the tangents at those points of K which have maximal and minimal distance from the plane are parallel to that plane. Transferred to the unit sphere this means that T is intersected (at least twice) by every great circle,[2] in other words, the center O of the unit sphere is an inner point of the convex hull to T. This condition has also been shown to be sufficient [6; 17; 9]. If it is satisfied, there exists a great variety of closed curves K which have the given tangent indicatrix, T. These curves have everywhere positive curvature if T has no cusps.

From the necessity of this condition several conclusions may be drawn. The "absolute length" $\int|ds_T|$ is greater than or equal to 2π. Hence, for the total absolute curvature of a closed curve K with length l we have

$$\int_0^l |\kappa|\, ds \geqq 2\pi. \tag{13}$$

This has been proved in different ways [5; 11; 15; 3; 4]. In B. Segre's proof (refound by Rutishauser and Samelson [12]) it is deduced from the following lemma: A spherical curve with length less than or equal to $2\delta < 2\pi$ is contained in a circle with spherical diameter δ. I mention that (13) also follows from a formula which is proved in integral geometry [2, p. 81]: Let L be the (absolute) length of a spherical curve and let $n(x)$ be the number of points in which the great circle with the pole x intersects the curve, then

$$L = \frac{1}{4}\int n(x)\,d\alpha \tag{14}$$

where $d\alpha$ denotes the area element described by x and where the integration is extended over the whole sphere. According to our condition for T we here have $n \geqq 2$.

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[2] As far as I know this remark is due to C. Loewner (see [5]).

By means of Schwarz's inequality we obtain from (13)

$$(2\pi)^2 \leqq \left(\int_0^l |\kappa| \, ds\right)^2 \leqq l \int_0^l \kappa^2 ds,$$

hence for the length l of an arbitrary closed space curve

$$l \geqq \frac{2\pi}{\max |\kappa|}$$

a theorem due to H. A. Schwarz.[3]

If the class of curves K or the class of spherical curves T considered is restricted, sharper or different results may be obtained.

Answering affirmatively a question by K. Borsuk [3], I. Fáry [4] recently proved that if the curve K is knotted we have

$$\int_0^l |\kappa| \, ds \geqq 4\pi. \tag{15}$$

His proof is based on the interesting remark that the total (absolute) curvature of a space curve K is the mean value of the total (absolute) curvatures of the orthogonal projections of K on the planes through a fixed point. Under our (stronger) assumptions (15) can also be derived from (14). Let P be any plane not intersecting a closed curve K without multiple points. Suppose that the distance from P to a point of K as a function of s has only 2 extremals. Then K may be divided into two arcs such that the distance increases on one of them and decreases on the other. Hence every plane P' parallel to P and intersecting K intersects in exactly two different points. Join these points by a segment. Then these segments for all planes P' considered make up a surface bounded by K and homeomorphic to a circular disk. Hence K cannot be knotted. For a knot K it follows that the distance of its points from P has at least 4 extremals, that is, at least 4 tangents of K are parallel to an arbitrary plane. This means that every great circle intersects T in at least 4 points. Hence $n \geqq 4$ in (14), which yields (15).[4]

7. We consider now the indicatrix B of the binormals. By means of

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[3] For references and related results see [1, §31].

[4] (Added December 1950.) Since this paper was written, important new results concerning the total curvature of knots have been published: J. W. Milnor, *On the total curvature of knots*; R. H. Fox, *On the total curvature of some tame knots*, Ann. of Math. vol. 52 (1950) pp. 248–257, 258–260.

the spherical polarity in which T and B correspond to each other we get at once as a necessary and sufficient condition that a closed spherical curve B be the binormal indicatrix of a closed curve K, the following: the tangent great circles of B cover the whole sphere, that is, from every point of the sphere a tangent great circle to B can be drawn.

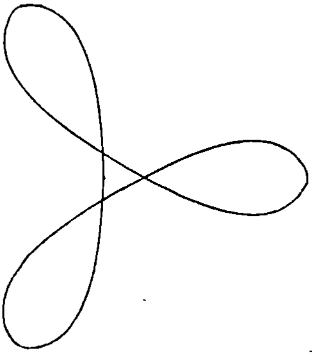

FIG. 2

Of course, there is no positive lower bound for the total (absolute) torsion

$$\int_0^l |\tau| \, ds = \int |ds_B|$$

of K, that is, the absolute length of B. But even if we assume that κ and τ are positive at all points of K, that is, that B has neither inflections nor cusps, there is no such bound. For, a small spherical curve of the shape sketched in Fig. 2 has these properties and its tangent great circles obviously cover the whole sphere. Hence there exist closed curves with everywhere positive curvature and everywhere positive torsion whose total torsions are arbitrarily small. The study of the example of Fig. 2 seems to indicate that if $\kappa > 0$ and $\tau > 0$ are assumed there may be a lower bound greater than 2π for the sum of the total curvature and the total torsion.

From his lemma mentioned in §6, B. Segre [16] has derived an estimate for the total absolute torsion. If B is contained in a hemisphere, let δ be the spherical diameter of the circumscribed circle of B; put $\delta = \pi$ otherwise. Then

$$\int_0^l |\tau| \, ds > 2\delta.$$

That this inequality is sharp for any given δ (even under the restrictions $\kappa > 0$, $\tau > 0$) follows from the fact that a spherical curve B of the shape sketched in Fig. 3 satisfies our conditions and may have a length which deviates arbitrarily little from 2δ.

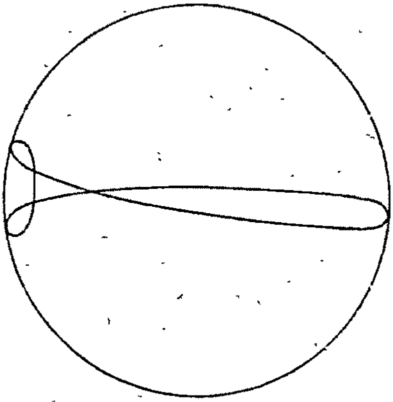

FIG. 3

8. Finally we consider the indicatrix N of the principal normals and, at the same time, its polar, the centrode C. Here a necessary condition results from the periodicity of κ and τ which implies

$$\phi(l) - \phi(0) = 2\pi\nu \tag{16}$$

for some integer ν, which is called the nutation of the curve K (W. Scherrer [14]). Hence, we conclude from (6) that the total length of the centrode C is $2\pi\nu$. This is just the condition that every point of a great circle rolling on C describes a closed curve, in other words, that the involutes of C, hence T and B, are closed. For the curve N we get from (8)

$$\int_0^{l_N} \gamma_N d\sigma = 2\pi\nu. \tag{17}$$

This means that if a tangent vector to the sphere at a point of N is displaced along N by Levi-Civita's surface parallelism it will return to its initial position after the curve has been traversed once. We see again that (16) implies that T and B are closed, and conversely.

An immediate consequence of (16) is that $\phi(s)$ for $0 \leqq s \leqq l$ passes a value congruent to $\pi/2$ (mod π) at least 2ν times and likewise a value congruent to 0 (mod π) at least 2ν times. Hence we have:

A closed space curve with the nutation ν has at least 2ν κ-inflections and at least 2ν τ-inflections.

If $\kappa > 0$ is assumed, we have $\nu = 0$. If, in addition, N has no multiple points, (17) and the Gauss-Bonnet formula yield Jacobi's theorem [8; 7]: The indicatrix of the principal normals of a closed space curve bisects the surface of the unit sphere. The connection of this theorem with his concept of parallelism has been observed by T. Levi-Civita [10], and the more general form (17) has been stated by W. Scherrer [14].[5]

In order to discuss the question whether condition (17) for N is sufficient I recall that if N is given, T (and B) is obtained as the envelope of a system of half great circles which are parallel along N. The direction of one of these circles may be prescribed arbitrarily according to the choice of the constant $\phi(0)$, $0 \leqq \phi(0) < 2\pi$. Hence there is a one-parameter family of curves each of which may be taken as T or B if it satisfies the corresponding condition discussed in §§6 and 7 respectively. Consequently the condition (17) for N will be sufficient if the following statement is true: For every closed spherical curve satisfying (17) at least one of these envelopes is intersected by every great circle; or (formulated for B) for every such curve a family of great circles exists which covers the whole sphere and is obtained from one of the great circles by parallel displacement along N.

I conclude my lecture with this open question. I hope that I have been able to show that the differential geometry of closed curves, though elementary, comprises interesting problems.

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[5] Scherrer's argument and his example showing that $\nu \neq 0$ may occur are in accordance with our assumptions, but are inconsistent with his own which exclude κ-inflections.

Scherrer aims at a formulation of (17) in terms of the spherical area bounded by N. To this purpose he gives a definition of the area bounded by a spherical curve with multiple points. However, this definition seems artificial and is open to objections. Compare S. B. Jackson's review of this paper [Mathematical Reviews vol. 8 (1947) p. 226]. A natural relation, the dual of (14), between the integral of the geodesic curvature and the areas of the regions into which the sphere is divided by the curve and its opposite curve is given by Blaschke [2, p. 83].

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THE UNIVERSITY OF SOUTHERN CALIFORNIA

THE OCTOBER MEETING IN NEW YORK

The four hundred sixty-first meeting of the American Mathematical Society was held at Columbia University on Saturday, October 28, 1950. The attendance was about 175, including the following 160 members of the Society.

M. Y. Aissen, E. J. Akutowicz, C. B. Allendoerfer, Helmut Aulbach, M. C. Ayer, Valentine Bargmann, F. P. Beer, E. G. Begle, Stefan Bergman, Lipman Bers, A. L. Blakers, Samuel Bourne, J. W. Bower, A. B. Brown, J. H. Bushey, W. R. Callahan, K. T. Chen, K. L. Chung, L. W. Cohen, M. L. Conlan, J. W. Cooley, Natalie Coplan, M. D. Darkow, P. H. Daus, Nelson Dunford, R. M. Exner, J. M. Feld, Werner Fenchel, F. G. Fender, W. E. Ferguson, Alvaro Ferlini, F. A. Ficken, Gerald Freilich, Bernard Friedman, Bent Fuglede, H. M. Gehman, Abe Gelbart, Leonard Gillman, Sidney Glusman, Samuel Goldberg, J. A. Greenwood, H. M. Griffin, Laura Guggenbuhl, E. J. Gumbel, Carl Hammer, G. A. Hedlund, Alex Heller, H. L. Herrick, Einar Hille, Abraham Hillman, A. J. Hoffman, T. R. Hollcroft, L. A. Hostinsky, S. T. Hu, E. M. Hull, T. R. Humphreys, Nathan Jacobson, W. S. Jardetzky, R. L. Jeffery, S. A. Joffe, R. A. Johnson, R. V. Kadison, Shizuo Kakutani, Aida Kalish, S. N. Karp, Stanley Katz, M. E. Kellar, B. C. Kenny, J. F. Kiefer, H. S. Kieval, M. S. Klamkin, J. R. Kline, E. G. Kogbetliantz, B. O Koopman, Marc Krasner, A W. Landers, R. E. Langer, Solomon Lefschetz, Benjamin Lepson, W. W. Leutert, M. E. Levenson, Howard Levi, Charles Loewner, E. R. Lorch, A. N. Lowan, Janet McDonald, L. A. MacColl, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, A. J. Maria, M. H. Maria, A. E. Meder, Jr., A. N. Milgram, K. S. Miller, W. H. Mills, Don Mittleman, L. J. Mordell, F. J. Murray, J. J. Newman, Morris Newman, L. R. Norwood, C. O. Oakley, A. C. O'Neill, J. C. Oxtoby, J. S. Oxtoby, T. E. Peacock, J. E. L. Peck, A. M. Peiser, R. S. Phillips, M. H. Protter, D. A. Quarles, Hans Rademacher, H. E. Rauch, H. W. Raudenbush, Giovanni Ricci, Moses Richardson, D. E. Richmond, C. E. Rickart, I. F. Ritter, J. E. Robinson, J. E. Rosenthal, Arthur Sard, Robert Schatten, J. A. Schatz, Samuel Schecter, Abraham Schwartz, Esther Seiden, D. B. Shaffer, I. M. Singer, P. A. Smith, H. H. Snyder, D. E. Spencer, V. E. Spencer, M. H. Stone, R. L. Swain, R. M. Thrall, A. W. Tucker, J. L. Walsh, J. V. Wehausen, J. H. Weiner, Louis Weisner, David Wellinger, J. G. Wendel, M. E. White, Albert Wilansky, M. A. Woodbury, Arthur Wouk, L. A. Zadeh, Antoni Zygmund.

The Council of the American Mathematical Society met at Columbia University on Saturday morning, October 28.

The Secretary announced the election of the following fourteen persons to ordinary membership in the Society:

Professor Pedro Abellanas, University of Madrid, Madrid, Spain;
Mr. Donald Lewis Arenson, Armour Research Foundation, Chicago, Ill.;
Mr. Solomon Isaac Ciolkowski, Washington University, St. Louis, Mo.;
Mr. David Robert Clutterham, D-31 Stadium Terrace, Champaign, Ill.;
Mr. Merlin Harry Dipert, Western Union College, LeMars, Iowa;
Mr. Donald Bruce Gillies, University of Illinois;
Mr. Edward Halpern, University of Massachusetts;
Mrs. Marjorie Halpern (Mrs. Edward), 427 North Pleasant St., Amherst, Mass.;
Mr. Lester LaVerne Helms, Bradley University, Peoria, Ill.;

Professor George Kurepa, Mathematical Institute, Zagreb, Yugoslavia;
Professor Lloyd Kenneth Jackson, University of Nebraska;
Professor Theophile Henri Lepage, University of Brussels, Brussels, Belgium;
Dr. Julio Rey-Pastor, University of Buenos Aires, Buenos Aires, Argentina;
Mr. Robert Findley Shaw, Eckert-Mauchly Computer Corporation, Philadelphia, Pa.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Dr. Jeffrey Dennis Weston, King's College, Newcastle-upon-Tyne, England; Société Mathématique de France: Professor Henri Cartan, University of Paris, Paris, France; Swiss Mathematical Society: Professor Walter Robert Baum, University of Tennessee, Knoxville, Tenn.; Unione Matematica Italiana: Professor Dario Graffi, University of Bologna, Bologna, Italy; Wiskundig Genootschap: Professor Johan Cornelis Hendrik Gerretsen, University of Groningen, Groningen, Netherlands.

The following appointments of representatives of the Society were reported: Professor John F. Randolph at convocation for the formal presentation of the Saint Bonaventure University charter on October 4, 1950; Professor J. M. Thomas at inauguration of Gordon Gray as President of The Consolidated University of North Carolina on October 8, 9, and 10, 1950; Professor R. R. McDaniel at inauguration of Earl Hampton McClenney, Sr., as President of St. Paul's Polytechnic Institute on October 12, 1950; Professor Euphemia L. Haynes at inauguration of Robert Prentiss Daniel as President of Virginia State College on October 14, 1950; Professor C. R. Wylie, Jr., at Diamond Jubilee of Brigham Young University on October 16 and 17, 1950; Professor J. S. Taylor at inauguration of John Christian Warner as President of Carnegie Institute of Technology and the Fiftieth Anniversary of Institution on October 27 and 28, 1950; Sister M. Laetitia Hill at dedication of the Science Hall at Incarnate Word College on November 3, 1950; Professor C. R. Wylie, Jr., at inauguration of Louis Linden Madsen as President of Utah State Agricultural College on November 3, 1950; Professor C. T. Bumer at inauguration of Spencer Miller, Jr., as President of American International College on November 8 and 9, 1950; Professor H. E. Bray at inauguration of Marion Thomas Harrington as President of The Agricultural and Mechanical College of Texas on November 9, 1950.

The following additional appointments by the President were reported: Professors H. A. Rademacher (Chairman), H. W. Brinkmann, C. C. MacDuffee as a Committee on the Cole Prize in Number Theory to be awarded at the 1951 Annual Meeting, for papers published in the period 1946–1950; Professors R. E. Langer (Chairman), J. W.

Green, W. T. Martin as a Committee to Nominate Representatives of Society on Policy Committee for Mathematics; Professors T. H. Hildebrandt (Chairman), P. A. Smith, J. W. T. Youngs as a Committee to Consider the Proposal to Add the President as an Ex-Officio Member of the Board of Trustees; Professors B. J. Pettis (Chairman), W. L. Duren, Jr., Harry J. Fleddermann, L. I. Wade, and W. M. Whyburn as a Committee on Local Arrangements for the New Orleans Meeting, 1951; Professor R. V. Churchill as Chairman of the Editorial Committee for the Proceedings of the Symposia in Applied Mathematics.

The Council voted to approve the recommendation of the Editorial Board of the American Journal of Mathematics that Professor P. A. Smith act for Professor Samuel Eilenberg as representative of the Society on that board during the present academic year. Professor Eilenberg is spending this period in France on a John Simon Guggenheim Memorial Fellowship.

The Council voted to accept the resignation of Professor L. M. Graves as a representative of the Society on the Editorial Board of the American Journal, effective December 31, 1950, and authorized and requested the Secretary to send a letter of appreciation to Professor Graves for his work.

The Council voted to accept the recommendation of the Committee on Arrangements for the Fourth Symposium on Applied Mathematics that the date of the meeting be changed from July 26–27, 1951, to June 22–23, 1951.

The Council voted that the following steps be taken in connection with the sale of the library and the housing of the permanent offices of the Society:

(a) That, subject to the approval of the Trustees, the Executive Director be authorized and requested to solicit firm bids for the library.

(b) That, in the event one or more bids of not less than $57,000 are received, the Executive Committee is authorized and requested, on behalf of the Council, to pass on the acceptability of such bids, to select from among several such bids the one best calculated to further the objectives of the Society and to make implementing recommendations concerning their findings to the Trustees.

(c) That, in the event no acceptable offer not less than $57,000 is received, the Executive Committee report their findings, if any, back to the Council.

(d) That, the Council favor continuation of the present exchanges as a matter of policy.

The Council voted that it favors the purchase of a building adequate to house all permanent offices of the Society as soon as an appropriate building, not necessarily in New York City, could be found and financed, and that it authorizes the Executive Committee to act in conjunction with the Board of Trustees in this matter.

Certain invitations to give addresses were announced: Professors G. B. Huff and M. H. Martin for the April, 1951 meeting in New Orleans; Professor Arthur Erdélyi for the April, 1951 meeting in Stanford; Professor Heinz Hopf for the February, 1951 meeting in New York; Professors L. H. Loomis and Herbert Federer for the April, 1951 meeting in New York.

The Council voted to propose to the members of the Society that the By-Laws be amended so as to add the President and the Treasurer of the Society to the Board of Trustees as ex-officio voting members.

The Council voted to accept a proposal of the Editorial Committee for the Proceedings of Symposia in Applied Mathematics, to appoint an editor for each volume for the Proceedings at the time the Symposium is arranged, it being understood that the appointee need not necessarily be a member of the committee.

At 2:30 P.M. Professor C. E. Rickart of Yale University gave an address on *Isomorphisms of infinite-dimensional analogues of the classical groups*. President J. L. Walsh presided at the address and at the business meeting which followed.

At 3:45 P.M. there was a business meeting of the Society. At this time the Secretary proposed amending the By-Laws as recommended by the Council for the purpose of:

1. Permitting a reduction in the number of representatives of the Society on the Editorial Board of the American Journal of Mathematics from three to two.

2. Changing the annual dues of individual members from $6.00 for the first two years of membership, $8.00 for the second two years, and $10.00 thereafter, to $9.00, $11.50, and $14.00 respectively.

3. Changing the minimum annual dues of contributing members from $15.00 to $20.00.

The proposed changes were approved.

There were two sessions for contributed papers, one at 1:00 P.M. in applied mathematics in which Professor Charles Loewner presided, and one at 4:15 P.M. in topology and analysis in which Professor L. W. Cohen presided.

Abstracts of papers presented in person at the meeting follow. Paper number 6 was read by Professor Richmond, paper number 4 by Professor Milgram, and paper number 12 by Professor Blakers. Abstracts of papers presented by title were printed as abstracts numbered 448–452 in the September 1950 issue of the Bulletin, and 454–456, 461–469 in the November 1950 issue of the Bulletin. Mr. Evans was introduced by Dr. B. H. Neumann.

ANALYSIS

1. M. I. Aissen: *A functional which depends upon the shape of a convex domain.*

A function $B_p(D)=\oint_C dS_q/h_{pq}$ is defined for all interior points, p, of a convex domain D. By q is meant a point of C, the boundary of D. By h_{pq} is meant the perpendicular distance (regarded as positive) from p to a supporting line to D at q. The functional $B(D)=\text{g.l.b.}_{p\in D}\, B_p(D)$ is also considered. It is shown that $B(D)$ is diminished (does not increase) by Steiner symmetrization. An explicit rational expression is obtained for the change in $B(D)$, which results from certain special affine transformations applied to domains with symmetries of the rectangle. For a fixed domain, D, it is shown that $B_p(D)$ is a strictly convex function of p, which becomes infinite as p approaches C. Hence there exists one and only one point p' in D, for which $B_{p'}(D)=B(D)$. A strong analogy is displayed between $B_p(D)$ and $2A/r_p^2$, where A is the area of D, and r_p is the inner conformal radius of D with respect to p. [G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 16.] (Received September 19, 1950.)

2. M. S. Klamkin: *A note on the Mittag-Leffler expansion theorem.*

For a function which satisfies the conditions of the Mittag-Leffler expansion theorem, $f(x)=f(0)+\sum_1^\infty b_n[1/(x-a_n)+1/a_n]$. If in addition $\int_{C_m}(f(z)/(z-x))dz\to 0$, then $f(0)=-\sum_{n=1}^\infty b_n/a_n$. This latter expansion is equivalent to the former one. For, if a function $\phi(x)=(f(x)-f(0))/(x-a)$ where a is not a pole of $f(x)$ is chosen, and $\phi(0)=-\sum_1^\infty b_n^1/a_n^1$ is evaluated, the former expansion is obtained. (Received September 13, 1950.)

3. Benjamin Lepson: *The maximum modulus of normalized polynomials on sets of positive transfinite diameter.*

A normalized polynomial is defined as a polynomial with complex coefficients for which the maximum of the moduli of the coefficients is 1. Consider polynomials in the complex variable z on certain point sets in the complex plane. It is known [Fekete, Math. Zeit. vol. 17 (1923) pp. 228–249] that if E is a closed bounded set of transfinite diameter d, then the maximum modulus on E of any polynomial of degree n with leading coefficient 1 is at least d^n. In the following theorem this result is extended to normalized polynomials: *Let E be a closed point set in the circle $|z|\leqq R\geqq 1$ for which there is a constant $c>0$ such that the maximum modulus on E of any polynomial of degree n with leading coefficient 1 is at least c^n. Then there is a constant $k>0$, depending only upon c and R, such that the maximum modulus on E of any normalized polynomial of degree n is greater than k^n.* A possible choice for k is $c/(3R)$, while the polynomial $az-1$ for small a shows that such a dependence on R is necessary. (Received September 14, 1950.)

4. A. N. Milgram and P. C. Rosenbloom: *Heat conduction on Riemannian manifolds in existence and uniqueness theorems.*

We consider exterior differential forms on a closed orientable Riemannian manifold whose coefficients depend on a parameter t. Using the generalization of the Laplace operator defined by de Rham, we study the analogue of the equation of heat conduction. From the fundamental solution for the Euclidean case we construct a parametrix, and derive an integral equation of Volterra type. Thus we prove the existence and uniqueness of the solution with prescribed initial values. If the form is closed initially, then it remains closed for all time, and its periods remain constant. As $t\rightarrow\infty$ a solution of the heat equation converges uniformly to a harmonic form. In this way we obtain a simple proof of Hodge's theorem on the existence of harmonic forms with prescribed periods from de Rham's theorem on the existence of closed forms with prescribed periods. (Received October 25, 1950.)

5. I. M. Singer: *Lie algebras of unbounded operators.*

Every strongly continuous unitary representation of an analytic group gives rise to a representation of the corresponding Lie algebra by unbounded skew-symmetric operators with a common, dense, invariant domain. [Gårding, Proc. Nat. Acad. Sci. U. S. A. vol. 33 (1947).] Using techniques involving the Cayley transform of a symmetric operator, one shows that the generators of the one-parameter groups restricted to the domain above (among others) have skew-adjoint closures. Conversely, it is shown that a condition of differentiability on the exponentials of a Lie algebra of unbounded skew-adjoint operators insures that a unitary representation of the local group can be obtained by exponentiating. A counter-example is produced to show that the existence of a common invariant domain for a Lie algebra of operators alone is not sufficient for a group representation. A sufficient condition for the existence of the group representation can also be obtained by demanding that on some common domain the exponential power series of the operators in the Lie algebra converge. This condition is shown to be necessary for nilpotent groups. Finally, it is proved that analytic groups which have faithful uniformly continuous unitary representations are direct products of abelian and compact analytic groups. (Received September 13, 1950.)

APPLIED MATHEMATICS

6. Ralph Gomory and D. E. Richmond: *Boundaries for the limit cycle of van der Pol's equation.*

The authors present a simple and natural method for constructing outer and inner boundaries for the limit cycle of van der Pol's equation, $d^2x/dt^2+\mu(x^2-1)dx/dt+x=0$. The method admits of unlimited improvement but even its simplest application gives results superior to those obtained by La Salle [Quarterly of Applied Mathematics vol. 7 (1949) pp. 1–19]. (Received September 14, 1950.)

7. S. N. Karp: *Natural charge distribution on a conical cup.*

The natural charge distribution for a conical cup has been obtained without approximation. Using spherical coordinates the potential U is expressed in the form $U=\int_{\sigma-i\infty}^{\sigma+i\infty}r^{-\nu}A_1(\nu)P_{\nu-1}(\cos\theta)d\nu$ for $0\leq\theta<\theta_0$, and $U=\int_{\sigma-i\infty}^{\sigma-i\infty}r^{-\nu}A_2(\nu)P_{\nu-1}(-\cos\theta)d\nu$ for $\theta_0<\theta<\pi$. The two-part boundary value problem in r at $\theta=\theta_0$ then gives rise to a single equation between two functions of ν. This equation is solved by Wiener-Hopf techniques in the complex ν plane. This involves: (a) factorization of $P_{-1/2+\mu}(\cos\theta_0)\cdot P_{-1/2+\mu}(-\cos\theta_0)/\sin\pi\nu$ in the form $K_+(\mu)\cdot K_-(\mu)$, where $K_+(\mu)$ is regular in a right half-plane and $K_-(\mu)$ in a left half-plane; (b) the asymptotic forms of $K_+(\mu)$, $K_-(\mu)$. The necessary results are obtained by employing: (a) the fact that the zeros of $P_{-1/2+\mu}(\cos\theta_0)$ considered as a function of μ are real, symmetric about the origin, and asymptotically in arithmetic progression; (b) comparison of $K_+(\mu)$ with the gamma function of a suitable argument. The method may also be applied in the case of incident fields. (Received August 30, 1950.)

8. W. W. Leutert: *A new method for the numerical solution of linear parabolic differential equations with constant coefficients.*

A three line difference equation is reduced to a two line difference equation by excluding particular solutions which might cause divergence and instability. The values on a line are obtained from those of the previous line by a simple process which involves only multiplication and addition of known quantities. Thus while the advantage of good convergence [error of order $(\Delta t)^2$] is retained from the three line equation, the new method will insure stability and convergence for all positive values of the mesh ratio $r = \Delta t/(\Delta x)^2$. It follows that von Neumann's test of stability is sufficient but not necessary. (Received July 31, 1950.)

9. F. J. Murray: *The Curie point for three-dimensional order-disorder.* Preliminary report.

The cubical lattice order-disorder problem for which van der Waerden has established the existence of a reference order at low temperatures (Zeitschrift für Physik vol. 118 (1941–1942) p. 473) is investigated in the region of the Curie point. Assuming that the possibility of perfect order has relative weight (Bolzman factor) one, one may consider all the possibilities for a single flaw in the reference order with a prescribed pattern in a fixed position on a face of the cubical lattice. The sum of the relative weights for all these possibilities can be considered as a weight for this pattern if the face pattern is simply connected. If the face pattern is disconnected, weights are assigned for each method of connecting the disconnected pieces. Removing a layer from the given face permits one to express the vector W whose elements are these weights in terms of matrix T and vector w_0 corresponding to the situation in which the fault is confined to the layer removed. $(1-T)W = w_0$. The temperature at which the maximum characteristic value of T is one corresponds to the disappearance of long range order. Methods for the numerical evaluation of the elements of T have been developed and a rough evaluation of the Curie point has been made, but it seems desirable to repeat this computation with automatic machinery. (Received September 26, 1950.)

10. Domina E. Spencer: *On separability in electromagnetic theory.*

The problem of solving Maxwell's equations can be reduced to the problem of solving the vector Helmholtz equation $\nabla^2 A^{\rightarrow} = k^2 A^{\rightarrow}$. Separability of the vector Helmholtz equation is studied in the eleven confocal quadric coordinate systems for which the scalar Helmholtz equation is separable. A method of solution is indicated, even when the differential equation for only one component of $A^{\rightarrow}$ is separable. It is shown that separability methods are applicable to electromagnetic problems in eight of the eleven coordinate systems. (Received September 13, 1950.)

11. L. A. Zadeh: *An operational calculus for time-dependent Heaviside operators.*

The operational relation $v(t) = H(p; t)u(t)$ is defined to mean $v(t) = \mathcal{L}^{-1}\{H(s; t)\,U(s)\}$ where $\mathcal{L}^{-1}$ represents the operation of inverse Laplace transformation (treating t in $H(s; t)$ as a fixed parameter), and $U(s)$ is the bilateral Laplace transform of $u(t)$. The time-dependent operator $H(p; t)$ is a generalization of the conventional Heaviside operator $H(p)$. Let N be a linear varying-parameter system and let $u(t)$ and $v(t)$ denote respectively the input and output of N. The system function of N is defined by the relation $H(j\omega; t) = v(t)/u(t)$ for $u(t) = e^{j\omega t}$ (Proceedings of the I.R.E. vol. 38 (1950) pp. 291–299). The fundamental property of the system function is that the output $v(t)$ corresponding to an arbitrary input $u(t)$ is related to $u(t)$ through $v(t) = H(p; t)u(t)$. Let $w(t) = H_2(p; t)\{H_1(p; t)u(t)\}$. This may be replaced by $w(t) = H_3(p; t)u(t)$ where $H_3(p; t)$ is given by the product relation $H_3(s; t) = H_2(p+s; t)H_1(s; t)$ which may be written symbolically as $H_3 = H_2 * H_1$. The inverse of an operator H is defined by the relation $H*H^{-1} = H^{-1}*H = 1$. If $v(t)$ and $u(t)$ are related to each other by a differential equation $L(p; t)v(t) = K(p; t)u(t)$, then $H(s; t)$ satisfies the differential equation $L*H = K$. Based on these relations, an operational calculus for time-dependent Heaviside operators can readily be constructed. (Received September 14, 1950.)

TOPOLOGY

12. A. L. Blakers and W. S. Massey: *The homotopy groups of a union of spheres with a single point in common.*

Let $\mathfrak{S}^n_k = S^n_1 \cup S^n_2 \cup \cdots \cup S^n_k$ be a topological space which is the union of k different n-dimensional ($n \geqq 2$) spheres S^n_i with a single point in common. By using a theorem of G. W. Whitehead (Ann. of Math. vol. 51 (1950) p. 210, Theorem 4.8) it is not difficult to prove that if $p < 3n-2$ then the homotopy group $\pi_p(\mathfrak{S}^n_k)$ is a direct sum of k subgroups each isomorphic to $\pi_p(S^n)$ and $k(k-1)/2$ subgroups each isomorphic to $\pi_p(S^{2n-1})$. The authors have now determined the structure of the group $\pi_{3n-2}(\mathfrak{S}^n_k)$. It is the direct sum of k subgroups each isomorphic to $\pi_{3n-2}(S^n)$, $k(k-1)/2$ subgroups each isomorphic to $\pi_{3n-2}(S^{2n-1})$, and a free abelian group on $(k+1)k(k-1)/3$ generators. As generators of this free abelian group, one may take the triple Whitehead products $[\iota_p, [\iota_q, \iota_r]]$, where $1 \leqq q < r \leqq k$, $1 \leqq p \leqq r$, and $\iota_t \in \pi_n(\mathfrak{S}^n_k)$ is the image of a generator of $\pi_n(S^n_t)$ under the injection $\pi_n(S^n_t) \rightarrow \pi_n(\mathfrak{S}^n_k)$, $1 \leqq t \leqq k$. (Received August 30, 1950.)

13. S. T. Hu: *On the realizability of homotopy groups and their operations.*

Whitehead's realizability theorem for homotopy groups, which gives no information about the Whitehead products of the higher homotopy groups, is strengthened in this paper as follows. Let $\pi_1, \pi_2, \cdots, \pi_n, \cdots$ be a given sequence of abstract groups. All groups except the first one are abelian and written additively, while π_1 is written multiplicatively; also π_1 operates on the left of every group π_n with $n \geqq 2$. Then the strengthened theorem is as follows: There exists an arcwise connected topological space B and a basic point $b_0 \in B$ satisfying the following conditions: (i) There exists, for each integer $n \geqq 1$, an isomorphism $h_n : \pi_n(B) \approx \pi_n$; (ii) For arbitrary elements $w \in \pi_1(B)$ and $a \in \pi_n(B)$, $n \geqq 2$, $h_n(wa) = h_1(w)h_n(a)$: and (iii) For arbitrary elements $a \in \pi_m(B)$ and $b \in \pi_n(B)$, $m \geqq 2$, $n \geqq 2$, the Whitehead product $a \circ b = 0$. The construction can be sketched as follows. To each integer $n \geqq 1$, consider the semi-simplicial complex $P_n = K(\pi_n, n)$ of Eilenberg and MacLane. P_n is an arcwise connected topological space whose homotopy groups are given by $\pi_n(P_n) \approx \pi_n$ and $\pi_i(P_n) = 0$ $(i \neq n)$. Let $X = P_1$ and Y be the topological product of the polytopes P_n $(n \geqq 2)$. Then the space B is constructed as a bundle space with X as base space and Y as fiber. (Received September 12, 1950.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN EVANSTON

The four hundred sixty-second meeting of the American Mathematical Society was held at Northwestern University on Friday and Saturday, November 24 and 25, 1950. In view of the fact that Northwestern University is celebrating its Centennial in 1951, this meeting of the Society was considered a portion of the centennial celebration. The attendance was about 160, including the following 132 members of the Society.

E. J. Akutowicz, L. U. Albers, A. A. Albert, W. R. Allen, J. J. Andrews, D. G. Austin, E. W. Banhagel, R. H. Bardell, D. Y. Barrer, R. G. Bartle, L. D. Berkovitz, H. R. Beveridge, R. H. Bing, L. M. Blumenthal, M. L. Boas, R. P. Boas, D. G. Bourgin, A. J. Brandt, Leonard Bristow, R. H. Bruck, R. C. Buck, E. L. Buell, J. W. Butler, Lamberto Cesari, Herman Chernoff, E. W. Chittenden, H. M. Clark, E. H. Clarke, M. D. Clement, E. G. H. Comfort, M. L. Curtis, Allen Devinatz, Flora Dinkines, N. J. Divinsky, W. F. Eberlein, M. H. M. Esser, G. W. Evans, W. A. Ferguson, W. H. Fleming, L. R. Ford, Evelyn Frank, J. C. Freeman, R. E. Fullerton, M. P. Gaffney, Cornelius Gouwens, L. M. Graves, Harold Greenspan, L. W. Griffiths, M. M. Gutterman, Franklin Haimo, P. R. Halmos, Melvin Henriksen, Fritz Herzog, D. L. Holl, T. C. Holyoke, A. S. Householder, C. C. Hsiung, H. K. Hughes, Meyer Jerison, L. W. Johnson, L. H. Kanter, Irving Kaplansky, William Karush, L. M. Kelly, D. E. Kibbey, E. H. Kingsley, Erwin Kleinfeld, Marc Krasner, M. Z. Krzywoblocki, H. G. Landau, Walter Leighton, D. J. Lewis, G. R. Livesay, A. P. Loomer, R. E. Lowney, Saunders MacLane, H. M. MacNeille, D. M. Merriell, J. M. Mitchell, C. W. Moran, E. J. Moulton, H. T. Muhly, S. B. Myers, E. A. Nordhaus, R. Z. Norman, T. E. Oberbeck, E. N. Oberg, Rufus Oldenburger, H. W. Oliver, E. H. Ostrow, O. G. Owens, Gordon Pall, George Piranian, D. H. Potts, G. B. Price, A. L. Putnam, W. T. Reid, Haim Reingold, Maxwell Rosenlicht, A. E. Ross, E. H. Rothe, J. C. Rothe, R. G. Sanger, R. H. Scherer, O. F. G. Schilling, J. E. Schubert, W. T. Scott, I. E. Segal, Esther Seiden, D. H. Shaftman, H. A. Simmons, M. L. Slater, D. M. Smiley, M. F. Smiley, E. H. Spanier, R. H. Stark, R. L. Sternberg, T. T. Tanimoto, H. P. Thielman, N. S. Thompson, E. F. Trombley, Bryant Tuckerman, W. R. Utz, Bernard Vinograde, D. R. Waterman, L. R. Wilcox, R. S. Wolfe, F. M. Wright, L. C. Young, J. W. T. Youngs, Daniel Zelinsky, J. L. Zemmer.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor D. G. Bourgin, of the University of Illinois, addressed the Society on *Classes of transformations and bordering transformations* at 2:15 p.m. Friday, November 24. Presiding officer at this lecture was Dr. H. M. MacNeille, Executive Director of the Society.

There were, in addition to the lecture by Professor Bourgin, four sessions for the presentation of contributed papers, two on Friday and two on Saturday. Presiding officers at these sessions were Professors W. T. Reid, L. M. Graves, W. T. Scott, and R. H. Bruck.

There was a tea for visiting members and their guests in the Faculty Lounge of the Technological Institute Building on Friday afternoon. For the first time in many years, a dinner was held in connection with a meeting of the Society in the Chicago area. This was a very pleasant affair in which Dean Tebbutt welcomed the Society on behalf of the University and your Associate Secretary replied. The Executive Director, Dr. H. M. MacNeille, was then called upon by the toastmaster, Professor L. R. Ford, to make a few remarks. He stated briefly some of the tasks of the New York office and concluded with a few comments in the nature of a report from foreign visitors to the recent International Congress of Mathematicians.

Abstracts of papers presented follow below. Abstracts followed by the letter "*t*" were read by title. Abstracts 446 and 447 in the September issue of this Bulletin and abstracts 453, 457–460 in the November issue were also read by title at this meeting. Paper number 27 was read by Professor Fullerton, 32 by Professor Herzog, 35 by Professor Piranian, and 52 by Professor Young. Mr. Nemmers was introduced by Dr. H. M. MacNeille and Mr. Chu was introduced by Professor Bernard Vinograde.

ALGEBRA AND THEORY OF NUMBERS

14*t*. R. H. Bruck: *The definitive associativity theorem for alternative rings.*

Let R be an alternative ring with associator $(x, y, z)=xy\cdot z-x\cdot yz$. A subset A of R (not necessarily closed under subtraction or multiplication) is called associative if $(A, A, A)=0$. The main theorem states: Every maximal associative subset of an alternative ring R is an associative subring. The proof holds if R is commutative or if $3(x, y, z)=0$ implies $(x, y, z)=0$, but these restrictions are probably irrelevant. Assuming the well-ordering postulate, the theorem contains as special cases the associativity theorems of Artin and Moufang, and the proof (aside from well-ordering) is exceedingly simple. A like theorem holds for commutative Moufang loops and presumably for noncommutative Moufang loops. (Received May 17, 1950.)

15*t*. Harvey Cohn: *On stable maxima for the product of homogeneous forms.* I.

For every real matrix a_{ij}, det $a_{ij}>0$, we define $\Phi(a)=\min |a\times p|$ where $a\times p =(\prod_{i=1}^{n}\sum_{j=1}^{n} a_{ij}p_j)/\det a_{ij}$ and p_j varies over all integer n-tuples, not all zero. Let $|a-a^*|=(\sum_{ijk} (a^*_{ij}a_{ik}-a^*_{ik}a_{ij})^2)^{1/2}$ be a measure of the difference between the matrices, vanishing when the rows are proportional in order. Henceforth the author calls the value $\Phi(a^*)$ a (relative) *stable maximum* if positive numbers ϵ, ρ and a set of q $(=n(n-1)+1)$ integral n-tuples $p^{(k)}$ exist such that $|a^*\times p^{(k)}|=\Phi(a^*)$ and, for $|a-a^*|<\epsilon$, and $k=k(a)$, this inequality holds: $|a\times p^{(k)}|<\Phi(a^*)-\rho|a-a^*|$. Referring to quadratic forms $a\times p=F(p_1, p_2)/(\text{disc } F)^{1/2}$, with coefficients proportional to integers, the author shows that $\Phi(a)$ provides a stable maximum if and only if the pair of diophantine equations $F(p_1, p_2) = \pm \min |F(p, q)|$ are both solvable in integers. (Received October 6, 1950.)

16t. Harvey Cohn: *On the finite determination of critical lattices for a convex body.*

The problem can be restricted without loss to lattices in d-space which have a set of d independent vectors α_i on the surface of the convex body. Considering first lattices where such a set is a basis, the author shows that the conditions that a convex body contains no lattice point in its interior except the origin can be reduced from the infinite set $\Phi(\sum_{i=1}^{d} p_i\alpha_i) \geqq 1$ where the integers p_i run over all n-tuples except the origin, to the finite set where $(\sum p_i^2)^{1/2} \leqq C_d d^{d/2}$ and where C_d is the constant defined in the author's previous abstract, *On finiteness conditions for a convex body in hyperspace* (Bull. Amer. Math. Soc. Abstract 56-3-104). In the event that the α_i do not form a basis, they are still shown to be of bounded index for each dimension, and a slight modification of the argument suffices. (Received October 6, 1950.)

17t. R. P. Dilworth and J. E. McLaughlin: *Distributivity.*

Let ϕ be an imbedding operator over a lattice L. L is said to be ϕ-distributive if $x \cap \phi(S) = \phi(x \cap S)$ for all elements x and all subsets of S of L. It is shown that L is ϕ-distributive if and only if L_ϕ is infinitely (meet) distributive. It is also shown that every Boolean algebra is ν-distributive where ν is the normal imbedding operator. Two important theorems on Boolean algebras—the Tarski-von Neumann Theorem and the Stone-Glivenko Theorem are consequences of these theorems. (Received October 5, 1950.)

18. Franklin Haimo: *Groups with a certain condition on conjugates.* Preliminary report.

Let G be a nilpotent group, and let M, a positive integer, be a uniform bound on the number of conjugates that any element of G may have. Then there are "large" integers n for which the operation $x \rightarrow x^n$ is a central endomorphism of G. (Received October 11, 1950.)

19t. H. T. Muhly: *Integral bases and varieties multiply of the first species.*

If V is an absolutely irreducible r-dimensional algebraic variety over a ground field k of characteristic zero, and if $\mathfrak{o} = k[x_0, x_1, \cdots, x_n]$ is the ring generated by the homogeneous coordinates of the general point of V; then a necessary and sufficient condition for the existence of an independent linear base for $\mathfrak{o}$ over the ring generated by a suitably chosen set of independent variables is that V be r times of the first species in the sense of Dubreil (*Sur la dimension des ideaux des polynomes*, Journal de Mathématiques (9) vol. 15 (1936)). It is shown that if $\mathfrak{a}$ is the ideal of V in the ring $R = k[X_0, X_1, \cdots, X_n]$ of $n+1$ indeterminates over k, and if $\mathfrak{a}_0 = \mathfrak{a}$, $\mathfrak{a}_i = \mathfrak{a}_0 + RX_1 + RX_2 + \cdots + RX_i$, $i = 1, 2, \cdots, r$, $\mathfrak{a}_{r+1} = \mathfrak{a}_0 + RX_0 + RX_1 + \cdots + RX_r$, then a necessary and sufficient condition for the $\mathfrak{a}$ residues $x_0, x_1, \cdots, x_r$ of $X_0, X_1, \cdots, X_r$ to be an admissible set of independent variables for $\mathfrak{o} \cong R/\mathfrak{a}$, and for $\mathfrak{o}$ to possess an independent linear base over $k[x_0, x_1, \cdots, x_r]$, is that the characteristic function $\chi(\mathfrak{a}, h)$ of $\mathfrak{a}$ satisfy the difference equations, $\Delta^j\chi(\mathfrak{a}, h) = \chi(\mathfrak{a}_j, h)$, $j = 1, 2, \cdots, r+1$, for all values of h, where Δ denotes the unit difference operator. (Received August 21, 1950.)

20*t*. F. E. Nemmers: *A theorem for alternative rings.*

Let a, b, c be elements of an alternative ring. The author proves the identity $(a, (a, b, c), c) = (a, b, c)(ca - ac) = (ac - ca)(a, b, c)$ and uses this identity to prove the theorem "An alternative division ring R satisfying $(a, (a, b, c), c) = 0$ for every $a, b, c \in R$, is a field." A corollary to this theorem is "In an alternative ring in which $6a = 0$ implies $a = 0$, the unit cannot be expressed as an associator." (Received October 9, 1950.)

21. G. B. Price: *Bounds for determinants with dominant principal diagonal.*

Let (a_{ij}) be an $n \times n$ matrix of complex numbers a_{ij}, and let D be its determinant. Assume that for each row the absolute value of the element in the principal diagonal is greater than the sum of the absolute values of all other elements in the same row (for references to the literature concerning these determinants see a paper by Olga Taussky in Amer. Math. Monthly vol. 56 (1949) pp. 572–676). Set $m_i = |a_{ii}| - \sum_{j=i+1}^{n} |a_{ij}|$ and $M_i = |a_{ii}| + \sum_{j=i-1}^{n} |a_{ij}|$ for $i = 1, 2, \cdots, n$. It is proved in this note that $0 < m_1 m_2 \cdots m_n \leqq |D| \leqq M_1 M_2 \cdots M_n$, and several generalizations and extensions are given. The lower bound for $|D|$ stated in the last sentence is an improvement of a similar lower bound given by Ostrowski (Bull. Sci. Math. (2) vol. 61 (1937) pp. 19–32). The proofs depend on a lemma which gives bounds for the solution of a certain system of linear equations. (Received October 2, 1950.)

22*t*. W. R. Utz: *Powers of a matrix over a lattice.*

In this note powers, under ordinary matrix multiplication, of a matrix with elements in a lattice are investigated. Among other results is a generalization of a theorem due to Sanders (*A linear transformation whose variables and coefficients are sets of points*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 442) to matrices with elements in a distributive lattice. (Received October 2, 1950.)

23*t*. N. A. Wiegmann: *A polar form for a matrix in a field not of characteristic two.*

It is known that every matrix A in the complex field has a polar representation and that if A is normal, the polar matrices commute. A matrix A with elements in a field $K \supseteqq F$ not of characteristic two is considered and a generalization of this polar form is obtained. It is shown that when U is an involution over F and A is defined to be normal if $AA^U = A^U A$, then A is normal if and only if $A^U = VA = AV$ where V is a U-orthogonal matrix. This is seen to include the complex case as a special instance. Possibilities for the modification of certain restrictions are considered as are special cases of this type of normal matrix. (Received October 4, 1950.)

ANALYSIS

24. L. U. Albers: *Use of Leray-Schauder methods on the existence and boundedness of solutions of quasi-linear partial differential equations of parabolic type.*

The quasi-linear partial differential equation of parabolic type is studied. The space variables range over the interior of a simply connected region in Euclidean n-space and the boundary of this region must have certain continuity properties. The time variable ranges over the interval $(0, T)$ where the size of T depends on the properties of the source function. Moreover the space region mentioned above is allowed to vary continuously with time in the interval $(0, T)$ in certain restricted ways. The partial differential equation problem is transformed into a mapping problem in Banach spaces of the type studied by Leray and Schauder, and the existence problem is solved at that level. The implications of the basic results for the conditions on boundary values and properties of the source function are studied. (Received October 10, 1950.)

25. R. P. Boas: *Completeness of cosines with phase shifts.*

It has been shown by V. A. Ditkin (Uspehi Matematičeskih Nauk N.S. vol. 5 (1950) pp. 196–197) that the set $\{\cos (nx+q_n)\}_{n=0}^{\infty}$ is complete on $(0, \pi)$ with respect to the class of integrable functions provided that $0 \leqq q_n < \pi/2$ for $n=0, 1, 2, \cdots$. The author extends this result, using complex-variable methods (different from the method used by Ditkin), to sets of the form $\{\cos (\lambda_n x+q_n)\}_{n=0}^{\infty}$. For example, if $|n-\lambda_n| \leqq 2\delta/\pi < 1/2$, $\delta > 0$, this set is complete if $\delta \leqq q_n < (\pi/2)-\delta$. (Received October 10, 1950.)

26. Lamberto Cesari: *On the absolute extremals for the integrals on parametric surfaces.*

The following existence theorem is obtained: *Each positive definite semiregular integral $J(S)$ has absolute minimum in the class of all surfaces of finite Lebesgue area, contained in a closed bounded convex part A of the (xyz)-space, and whose boundary curve is a given closed Jordan curve C provided that C is boundary curve of at least one surface of finite Lebesgue area in A.* This theorem, in which the condition of the boundedness of A can be removed, extends to general parametric surfaces a theorem of Tonelli for curves. The same theorem extends an analogous statement due to McShane for integrals $J(S)$ in which the function involved does not depend upon the point in the space. Another particular case is the problem of Plateau. The proof of the above existence theorem is based only on the concept of lower semicontinuity, on geometrical considerations which arise in the theory for Lebesgue area, on the Eilenberg inequality applied to polyhedral surfaces, and on a detailed study of the known smoothing processes and especially of a new one for polyhedral surfaces. Analogous results have been communicated recently by J. M. Danskin and A. G. Sigalov as obtained by different methods. (Received November 25, 1950.)

27. Lamberto Cesari and R. E. Fullerton: *On regular representations of surfaces.*

Let S be a Frechet surface. A representation Φ: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$; $0 \leqq u, v \leqq 1$, of S is said to be regular if there exists a countable set of horizontal line segments $\{\eta_i\}$, $0 \leqq u \leqq 1$, $v=k_i$, and a countable set of vertical segments $\{\xi_j\}$, $0 \leqq v \leqq 1$, $u=c_j$, each dense in the unit square and such that the projections of the image under Φ of each segment on the coordinate planes are of Lebesgue two-dimensional measure zero. It is shown that any nondegenerate surface of finite Lebesgue area possesses a regular representation. The methods used are geometric and make no use of the Dirichlet integral. (Received October 11, 1950.)

28. Herman Chernoff: *An extension of a result of Liapounoff on the range of a vector measure.*

Liapounoff proved that if $\mu_1, \mu_2, \cdots, \mu_n$ are countably additive finite measures, the range of values of the vector $\mu(E)=(\mu_1(E), \mu_2(E), \cdots, \mu_n(E))$ is a bounded closed set and in the nonatomic case convex. If μ_{it}, $i=1, 2, \cdots, k$: $t=1, 2, \cdots, n_i$, are countably additive finite measures and $\{(E_1, E_2, \cdots, E_k)\}$ is the totality of decompositions of a space X into k pairwise disjunct measurable sets, then the range of $\psi=(\mu_{11}(E_1), \mu_{12}(E_1), \cdots \mu_{1n_1}(E_1), \mu_{21}(E_2), \cdots, \mu_{kn_k}(E_k))$ is bounded and closed and in the nonatomic case, convex. (Received October 18, 1950.)

29. J. T. Chu: *Generalized hermitian operators in Hilbert space.* Preliminary report.

Let X be an operator in Hilbert space S characterized by: (1) X^* exists, (2) $D(X) \cap D(X^*)$ generates S, and (3) $X=X^*$ over $D(X)\cap D(X^*)$. X is a generalization of a Hermitian operator. If linearity and boundedness are also assumed, then X is Hermitian. So to get nontrivial results, we must assume that X is unbounded. In some respects X is like a Hermitian operator. In fact it is a nonsymmetric extension of a Hermitian operator. We prove some sets of necessary and sufficient conditions that a nonsymmetric extension of a Hermitian operator be an operator of this kind. A more concrete treatment is obtained by using infinite matrices. However here we must assume that X is closed and linear. Then X is represented by matrices which are "almost" Hermitian. And conversely, unitary transformations of Hermitian matrices under certain conditions yield operators of this kind. (Received October 12, 1950.)

30. Allen Devinatz: *Integral representations of positive definite functions.*

Following the ideas introduced by N. Aronszajn (Proc. Cambridge Philos. Soc. vol. 39 (1943) p. 133; Trans. Amer. Math. Soc. vol. 68 (1950)) the author investigates transformations in reproducing kernel spaces. Necessary and sufficient conditions are obtained for a function $f(x)$, $0 \leqq x < \infty$, to be represented uniquely as the integral $\int_0^\infty t^x dV(t)$, where $V(t)$ is a bounded monotone increasing function. For $-\infty < x < \infty$, necessary and sufficient conditions are given for $f(x)$ to be represented by the integral $\int_\epsilon^\infty t^x dV(t)$, where $V(t)$ is as before and $\epsilon > 0$. In this case it is shown that the representation is always unique. Necessary and sufficient conditions are also obtained for uniquely representing a function $f(x, y)$ as the integral $\int_0^\infty\int_{-\infty}^\infty t_1^x e^{it_2 y} dV(t_1, t_2)$. The main condition on these functions is that $f(x_1+x_2)$ or $f(x_1+x_2, y_1-y_2)$ are positive definite functions. The main idea of the proof is to set up Abelian systems of self-adjoint or normal transformations and use these systems in much the same manner as groups of unitary transformations are used to prove the Bochner theorem from Stone's theorem. Using discrete sequences as the domains of these functions, various types of moment problems may be solved. Extensions to higher-dimensional cases present no difficulties. (Received October 12, 1950.)

31. M. P. Gaffney: *The harmonic operator for exterior differential forms.*

For Riemannian manifolds, de Rham and Kodaira have defined the generalized harmonic operator $\Delta (=d\delta+\delta d)$. If its domain is properly chosen, Δ can be viewed as a Hilbert space transformation. Following a suggestion of M. H. Stone's, the author has used the Friedrichs mollifier to develop the properties of Δ from this point of view (thus avoiding the use of integral equation theory). To obtain results in the noncompact case the concept of "negligible boundary" is introduced: if α^{p-1}, $d\alpha$, β^p, and

$\delta\beta$ are all square integrable then $(d\alpha, \beta) = (\alpha, \delta\beta)$. Sufficient conditions are given for negligible boundary. The closure of Δ (with suitable domain) is proved self-adjoint (a necessary prelude to the use of the spectral theorem). All characteristic forms of Δ are proved to be of class C^2. In the compact case a direct proof is given that the inverse of Δ is completely continuous. (Received October 12, 1950.)

32. Fritz Herzog and George Piranian: *On the univalence of functions whose derivatives have a real positive part.*

A domain D is said to have property (U) provided each function whose derivative has a positive real part throughout D is schlicht in D. K. Noshiro showed that every convex domain has property (U) (Journal of the Faculty of Science, Hokkaido Imperial University, Series I, vol. 2 (1934–1935) p. 151). A domain is defined to be *almost convex* provided each pair of circles in D can be connected by a straight line segment in D. The *set of deficiency* of a domain is its complement relative to the interior of its convex hull. The authors prove the following: For a domain D to have property (U), it is sufficient that D be almost convex and necessary that the set of deficiency of D be totally disconnected. (Received August 23, 1950.)

33. A. S. Householder: *Polynomial iterations to roots of algebraic equations.*

By an iteration to a root ξ of an equation $f(x) = 0$ is meant a rule $\phi(x)$ such that for any x_0 in some neighborhood of ξ the sequence defined by the recursion $x_{i+1} = \phi(x_i)$ converges to ξ. The iteration is of order r in case $\phi^{(r)}$ is the derivative of lowest order which does not vanish at ξ. It is known that when f is analytic, an iteration ϕ of any specified order r can be constructed, and Domb, Proc. Cambridge Philos. Soc. vol. 45 (1949) pp. 237–240, has shown that when f is a polynomial one can make ϕ also a polynomial. The purpose of this note is to establish a simple algorithm for this. If d is the highest common divisor of f and f', $g = f/d$, and p and q are any polynomials satisfying $pg' + qg = 1$, then $\phi_1 = x - gp$ is a polynomial iteration of second order, and $\phi_r = \phi_{r-1} + (-)^r p_r g^r / r$ is a polynomial iteration of order $r+1$ provided $p_1 = p$ and $p_r = pp'_{r-1} - (r-1)qp_{r-1}$. (Received October 9, 1950.)

34. L. H. Kanter: *On the zeros of the parametric derivatives of the ultraspherical polynomials.*

The notation used is that of G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23. Let $\partial P_n(x, \lambda)/\partial\lambda = G_n(x, \lambda)$. Since $P_n(x_\nu, \lambda) \equiv 0$ in λ, it follows that $[\partial P_n(x, \lambda)/\partial\lambda + (\partial P_n/\partial x)x'(\lambda)]_{x=x_\nu} = 0$. From the fact that $R_n(x, \lambda)/P_n(x, \lambda)$ has at most one zero between two successive zeros of $P_n(x, \lambda)$ of the same sign, and the relation $R_n(x_\nu, \lambda) + P_n'(x_\nu, \lambda)x_\nu'(\lambda) = 0$, the following theorem is deduced: the positive zeros of $R_n(x,\lambda)$ interlace with the positive zeros of $P_n(x, \lambda)$; a similar theorem holds for the negative zeros. Also, like theorems hold for the polynomials $G_{n-2}(x, \lambda) = \partial(k_n^{-1} P_n)/\partial\lambda$ and for $F_n(x, \lambda) = R_n(x, \lambda) + cP_n(x, \lambda)$, where c is an arbitrary constant. (Received October 9, 1950.)

35. A. J. Lohwater and George Piranian: *Note on certain sets of harmonic measure zero.* Preliminary report.

This paper continues the search for conditions on a set E on the boundary of a Jordan region R that are necessary or sufficient in order that, for the conformal mappings f of R upon the unit disc, $f(E)$ should be a set of one-dimensional Lebesgue measure zero. It is known that the two-dimensional measure of a Jordan curve need not be zero [Osgood (Trans. Amer. Math. Soc. vol. 4 (1903) pp. 107–112), R. L. Moore and J. R. Kline (Ann. of Math. (2) vol. 20 (1918) pp. 218–223)]. The authors construct a Jordan curve that has a subset whose two-dimensional measure is positive, and whose harmonic measure is zero. (Received October 2, 1950.)

36. R. E. Lowney: *A boundary value problem involving an exponential turning point.*

The differential system $u''(x)+\rho^2q(x)u(x)=0$; $u(-a)=u(b)=0$; a, $b>0$; has been studied by R. E. Langer and J. Barron wherein $q(x)$ vanishes as a power of x. In the present paper, $q(x)$ is taken as $e^{-2/|x|}/x^4$, $x\neq 0$; $q(0)=0$, and $a\neq b$. The removeable discontinuity in $q(x)$ is reflected in the second independent solution of the differential equation. The eigenvalues are found to consist of two real infinite sets asymptotically approaching a regularly spaced set, one from above and the other from below. The expansion of a function in terms of eigenfunctions is discussed, and conditions for convergence determined. It is found that conditions for convergence at the point $x=0$ are slightly more restrictive than those for convergence at any other point of the interval. (Received October 9, 1950.)

37. O. G. Owens: *Homogeneous Dirichlet problem for inhomogeneous ultrahyperbolic equation.*

Let λ_n $(n=1, 2, 3, \cdots)$ denote the characteristic values associated with the wave-equation, $\phi_{x_1x_1}+\phi_{x_2x_2}+\lambda\phi=0$, and with the region $G(X)$ of cartesian (x_1, x_2)-space. Similarly, let μ_m $(m=1, 2, 3, \cdots)$ denote the characteristic values associated with the region $G(Y)$ of cartesian (y_1, y_2)-space. Then, the solution of the Dirichlet problem for the cross product region $G=G(X)\times G(Y)$ and the ultrahyperbolic equation $u_{x_1x_1}+u_{x_2x_2}-u_{y_1y_1}-u_{y_2y_2}=f(x_1, x_2, y_1, y_2)$ is unique, provided $|\lambda_n-\mu_m|$ is never zero. Furthermore, if f vanishes on the boundary of G and if $|\lambda_n-\mu_m|$ has a positive lower bound, then there is a unique solution of the ultrahyperbolic equation vanishing on the boundary of G. An example shows that there are domains $G=G(X)\times G(Y)$ for which $|\lambda_n-\mu_m|$ is bounded away from zero. (Received October 11, 1950.)

38*t*. A. R. Schweitzer: *Functional equations associated with Grassmann's extensive algebra.*

Two classes of functional equations are discussed. I. Equations in iterative compositions of functions (including the author's quasi-transitive functional equations) generalizing Grassmann's relation $a \leftarrow b=(a-r)-(b-r)$, *Gesammelte Werke*, vol. 1, part 1, p. 373. II. Equations connected with Grassmann's inner product; ibid. pp. 11, 345, 349. Let $E_1, E_2, \cdots, E_n$ $(n=2, 3, \cdots)$ be independent vectors in Grassmann space; let $X_i=x_{i1}E_1+x_{i2}E_2+\cdots+x_{in}E_n$ $(i=1, 2, 3)$ and assume $X_i^2=f(x_{i1}, \cdots, x_{in})$. Then the identity $(X_1+X_2+X_3)^2=X_1^2+X_2^2+X_3^2+2X_1X_2+2X_2X_3+2X_3X_1$, where $2X_iX_j=(X_i+X_j)^2-X_i^2-X_j^2$, leads to the relation $(X_1+X_2+X_3)^2-(X_1+X_2)^2-(X_2+X_3)^2-(X_3+X_1)^2+X_1^2+X_2^2+X_3^2=0$. Replacing the indicated inner squares by corresponding functional expressions, the author obtains a functional equation which is interpreted as a special case of a functional equation due to M. Fréchet, Nouv. Ann. Math. (4) vol. 9 (1909) pp. 145–162. Reference is made to G. van der Lijn, Bull. Sci. Math. vol. 64 (1940), part 1, pp. 55–80, 102–112, 163–196. (Received October 9, 1950.)

39. E. F. Trombley: *Some boundary value problems involving functions of two complex variables.*

If a function of two complex variables $f(z_1, z_2)$ is (i) regular when the imaginary parts of z_1 and z_2 are both greater than zero (call this region E) and (ii) $\sup_{y_1, y_2 > 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x_1+iy_1, x_2+iy_2)|^p dx_1 dx_2 = M^p < \infty$ where $p > 0$ and M is a constant, then the iterated limit $\lim_{y_1 \to 0} \lim_{y_2 \to 0} f(x_1+iy_1, x_2+iy_2)$ $(=F(x_1, x_2))$ exists almost everywhere and is almost everywhere equal to the limit of $f(w_1, w_2)$ as (w_1, w_2) approaches (x_1, x_2) along any path nontangential to the distinguished boundary of E. Further $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | F(x_1, x_2)|^p dx_1 dx_2 = M^p$. If $p \geqq 1$, then $f(z_1, z_2)$ is represented by both the Cauchy and Poisson integral of $F(x_1, x_2)$. In this case $\sup_{y_1 > 0} \int_{-\infty}^{\infty} |f(x_1+iy_1, z_2)|^p dx_1 < \infty$ for almost every z_2. (Received October 12, 1950.)

40. D. R. Waterman: *On some high-indices theorems.*

The Hardy-Littlewood "high-indices" theorem asserts that for lacunary series Abel summability implies convergence. That is to say that if $f(s) = \sum_1^{\infty} a_n e^{-\lambda_n s}$ where $\lambda_{n+1}/\lambda_n \geqq q > 1$ for $n = 1, 2, \cdots$, and $\lim_{s \to +0} f(s)$ exists, then $\sum_1^{\infty} a_n$ converges. In his proof of this theorem Ingham demonstrates the basic result $|a_n| \leqq A_q \sup f(s)$ where A_q depends on q alone. Absolute Abel summability is defined by the condition that $f(s)$ be a function of bounded variation in $[0, \infty]$. Zygmund has shown that absolute Abel summability implies absolute convergence for lacunary series. Important there is the inequality $\sum_1^{\infty} |a_n| \leqq A_q \int_0^{\infty} |f'(s) ds$. Here the author shows that $\sum_1^{\infty} |a_n|^m \lambda_n^{m-1} \leqq A_{q,m} \int_0^{\infty} |f'(s)|^m ds$ where $A_{q,m}$ depends on q and m alone. The Hardy-Littlewood theorem and the Zygmund theorem are the cases corresponding to $m = \infty$ and $m = 1$ respectively. Also given are the results $\sum_1^{\infty} |a_n|^m \leqq A_{n,q,m} \int_0^{\infty} (1-e^{-s})^{m-1} |f'(s)|^m ds$ where $0 < h \leqq \lambda_1$, and $\sum_1^{\infty} |a_n|^m \lambda_n^{m-\beta} \leqq A_{n,q,m,\beta} \cdot \int_0^{\infty} (1-e^{-s})^{\beta-1} |f'(s)|^m ds$ for $\beta \leqq m$. (Received August 22, 1940.)

APPLIED MATHEMATICS

41. E. J. Akutowicz. *On the so-called phase problem for Fourier integrals.*

In the present note the following theorem is proved: Let ρ be a given real or complex-valued function on $-\infty < x < +\infty$ belonging to $L^1 \cap L^2$ and vanishing for $x > x_0$. Let $\mathcal{C}$ be the class of all such functions, where x_0 may vary with the function. Let $\hat{\sigma}$ denote the Fourier transform of σ. Then the only functions $\sigma \in \mathcal{C}$ such that $|\hat{\sigma}| = |\hat{\rho}|$ are of the form $\exp(ic) \cdot \rho(x+\text{const.})$, where c is a real constant. This result is related to a theoretical question of some urgency in crystallography. Namely, to what extent do the absolute values of the Fourier coefficients of a positive periodic function determine that function? (Received September 25, 1950.)

42. M. Z. Krzywoblocki: *On the generalization of Friedrichs' theorem on the nonvanishing of the Jacobian in transonic flow.*

Nikolsky and Taganov proved the theorem on the non-occurrence of a limiting line at the boundary or in the interior of the supersonic irrotational flow region of an inviscid fluid. Friedrichs presented a more complete theorem in which he not only had shown these facts in a different way, but in addition, he proved that the limiting line cannot first appear on the curve of sonic flow of such a fluid. Friedrichs' proof shows precisely that the Jacobian cannot vanish somewhere in the flow domain. In the present paper the author extends Friedrichs' proof on the nonvanishing of the Jacobian to rotational and viscous fluid flows. The relation between Jacobian and the limiting line in such fluids is treated in another paper by the author. (Received September 25, 1950.)

43t. H. E. Salzer: *Formulas for finding the argument for which a function has a given slope.*

The problem of finding where a function $f(x)$ that is tabulated at equal intervals h has a given slope, that is, given $f'(x) \equiv f'(x_0+ph)$, to find p, arises very frequently in numerical computation, especially when locating a maximum or minimum point where $f'(x_0+ph)=0$. Reversion of the series in p which is obtained by differentiating any one of the direct interpolation formulas for $f(x_0+ph)$ is more convenient than (1) calculation of the derivatives at the tabular points followed by inverse interpolation or (2) subtabulation of the derivative around an estimated value of p. These formulas give the coefficients of the reverted series for p directly in terms of the tabular entries so that no differences are necessary. The formulas cover the cases where n-point Lagrangian interpolation formulas are needed in direct interpolation, for $n=3$, 4, 5, 6, and 7. (Received October 10, 1950.)

GEOMETRY

44t. A. R. Schweitzer: *On the derivation of the regressive product in Grassmann's geometrical calculus.* II.

Regressive products are considered only in association with an n-simplex S_n ($n=2, 3, \cdots$) with vertices $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ derived numerically from the independent points $\epsilon_1, \epsilon_2, \cdots, \epsilon_{n+1}$. Then by $P(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$ the author denotes a regressive product and by $V(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ its value; and $P(\alpha_1, \alpha_2, \cdots, \alpha_{n+1}) = V(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})$ is assumed to be the complete statement governing P. The author discusses outer products whose factors are outer products in analogy with determinants having determinants as constituents (including reciprocal determinants). The special case $n=3$ is considered with $P_1=\alpha_1\alpha_2\alpha_3\cdot\alpha_1\alpha_2\alpha_4$ and $P_2=\alpha_1\alpha_2\alpha_3\cdot\alpha_1\alpha_2\alpha_4\cdot\alpha_1\alpha_3\alpha_4$. Let V_1 and V_2 be the values of P_1 and P_2 respectively. Extending a suggestion of Grassmann (*Gesammelte Werke*, vol. 1, part 1, p. 335) the author assumes $V_1=r_1\cdot\alpha_1\alpha_2$ and $V_2=r_2\cdot\alpha_1$ where r_1 and r_2 are numerical functions of the α's and the ϵ's which remain to be specialized. With the aid of elementary properties of determinants it is found that $r_1=c_1q$ and $r_2=c_2q^2$ where $q=[\alpha_1\alpha_2\alpha_3\alpha_4]/[\epsilon_1\epsilon_2\epsilon_3\epsilon_4]$ and c_1 and c_2 are undetermined numerical functions of the ϵ's. (Received October 9, 1950.)

45. Esther Seiden: *Some theorems in finite projective geometry.*

The purpose of this paper is to investigate the properties of a set consisting of a maximum number of no three collinear points in finite projective plane $PG(2,2^n)$—with three homogeneous coordinates, each taking on 2^n values. It is shown that $2^{n-1}+2$ points of the configuration determine uniquely the remaining 2^{n-1} points. The problem studied is: Given 4 no three collinear points to determine all the set of no three collinear points containing these four points. The number of such sets consisting of a conic and the point of intersection of the tangents to the conic is found to be 2^n+2. For $n=3$, the conics form the only sets of no three collinear points. However, for n greater or equal to 4 it is proved that there always exist sets of no three collinear points which do not form a conic. (Received October 11, 1950.)

STATISTICS AND PROBABILITY

46t. J. J. Andrews: *On the admissibility of time series.*

In two papers of Regan (Trans. Amer. Math. Soc. vol. 36 (1934) pp. 511–529 and Amer. J. Math. vol. 43 (1936) pp. 867–873) it was shown that the laws pertaining to the distribution function of time series were consistent. In the first paper an ideal series having constant probability was constructed by the use of the admissible numbers of Copeland. In the problem of time series with variable probability Regan transforms the series obtained in his first paper. This paper combines the results of these two papers. An ideal time series is constructed so that the laws pertaining to the case of variable probability are satisfied and then with suitable restrictions on this series with its corresponding distribution function the case where the probability is constant is solved. The number of essential theorems used here to establish the same results can be reduced to two, with corollaries forming the other results. Regan considered intervals of the form 2^{-r+1} and for the probability of at least one point he also considered rational intervals not of the form 2^{-r+1} and irrational intervals. Besides these intervals, this time series is shown to be valid for a set of points E whose frontier points are of measure zero when applied to the probability of at least one point in the set E. (Received October 9, 1950.)

47t. J. H. Curtiss: *Sampling methods applied to differential and difference equations with special reference to equations of the elliptic type.*

The paper discusses at some length the relationship between random walks in bounded regions and elliptic differential and difference operators, stressing statistical consideration useful in applying the "Monte Carlo method" to the numerical solution of boundary value problems. Although the basic theorems are due to Courant-Friedrichs-Lewy and Petrowski, a number of results believed to be new are presented. These include a uniform bound for the mean duration of general random walks starting anywhere in the region; derivation of the nonhomogeneous difference equation satisfied by the mean duration of a walk on a rectangular lattice; various explicit formulas and special bounds for the mean duration; an estimate of the degree of approximation of the solution of the difference equation to that of the differential equation (the estimate is in terms of the mean duration of the walk); formulas for the dispersion of the random walk statistics (including the dispersion of the number of returns to a given point); a discussion of "importance sampling" and bookkeeping procedures. In connection with the last item a result due to Wasow is proved which shows that if a random walk starts at P and the particle visits a second point P' one or more times, then these visits can profitably be considered as providing a bonus of one, but only one, walk from P'. Connections with sequential analysis in statistics are pointed out. (Received October 9, 1950.)

48t. W. A. Vezeau: *On the product distribution of normally distributed variables.*

A study is made of the product, $z=xy$, where x and y are independently distributed according to the bivariate normal law. The Fourier inversion method is used to determine the frequency function, $f(z)$. For all practical purposes the function, $f(z)$, is difficult to determine. The discussion of $f(z)$ is taken up accordingly as r and s (where r and s are ratios of mean to standard deviation for x and y) are small, not so small, and large. For small r and s, $f(z)$ is expressed as an infinite set of Bessel functions, $K_n(z)$. For medium r and s, an approximation method is given with its error. Extensive calculations are made for $r=1$, $s=4$ and $r=3$, $s=5$. For larger r and s an approximation for $f(z)$ by a normal distribution is given and limits of error established. Instances of statistical problems involving the product frequency function are given. (Received October 6, 1950.)

TOPOLOGY

49. M. L. Curtis: *Homotopy-regular convergence.* Preliminary report.

The notion of regular convergence of a sequence of sets as defined by G. T. Whyburn utilizes homology. By analogy one can define a regular convergence in terms of homotopy. A sequence $\{M_i\}$ of closed sets converging to a set M is said to converge *homotopy-p-regularly* provided that: Given $\epsilon>0$ there exist $\delta>0$ and an integer i_0 such that any continuous p-sphere of dia$<\delta$ in M_i, with $i>i_0$, is homotopic to a point on a set of dia$<\epsilon$ in M_i. It is shown that if M and all M_i are n-dimensional, M is LC^n, and $\{M_i\}\rightarrow M$ homotopy-p-regularly for $p=0, 1, \cdots, n$, then, for sufficiently large i, M_i has the homotopy type of M. The special case in which M is a continuum bounding a domain A in S^n is considered. In this case if the M_i separate A are LC^{q-1}, and converge homotopy-p-regularly to M for $p=0, 1, \cdots, q$, then A is ULC^q and M is LC^q. R. L. Wilder has shown that if a locally contractible continuum separates S^n and is "free in the strong sense" into complementary domain A, then A is ulc^n. Application of our last-mentioned result to this situation shows that A is ULC^n. (Received October 11, 1950.)

50*t*. M. K. Fort: *Subsets of hyperspaces and local connectivity.*

Let 2^X be the set of all nonempty compact subsets of a metric space X. Let H be the topology for 2^X which is induced by the Hausdorff metric, and let R be the topology induced by 0-regular convergence. It is shown that if A is a locally connected continuum in 2^X with respect to the R topology and each member of A is a locally connected continuum in X, then the union of the members of A is a locally connected continuum in X. An example shows that the above theorem is not true if the R topology is replaced by the H topology. (Received October 13, 1950.)

51*t*. W. S. Massey: *A new cohomology invariant of topological spaces.*

Let X be a finite simplicial complex and let H^n denote the n-dimensional cohomology group of X with coefficients in an associative ring. Choose elements $a\in H^p$ and $b\in H^q$. Define $K^m(a, b)$ to be the subgroup of H^m consisting of those elements u, such that $a\cup u=0$ and $u\cup b=0$ (Alexander-Čech-Whitney product). Let $L^m(a, b) =(H^{m-q})\cup b)+(a\cup H^{m-p})$; then $L^m(a, b)$ is a subgroup of H^m. Define a homomorphism $\phi_{a,b}:K^r(a, b)\rightarrow H^s/L^s(a, b)$, where $s=p+q+r-1$, as follows. Given $u\in K^r(a, b)$, choose cocycles a', b', and u' representing a, b, and u respectively. There exist cochains x' and y' such that $a'\cup u'=\delta x'$, and $u'\cup b'=\delta y'$, where δ denotes coboundary. Then the cochain $z'=x'\cup b'+(-1)^{p-1}a'\cup y'$ is a cocycle. Let z denote its cohomology class. Define $\phi_{a,b}(u)$ to be the coset of z mod $L^s(a, b)$. The homomorphism $\phi_{a,b}$ is an invariant of the homotopy type of X, and hence, a fortiori, a topological invariant. It is possible to exhibit complexes X and Y which have isomorphic cohomology rings, no matter what the choice of coefficients, but which can be distinguished by means of this new operation. (Received September 28, 1950.)

52. E. E. Moise and G. S. Young: *On imbedding spaces in 2-manifolds.* II.

Theorem: If every point of a Peano space M has a neighborhood that can be imbedded in the plane, then M can be imbedded in some compact 2-manifold. This generalizes a previous result of the authors [Bull. Amer. Math. Soc. Abstract 54-1-82]. (Received October 11, 1950.)

53*t*. Deane Montgomery and Leo Zippin: *Two-dimensional subgroups.*

Let G be a separable metric connected locally compact n-dimensional group which is not compact. The authors have recently shown that G contains a closed subgroup isomorphic to the real numbers. This paper shows for n at least two that G contains a closed noncompact connected two-dimensional subgroup. (Received October 11, 1950.)

54. Jane C. Rothe: *A definition of index for certain mappings in Hilbert space.*

A topological index is defined for certain mappings of Hilbert space into itself, that is, mappings of the form $I+C+T$ where I is the identity transformation, C is linear, completely continuous, and self-adjoint and T is a higher order transformation which need not be completely continuous. This index has the usual properties of a topological index. It is shown that if T is completely continuous so that the Leray-Schauder index is defined for $I+C+T$, then the Leray-Schauder index and the index defined here are equal. Examples of mappings $I+C+T$ such that T is not completely continuous are exhibited. So it is shown that this definition of index is a nonvacuous extension of the Leray-Schauder definition. (Received October 5, 1950.)

55*t*. M. E. Shanks: *On semi-continuous decompositions of T_1-spaces.*

A decomposition of a T_1-space is upper semi-continuous (usc) if and only if the natural mapping on the decomposition space is closed. Dually, a similar statement is valid for lower semi-continuity (lsc) and open mappings. Different definitions of upper (lower) semi-continuity are given in terms of limits of directed sets, dusc (dlsc). The role of dusc, which differs from usc, is to preserve separation properties under the natural mapping onto the decomposition space. The two definitions of lower semi-continuity are equivalent. (Received October 11, 1950.)

J. W. T. YOUNGS,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The four hundred sixty-third meeting of the American Mathematical Society was held at the University of California, Los Angeles, on Saturday, November 25, 1950. The attendance was approximately 100, including the following 76 members of the Society:

T. M. Apostol, Richard Arens, L. A. Aroian, W. G. Bade, E. F. Beckenbach, Clifford Bell, Gertrude Blanch. H. F. Bohnenblust, J. V. Breakwell, F. H. Brownell, A. S. Cahn, Luther Cheo, T. M. Cherry, L. M. Coffin, J. G. van der Corput, A. C. Davis, E. A. Davis, C. R. DePrima, R. J. Dickson, R. P. Dilworth, H. A. Dye, Arthur Erdélyi, Harley Flanders, G. C. Forsythe, J. W. Green, W. T. Guy, G. J. Haltiner, A. R. Harvey, Emma Henderson, J. G. Herriot, M. R. Hestenes, P. G. Hoel, Alfred Horn, R. E. Horton, D. G. Humm, Rufus Isaacs, J. R. Jackson, Fritz John, P. B. Johnson, P. J. Kelly, Cornelius Lanczos, G. E. Latta, R. B. Leipnik, G. F. McEwen, J. C. C. McKinsey, A. V. Martin, A. B. Mewborn, W. E. Milne, T. S. Motzkin, Fritz Oberhettinger, E. E. Osborne, R. H. Owens, L. J. Paige, George Pólya, W. T. Puckett, H. R. Pyle, W. P. Reid, Edgar Reich, R. R. Reynolds, Seymour Sherman, G. E. F. Sherwood, E. S. Sokolnikoff, R. H. Sorgenfrey, M. L. Stein, Robert Steinberg, E. G. Straus, A. C. Sugar, J. D. Swift, A. E. Taylor, F. G. Tricomi, F. A. Valentine, D. D. Wall, S. S. Walters, L. F. Walton, Morgan Ward, W. R. Wasow, G. M. Wing, F. H. Young.

In the morning there was a general session for contributed papers at which Professor E. F. Beckenbach presided. At the conclusion of this session, brief reports were made by Professor E. F. Beckenbach and Professor Frantiček Wolf on the progress of the Pacific Journal of Mathematics. In the afternoon Professor Erdélyi introduced Professor van der Corput, who presented the invited address, *Asymptotic expansions*. This was followed by two sessions for contributed papers, at which Professors A. E. Taylor and L. F. Walton presided.

Following the meeting there was a tea at the Institute for Numerical Analysis of the National Bureau of Standards. On display for the visitors were the extensive computing machinery of the Institute and the offices of the Pacific Journal.

Abstracts of papers presented at the meeting follow. Papers presented by title are indicated by the letter "*t*." Paper number 56 was read by Professor Apostol.

Algebra and Theory of Numbers

56. T. M. Apostol and H. S. Zuckerman: *On magic squares constructed by the uniform step method.*

If the numbers 1, 2, $\cdots$, n^2 are placed in a square array such that any row or column contains a complete residue system mod n, but no two numbers between the same two multiples of n, then the square is called regular. Every regular square is magic. Let (A, B) be the coordinates of the square in the Ath column and Bth row. To construct the square, 1 is placed in cell (p, q). If x is in (A_x, B_x), then in general $x+1$ is placed in $(A_x+\alpha, B_x+\beta)$ by making the "step" (α, β) from (A_x, B_x). If this cell is already occupied, then a "break-step" (a, b) is used to put $x+1$ in $(A_x+\alpha+a, B_x+\beta+b)$ instead. If this cell is also occupied then a second break-step (c, d) is used. Let $D(\alpha, n)=D(\beta, n)=\delta$, $D(a, n)=D(b, n)=\epsilon$, $D(\alpha b-a\beta, n)=\delta\epsilon\theta$, $D(\delta, \epsilon)=1$, where D denotes the g.c.d. Then the method above is described by the congruences $A_x\equiv p+\alpha(x-1)+a[(x-1)\delta/n]+c[(x-1)\delta\epsilon\theta/n^2]$, $B_x\equiv q+\beta(x-1)+b[(x-1)\delta/n]+d[(x-1)\delta\epsilon\theta/n^2]$ (mod n). From these congruences the authors prove: The square will be filled if and only if $D(bc-ad, \delta\epsilon\theta)=\epsilon$, $D(\beta c-\alpha d, \delta\epsilon\theta)=\delta$. If $\theta=1$, the resulting square will be regular if and only if $D(c, \delta\epsilon)=D(d, \delta\epsilon)=1$. The case $\delta=\epsilon=\theta=1$ was considered by D. N. Lehmer (Trans. Amer. Math. Soc. vol. 31 (1929) pp. 529–551). (Received October 11, 1950.)

57. Anne C. Davis: *A fixpoint criterion for completeness of a lattice.* Preliminary report.

Let L be a lattice with the inclusion relation $\leqq$. As is known (Birkhoff, *Lattice Theory*, rev. ed., p. 54; Tarski, Bull. Amer. Math. Soc. Abstract 55-11-496), for L to be complete it is necessary that every increasing function on L to L have a fixpoint. Tarski asked the question whether this condition is also sufficient. The answer is affirmative. In fact, if L is incomplete, there are sets $X\subseteq L$ with no least upper bound; let M be such a set with the smallest possible power. Using the well-ordering principle, two transfinite sequences, α and β, are constructed such that: (i) α is increasing, every term of α is the least upper bound of some set $X\subseteq M$, and every upper bound of α is an upper bound of M; (ii) β is decreasing, every term of β is an upper bound of M, and no lower bound of β is an upper bound of M. Now let $x\in L$. If x is an upper bound of M, let $f(x)$ be the first term β_η of β with x not $\leqq \beta_\eta$; otherwise, let $f(x)$ be the first term α_ξ of α with α_ξ not $\leqq x$. The function f thus defined is increasing and has no fixpoint. (Received October 12, 1950.)

58t. D. G. Duncan: *A problem in modular lattices.* (*Problem 29, G. Birkhoff, Lattice theory, rev. ed.*)

Denote by C the lattice obtained by embedding the lattice $1+1=FL(2)$ in a finite chain. The free modular lattice generated by $2+C$ is determined by constructing the meet and join tables of its elements. The free modular lattice generated by $2+1+1$ (Problem 29) then appears as a special case of this more general problem. (Received October 12, 1950.)

59. Harley Flanders: *Some matrix theorems.*

First result: If A is a normal matrix, then A^* is expressible as a polynomial in A. More generally, if $A_1, \cdots, A_m$ are normal transformations on a finite-dimensional space, then there is a complex polynomial $f(x)$ such that $A_i^*=f(A_i)$. Second result: Let A be an m by n and B an n by m matrix with coefficients in an abstract field k. Then the nonzero characteristic roots of AB and BA coincide with the same *geometric* multiplicities. What is more, the corresponding elementary divisors are the same. (Received November 25, 1950.)

60. L. J. Paige: *Complete mappings of finite groups.*

A biunique mapping $x \rightarrow \theta(x)$ of a group G upon G is said to be a complete mapping if $x \cdot \theta(x) \equiv \eta(x)$ is a biunique mapping of G upon G. A necessary condition that there exist a complete mapping for a group G is that there exist an ordering of the elements of G such that the product of the elements in this order is the identity. If H is an invariant subgroup of G such that H and G/H have complete mappings, then G has a complete mapping. (Received October 10, 1950.)

61*t*. Alfred Tarski: *On homomorphic images of complete Boolean algebras*. Preliminary report.

Let $\mathfrak{P}(X)$ denote the power of the set X. Given a cardinal $\aleph$, a Boolean algebra B is $\aleph$*-complete* if the join $\sum X$ exists for every $X \subseteq B$ with $\mathfrak{P}(X) \leqq \aleph$; an ideal I in B is $\aleph$*-complete* if $\sum X \in I$ for every $X \subseteq I$, $\mathfrak{P}(X) \leqq \aleph$. Let B be an $\aleph_0$-complete Boolean algebra, I any ideal in B, and B/I the corresponding coset algebra. Then the following results hold: I. *If $X \subseteq B/I$ is a set of disjoint elements, $\mathfrak{P}(X) \leqq \aleph_0$, and $\sum X$ exists (in B/I), then $\sum Y$ exists for every $Y \subseteq X$.* II. *If $\mathfrak{P}(X) \leqq \aleph_0$ for every set $X \subseteq B/I$ of disjoint elements, then B/I is complete.* III. *If m is any measure on B, and I the set of all $x \in B$ with $m(X) = 0$, then B/I is complete.* IV. $\mathfrak{P}(B/I) \neq \aleph_0$. II and III were previously obtained under additional assumptions: I is $\aleph_0$-complete, m is countably additive (for references see Tarski, Comtes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie vol. 30, Class III, pp. 175 ff.). IV implies that no countably infinite Boolean algebra is a homomorphic image of a complete atomistic Boolean algebra (a solution of a problem of Dilworth). Assuming that B is $\aleph_{\xi+1}$-complete and I is $\aleph_\xi$-complete (ξ an arbitrary ordinal), I and II remain valid if $\aleph_0$ is replaced by $\aleph_{\xi+1}$. (Received October 13, 1950.)

62. Morgan Ward: *A class of soluble diophantine equations.*

The diophantine equation (D): $Z^m = F(x)$ is considered over a commutative ring R with unit element. Here $F(x)$ is a homogeneous form in k indeterminates $x_1, x_2, \cdots, x_k$ with coefficients in R of degree n, m is a positive integer, and z is another indeterminate. In case m is prime to n, there exists in R a k-parameter family of solutions of (D) which may be explicitly exhibited given m, r, and $F(x)$. In case R is a field, this family essentially includes all solutions of (D) in R with $z \neq 0$. For example, the diophantine equation $x^n + y^n = z^m$ has a two-parameter family of integral solutions for every m prime to n. (Received November 6, 1950.)

ANALYSIS

63. W. G. Bade: *The inversion of generalized convolution transforms.*

Let $E(s) = e^{bs} \prod_1^\infty (1 - s/a_k) e^{s/a_k}$, a_k real, $\sum_1^\infty 1/a_k^2 < \infty$, and let $G(t)$ be the inverse bilateral Laplace transform of $1/E(s)$. Widder and Hirschman (Trans. Amer. Math. Soc. vol. 66 (1949) pp. 135–201) have shown that in the space $C[-\infty, \infty]$ (and more generally) the operator $\int_{-\infty}^{\infty} G(\xi) x(t - \xi) d\xi$ is inverted by $\lim_{n \to \infty} e^{bD} \prod_1^n (1 - D/a_k) e^{D/a_k}$ where $D = d/dt$, $e^{aD} x(t) = x(t + a)$. Essential use is made of the fact $G(t) \geqq 0$, and various methods are needed for different distributions of the $\{a_k\}$. Their results for $C[-\infty, \infty]$ are covered by the following theorem, which does not require that $G(\xi)$ be real. Theorem: Let X be a complex Banach space. Let T be a closed distributive (generally unbounded) operator satisfying (a) the domain $D(T)$ is dense in X, (b) $\sigma(T)$ is confined to the strip $-\alpha < \sigma < \alpha < \infty$, $s = \sigma + i\tau$, (c) the resolvent $(sI - T)^{-1} = R_s(T)$

satisfies $\|R_s(T)\| \leqq 1/|\sigma-\alpha|$. Let $E(s)$, $P_n(s)$ $(P_0 \equiv 1)$ satisfy (1) $P_n(s)$ are polynomials in s and e^{as}, (2) $P_n(s) \rightarrow E(s)$ and $|P_n(s)/E(s)| \leqq M$ for s in $-r<\sigma<r$, $r>\alpha$, (3) $P_n(s)/E(s)$ are absolutely convergent bilateral Laplace transforms in $-r<\sigma<r$ of functions $G_n(\xi)$, $-\infty<\xi<\infty$, $(G_0=G)$, (4) the transformations defined by $\int_{-\infty}^{\infty} G_n(\xi)e^{-\xi T}x d\xi$, $x \in X$, have uniformly bounded norms. Then if $y=\int_{-\infty}^{\infty} G(\xi)e^{-\xi T}x d\xi$, $x \in X$, $\lim_{n\rightarrow\infty} P_n(T)y=x$. For definition of $e^{\xi T}$ see Hille, Amer. Math. Soc. Colloquium Publications, vol. 31, chap. 12. Convergence is proved first for $x \in D(T^2)$ and extended using the Banach-Steinhaus Theorem. (Received November 7, 1950.)

64. F. H. Brownell: *Asymptotically ergodic output under ergodic input of delay differential machines.*

Consider here the behavior of a linear autonomous delay differential machine subjected to an ergodic input, the governing equation thus being (1) $x^{(n)}(t) + \sum_{k=0}^{n-1} \int_0^{\infty} x^{(k)}(t-h)dF_k(h) = y(t)$ where $y(t)$ is the input and $x(t)$ the output. If the machine coefficient functions $F_k(h)$ satisfy a few reasonable conditions, and if the resulting system is stable, one can show that if $y(\xi, t)$ is an ergodic process for ξ over a probability space, then $x(\xi, t) = x_1(\xi, t) + x_2(\xi, t)$ where $x_1(\xi, t)$ decays exponentially for each ξ and $x_2(\xi, t)$ is an ergodic process. Also the power spectrum of $x_2(\xi, t)$ can be obtained as one would expect formally from that of $y(\xi, t)$. The proof uses the Laplace transform representation of the solution of (1), which can easily be justified, and from which the conclusion is almost trivial. The result is implicit for the significance of much of Wiener's "time series." Doob in the Berkeley Symposium, 1945–1946, proves for equation (1) without delay terms that a stationary input implies an asymptotically stationary output, but does not discuss ergodicity. (Received October 23, 1950.)

65. H. A. Dye: *A Radon-Nikodým theorem for operator algebras.*

Let $\mathfrak{A}$ be a W^*-algebra of operators on a complex Hilbert space, $\mathfrak{A}$ being of finite class and satisfying the condition that any family of mutually orthogonal nonzero projections is at most countably infinite. A positive functional ρ on $\mathfrak{A}$ is called countably additive in case $\rho(\sum P_n) = \sum \rho(P_n)$, for any sequence of mutually orthogonal projections in $\mathfrak{A}$. An absolute continuity relation $<$ is introduced between countably additive positive functionals by defining $\rho<\sigma$ in case $\sigma(P)=0$ implies $\rho(P)=0$, for any projection P in $\mathfrak{A}$. Under these conditions the following Radon-Nikodým theorem holds: $\rho<\sigma$ if and only if $\rho(A)=\sigma(T^*AT)$, for some closed densely-defined operator $T(\eta\mathfrak{A})$ and all A in $\mathfrak{A}$. A justification of this formal equation is contained in the key result that any such functional has the form $\sum_{i=1}^{\infty} (Ax_i, x_i)$, for appropriate vectors x_i. As corollaries, one proves the following: (1) the trace T on a finite ring satisfies an equation $\sum_{i=1}^{\infty} (T(A)x_i, x_i) = \sum_{i=1}^{\infty} (Ax_i, x_i)$; (2) the strongest and the strong neighborhood topologies coincide on any finite ring; (3) any algebraic *-isomorphism between finite rings is bicontinuous in the weak neighborhood topology. (Received November 6, 1950.)

66. Alfred Horn: *Symmetrisable operators.*

A bounded linear operator T in Hilbert space is said to be symmetrisable if there exists a positive symmetric bounded operator G such that GT is symmetric. It is proved that such an operator T always has values different from 0 in its spectrum provided that $GT \neq 0$. The proof carries over to the case of *-algebras. In case T is completely continuous, and $Gx=0$ implies $Tx=0$, A. C. Zaanen has given an expansion for Tx in terms of the eigen-vectors of T. Another proof is obtained here. (Received October 12, 1950.)

67. Rufus Isaacs: *The convergence of monodiffric power series.*

Monodiffric functions of a complex variable are those which satisfy partial difference equations analogous to the Cauchy-Riemann equations and thus constitute a finite difference analogue of classical analytic functions. They include a class of series corresponding to power series which can also be viewed as a complex extension of the Newton interpolatory series. Their region of convergence is as follows: A certain point $z_0=x_0+iy_0$ in the strip $|x-y|\leqq 1$ is called the vertex of convergence. The series converges in the quarter-plane $x>x_0$, $y>y_0$; also (if they are not already included) the half-lines x=nonnegative integer, $y>y_0$ and y=n.n.i., $x>x_0$, and the points where both x and y=n.n.i.'s. The series diverges in the interior of the remaining region with two possible exceptions. There is a segment extending downwards and leftwards from z_0 at 45° of length always less than $3/2$ which is not covered by the present proof; certain special series where only every fourth coefficient is not zero may converge at every fourth integral lattice point on the line $x=y$. (Received November 22, 1950.)

68. R. B. Leipnik: *Size functions.*

Let S be a set, and for each cardinal a, let $S(a)$ be the set of a-membered subsets of S. A function f is called an a-size function in S if and only if f is on S (successor a) to the non-negative finite reals, $\sum_{x\in T} f((T-\{x\})\cup\{y\})\geqq f(T)$ whenever $y\in S$ and $T\in S$ (successor a), and $f(U)>0$ if and only if cardinal (U)=cardinal (distinct (U)), where distinct (U) is the set of distinct elements of U. (A 1-size function is a metric.) The relation of size functions to measure and dimension theory is developed by metric techniques. (Received October 12, 1950.)

69. J. J. Newman: *Remark on some papers by H. Pollard.*

This note gives a short proof that there exists a function $f(x)$ of class L^4 such that the Legendre series of $f(x)$ fails to converge in L^4 mean. This settles a question raised by H. Pollard, who showed (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 387–403) that the same property obtains for L^p if $p>4$ but not if $2\leqq p<4$. The analogous questions for ultraspherical and Jacobi series (Pollard, loc. cit. vol. 63 (1948) pp. 355–367, and Duke Math. J. vol. 16 (1950) pp. 189–191) have a similar answer. (Received October 16, 1950.)

70. F. G. Tricomi: *Expansion of the hypergeometric functions in series of confluent ones.*

Putting $x=2z/(c-2a-z)$ Gauss hypergeometric function $F(a, b; c; x)$ may be developed in an infinite series of confluent hypergeometric functions, $F_1(c-b, c+m; z)$, $m=0, 1, 2, \cdots$, whose coefficients are connected in a very simple manner with the coefficients A_m of this author's known expansion of the confluent functions in series of Bessel functions. The series converges inside the circle $|z|<|c-2a|$. Applied to the particular case of the Jacoby polynomials the new expansions furnish very good and simple asymptotic approximations of the *first* and *last* zeros of these polynomials, which even in the case of the Legendre polynomials are better than those given in the past. (Received October 4, 1950.)

71. F. H. Young: *Matrix transformations of Fourier coefficients.*

Let $T=(a_{kj})$ be an $\infty\times\infty$ matrix of real constants, where $\sum_{j=0}^{\infty}|a_{kj}|<M$ for all k. Let $f\sim(a_k, b_k)$, f in L_p for some p, $1\leqq p\leqq\infty$. T is said to belong to the class (L_p) if $(\sum_{j=0}^{\infty}a_{kj}a_j, \sum_{j=0}^{\infty}a_{kj}b_j)\sim F$ in L_p. Necessary and sufficient conditions are developed for T to belong to (L_p), with special emphasis on the end points (L_1) and (L_∞). Certain special results are obtained from the use of symmetric matrices. Matrix transformations are generalizations of those effected by factor sequences, which correspond to diagonal matrices. (Received October 2, 1950.)

GEOMETRY

72. F. A. Valentine: *A characterization of simply-connected closed arcwise convex sets.*

Let S be a set of points in the Euclidean plane E_2. A set $S\subset E_2$ is said to be *unilaterally connected* if for each pair of distinct points x and y in S, there exists a continuum $S_1\subset S$ which contains x and y, and which lies in one of the closed half-planes determined by the line $L(x, y)$. (The line $L(x, y)$ is determined by x and y.) A set $S\subset E_2$ is said to be *arcwise convex* if each pair of points in S can be joined by a convex arc lying in S. (A convex arc is one which is contained in the boundary of its convex hull.) *Theorem: A necessary and sufficient condition that a simply-connected closed set* $S\subset E_2$ *be arcwise convex is that it be unilaterally connected.* (Received October 11, 1950.)

LOGIC AND FOUNDATIONS

73*t*. Alfred Tarski: *On a statement related to the principle of choice.* Preliminary report.

Let A be any set, B the set of all its nonempty subsets, and C the set of all functions on B to A. The fact that the principle of choice does not apply to subsets of A can be expressed as follows: I. *For every* $f\in C$ *there is an* $X\in B$ *such that* $f(X)\notin X$. Consider the statement: II. *There is a function* G *on* C *to* B *such that* $f(G(f))\notin G(f)$ *for every* $f\in C$. II has the form of a special case of the principle of choice. Clearly II implies I. Henkin, who formulated II, raised the problem whether II can be derived from I (without the help of the principle of choice). The solution is affirmative. In fact, given an $f\in C$, we define by transfinite induction: $x_0=f(A)$, $x_\beta=f(A-\{x_\xi\}_{\xi<\beta})$ unless $A-\{x_\xi\}_{\xi<\beta}$ is empty. By I we conclude that there is smallest α such that $A-\{x_\xi\}_{\xi<\alpha}$ is not empty and $x_\alpha\notin A-\{x_\xi\}_{\xi<\alpha}$. We let $G(f)=A-\{x_\xi\}_{\xi<\alpha}$. (Transfinite ordinals can be eliminated from this argument by the method used by Zermelo in the proof of the well ordering principle.) Thus, II *is equivalent to* I, and *the statement that* II *holds for no set* A *is equivalent to the axiom of choice.* (Received October 13, 1950.)

74*t*. Alfred Tarski: *Remarks on the formalization of the predicate calculus.* Preliminary report.

The lower predicate calculus with identity can be formalized as indicated in Quine's *Mathematical logic*, chap. 2, by adding: *I. $\vdash\ulcorner\alpha=\alpha\urcorner$, and *II. $\vdash\ulcorner\alpha=\beta\supset(\Phi\supset\Psi)\urcorner$ *where* Φ *is atomic and* Ψ *is obtained from* Φ *by replacing an occurrence of* α *by* β. This formalization can be equivalently simplified by replacing (Quine's) *103 by *103′: $\vdash\ulcorner\Phi\supset(\alpha)\Phi\urcorner$ *for* α *not occurring in* Φ; *104 by *104′: $\vdash\ulcorner(\alpha)\Phi\supset\Phi\urcorner$; *I by *I′: $\vdash\ulcorner\sim(\alpha)\sim(\alpha=\beta)\urcorner$. In this new version the notion of proper substitution of free variables is not involved. The notion of free variables can be dispensed with entirely by agreeing to consider as theorems all formulas whose closures are theorems in the old sense; to formalize the predicate calculus thus modified, omit all mention of closure in *100, *101, *102, *103′, *104′, *I′, *II, and accept (besides modus ponens, *105) a new rule of inference: *If ϕ is a theorem, so is* $\ulcorner(\alpha)\Phi\urcorner$. Intuitively underlying the preceding remarks is the following simple fact: *If Ψ is like Φ except for containing free occurrences of β whenever Φ contains free occurrences of α, then* $\vdash\ulcorner\Psi\equiv(\alpha)(\alpha=\beta\supset\Phi)\urcorner$, that is, $\ulcorner(\alpha)(\alpha=\beta\supset\Phi)\urcorner$ is an equivalent expression for Ψ. The above observations are relevant for the foundations of many-dimensional projective algebras. Compare Abstract 57-1-75 by Thompson. (Received October 23, 1950.)

75*t*. F. B. Thompson: *Some problems concerning the predicate calculus.* Preliminary report.

K is the class of theorems of the predicate calculus which remain theorems when atomic formulae are uniformly replaced by atomic formulae with the same variables. Let $L\subset K$ contain those theorems which remain theorems when atomic formulae are uniformly replaced by arbitrary atomic formulae. Among axioms for predicate calculus in Quine's *Mathematical logic*, *100–*102 are in L, *103 is in K but not in L, *104 is in neither. Let *103′ be $\vdash\ulcorner(\alpha)\Phi\supset(\alpha)(\alpha)\Phi\urcorner$ and $\vdash\ulcorner\sim(\alpha)\Phi\supset(\alpha)\sim(\alpha)\Phi\urcorner$, and *104′ be $\vdash\ulcorner(\alpha)\Phi\supset\Phi\urcorner$. *103′ and *104′ are in L. Tarski asked the questions: Are all theorems (i) in K, (ii) in L, derivable from (i) *100–*103, *104′, (ii) *100–*102, *103′, *104′? The theorems which are closures of $\ulcorner(\alpha)\Phi\wedge(\beta)\Theta\wedge(\gamma)\Psi\supset(\alpha)(\beta)(\gamma)\cdot\{(\alpha)[(\beta)\Psi\wedge(\gamma)\Theta]\wedge(\beta)[(\gamma)\Phi\wedge(\alpha)\Psi]\wedge(\gamma)[(\alpha)\Theta\wedge(\beta)\Phi]\}\urcorner$, α, β, γ being free in Φ, Θ, and Ψ, are counter-examples for both cases. A problem formulated by Jaśkowski, Studia Societatis Scientiarum Torunensis. Sect. A vol. 1, p. 19 (see Henkin, Journal of Symbolic Logic vol. 14, p. 65), is equivalent to problem (ii), and thus has a negative solution. In algebraic translation, the above result gives a negative solution of the representation problem for 3-dimensional projective algebras. (Many-dimensional projective algebras were defined by Chin-Tarski. For 2 dimensions see Everett-Ulam, Amer. J. Math. vol. 68, and Chin-Tarski, Bull. Amer. Math. Soc. Abstract 54-1-90.) (Received October 13, 1950.)

STATISTICS AND PROBABILITY

76. Edgar Reich: *On the definition of information rate.*

Consider a two-point unidirectional communication system where $p(x, y)$ is the probability that a random variable x is transmitted, and as a result of this, a random variable y, not necessarily the same as x if there is "noise," is received. We have $p_0(x)=\int p(x, y)dy$ as the a priori probability that x *will* be transmitted, and $p_y(x)=p(x, y)/\int p(x, y)dx$ as the a posteriori probability that x *was* transmitted. Postulate the existence of $F(u, v)$ such that $U\{p\}=\int F[p(x), x]dx$ is a measure of the uncertainty associated with the distribution $p(x)$. Then $I=U\{p_0(x)\}-U\{p_y(x)\}$ is a measure of the information gained on receiving y, and $R=E[I]$ is the rate at which information is received. Require the invariance of R under any one-to-one relabeling of the variables $\xi=f(x)$, $\eta=f(y)$. Under mild restrictions, it follows that $R=\int\int p(x, y)\cdot\log[p(x, y)/p_0(x)q(y)]dxdy$, where $q(y)=\int p(x, y)dx$. This is the formula proposed by C. E. Shannon from other considerations. (Received September 18, 1950.)

TOPOLOGY

77*t*. S. T. Hu: *Reduction of the homotopy sequences of the spaces of*

paths to the exact sequences concerning the triad homotopy groups.

A path σ in a topological space X is a map σ: $I \to X$ of the unit interval I into X. Let Ω denote the space of all paths in X with the compact-open topology. Let A, B be subsets of X with nonvacuous intersection $C = A \cap B$ and $x_0 \in C$. Denote by $[X, A, B]$ the subspace of Ω consisting of the paths σ with $\sigma(0) \in A$ and $\sigma(1) \in B$ and analogously for analogous notations. The point path $\sigma_0(I) = x_0$ is used as the basic point for all homotopy groups concerned. First, the homotopy sequence of the pair $[X, x_0, B]$ and $[A, x_0, C]$ reduces to the exact sequence $\cdots \to \pi_n(A, C) \to \pi_n(X, B) \to \pi_n(X; A, B) \to \pi_{n-1}(A, C) \to \cdots$ of Blakers and Massey. Next, let Γ denote the subset of $[X, x_0, x_0]$ consisting of the paths σ with $\sigma(t) \in A$ if $t \leqq 1/2$ and $\sigma(t) \in B$ if $t \geqq 1/2$. The homotopy sequence of the pair $[X, x_0, x_0]$ and Γ reduces to the exact sequence $\cdots \to \pi_n(A/B) \to \pi_n(X) \to \pi_n(X; A, B) \to \pi_{n-1}(A/B) \to \cdots$ of Blakers and Massey (Bull. Amer. Math. Soc. Abstract 56-2-208). Last, assume that A and B be contractible. Then the homotopy sequence of the pair $[X, A, B]$ and $[C, C, C]$ reduces to a new exact sequence $\cdots \to \pi_{n+1}(X; A, B) \to \pi_{n-1}(C) \xrightarrow{s} \pi_n(X) \to \pi_n(X; A, B) \to \cdots$. The homomorphism s: $\pi_{n-1}(C) \to \pi_n(X)$ is a generalisation of Freudenthal's suspension operation. (Received October 11, 1950.)

78*t*. S. T. Hu: *A fibering of the space of paths over the space of curves in a metric space.*

Consider a metric space X with metric ρ and two subsets A and B. Let Ω denote the totality of the paths (equal to p-curves) f: $I \to X$ in X with $f(0) \in A$ and $f(1) \in B$. Ω forms a metric space with metric ρ defined by $\rho(f, g) = \sup_{t \in I} \rho(f(t), g(t))$. Let $d(f, g)$ denote the Fréchet distance between f and g. d defines an equivalence relation $f \sim g$ if and only if $d(f, g) = 0$. Denote by Γ the set of all the equivalence classes called curves and by $[f]$ the class containing f. Γ form a metric space with metric ρ defined by $\rho([f], [g]) = d(f, g)$. The association $f \to [f]$ defines a continuous map p: $\Omega \to \Gamma$. It is proved that Ω is a fibre space over Γ with p as projection. The fibres of Ω are all contractible to a point in itself. There is a natural cross-section q: $\Gamma \to \Omega$ which is isometric. If Γ is imbedded into Ω by means of q, then Γ is a deformation retract of Ω. The last assertion is known to E. Pitcher when A and B are single points. This enables one to compute the homology and homotopy invariants of Γ and its subsets by means of those of Ω and its corresponding subsets. (Received October 11, 1950.)

79*t*. S. T. Hu: *Paths of class C^k in a differentiable manifold.*

Let M be a compact differentiable manifold of class C^r and A, B be two nonvacuous subsets of M. Consider the space Ω of the paths (p-curves) f: $I \to M$ with $f(0) \in A$ and $f(1) \in B$ provided with the compact-open topology. Denote by Ω^k, $k \leqq r$, the subspace of Ω which consists of the paths f of class C^k. It is proved that the spaces Ω and Ω^k have the same homology and homotopy groups. More precisely, the singular polytope (equal to the topologized singular complex) $P(\Omega^k)$ of Ω^k is a deformation retract of the singular polytope $P(\Omega)$ of Ω. (Received October 11, 1950.)

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BOOK REVIEWS

Einführung in die Differentialgeometrie. By W. Blaschke. Berlin, Springer, 1950. 8+146 pp. 16 DM.

This new book of Blaschke's is a text covering the metric differential geometry of curves and surfaces lying in ordinary Euclidean space. It covers roughly the same material as the first volume of his *Vorlesungen über Differentialgeometrie,* Berlin, Springer, 3d ed., 1930, but the two books are remarkably different both in content and approach.

The most striking innovation in the present work is the systematic use of E. Cartan's exterior differential forms to which Blaschke is a fairly recent convert. The adoption of this technique has caused him to rewrite large portions of his older book completely. There is more on minimal surfaces here than there was earlier, but the former sections on differential geometry in the large have been condensed or omitted and the earlier treatment of line geometry is omitted entirely. On the whole these changes are for the better, and this book is an excellent introduction to the subject with appropriate emphasis on its classical and modern aspects.

The book is compactly written and contains more information than one would anticipate from so small a number of pages. To a large extent this is made possible by the author's familiar custom of presenting as "exercises" brief outlines of numerous theorems together with appropriate references. These "exercises" are one of the most valuable features of the book. It is unlikely that universities in the United States could use such a book as a classroom text (even if it were in English), but it would serve as an excellent supplement to more usable texts such as the recent outstanding one by D. J. Struik, *Differential geometry*, Addison-Wesley, 1950.

After an initial chapter on vector and matrix algebra, the author turns to a treatment of strips and curves. His emphasis on strips rather than on curves is quite unorthodox and will not find universal favor. Indeed it is surprising (to say the least) to find that the Frenet formulas for a curve are introduced as a lemma preparatory to the Four-Vertex Theorem. The exercises include references to curves of constant width. helices, and the isoperimetric property of the circle (seven proofs).

The third chapter considers the calculus of exterior differential forms. Although the treatment is clearer than those in Cartan's books, there is still an air of mystery about this beautiful technique. An elementary treatment of this subject which explains the numerous subtleties involved would be very welcome.

In the fourth chapter the author turns to the theory of surfaces and discusses their intrinsic properties. The usual topics are discussed including total curvature, the Gauss-Bonnet Theorem, parallel displacement, and special nets. This treatment continues in the fifth chapter which discusses geodesics, surfaces of constant curvature, parallel curves, Liouville nets, and conformal mapping.

The extrinsic properties of a surface such as lines of curvature, asymptotic lines, Meusnier's Theorem and Dupin's Theorem appear in chapter six. All of this is done by Cartan's methods and the exercises present the same theory in Gauss's notation and then in tensor notation. The chapter also includes a discussion of the rigidity and bending of surfaces.

The final chapter treats minimal surfaces and the problem of Plateau. Appropriately Blaschke introduces complex coordinates and complex geometric elements, so that an understanding of this chapter requires a knowledge of the theory of functions of a complex variable.

CARL B. ALLENDOERFER

Supersonic flow and shock waves. By R. Courant and K. O. Friedrichs. New York, Interscience, 1948. 16+464 pp. $7.00.

This book is an excellent up-to-date account of the related problems of supersonic flow and non-linear wave propagation. Its content ranges from theory of hyperbolic partial differential equations to the practical problems of flow in nozzles and jets.

The point of view with which this book is written is best described by the words of the authors: "The book has been written by mathematicians seeking to understand in a rational way a fascinating field of physical reality, and willing to accept compromise with empirical approach." This rational approach is extremely valuable in such a field, where convenient but inaccurate concepts are often found to creep into existing literature. For example, in other discussions, the concept of a Mach line or a characteristic is sometimes introduced by associating it with a small disturbance. While this is no doubt a convenient way to discuss many properties associated with the characteristic, it sometimes leads to the erroneous concept that a characteristic is a line of disturbances. Again, in other discussions, the use of characteristics is often treated so closely with the numerical method of step-by-step integration of a supersonic flow field, that there is a danger of taking the latter as an essential part in the method of characteristics. In this book, all these misleading discussions are absent. The role of characteristics as convenient coordinate curves, the propagation of discontinuities (of higher order) along characteristics, and the Mach line as the limiting position of the weak shock line are treated separately and logically. The numerical method of step-by-step integration of supersonic flow field is discussed briefly toward the end of Chapter II, but it contains the essence of the method. The practical advantage of the present treatment can be seen from the conciseness of the discussion of the problem of designing a perfect nozzle (§113 and pp. 85–86).

Since this book treats the more interesting case of non-linear phenomena, rather than the classical problems of linear wave propagation, it is very much limited to the case of two independent variables. The limited usefulness of the notion of characteristics in the case of more than two independent variables is discussed at the end of Chapter II. [Perhaps it would not have been out of place here to include a brief discussion of the connection between characteristic surfaces and the wave front, such as that in §3, Chapter VI of vol. II of Courant-Hilbert's *Methoden der Mathematischen Physik.*]

The book begins with a discussion of the basic physical notions and the mathematical formulation of basic physical laws. Thus, the book is readable by any mathematician who is familiar with mechanics in general. The second chapter gives the main body of the mathematical basis of later developments in the book. Here are discussed the theory of characteristics and in particular the theory of simple waves. The simple wave theory includes the one-dimensional progressive waves and the Prandtl-Meyer flow in the two-dimensional steady case. It shows more clearly why such simple flows are possible than conventional ad-hoc treatments.

Beginning with the third chapter, this book deals with specific problems. The discussions of the basic aspects are carried out in remarkable detail. "On the other hand, no attempt has been made . . . to provide summaries of results which could be used as recipes for attacking specific engineering problems" (from Authors' Preface).

One-dimensional problems of gas dynamics, including simple waves of rarefaction and compression, shock waves of varying strength, interaction of shock waves, are treated in the third chapter. The discussion of wave interaction is very extensive and systematic, and is perhaps the first time that such work is published in an organized presentation. This chapter also includes a discussion of detonation and deflagration waves and a brief discussion of wave propagation in elastic-plastic material.

Chapter IV treats isentropic steady plane flow. The authors have made the development in this part parallel to but independent of those in Chapter III. This brings out very clearly the analogy and differences between the problems of one-dimensional waves and those of steady plane flows. Besides such topics as the hodograph transformation, limiting lines, simple waves, there is an extensive discussion of shock interaction and reflection. The last part of this chapter deals with the exact nature of problems of steady flow past an obstacle, including an extensive and systematic discussion of the perturbation theory. This part should be highly recommended to the inquisitive workers in this field.

Two more short chapters follow. Chapter V deals with flow in nozzles and jets. Chapter VI deals with three-dimensional flows having suitable symmetry properties, so that there are still two independent variables. The book ends with an extensive and valuable list of references to books and other publications.

In conclusion, the reviewer wishes to recommend this book to every worker in the field of dynamics of a compressible fluid, whether he is studying it from the point of view of a mathematician, a physicist, or an engineer.

C. C. LIN

Moderne algebraische Geometrie. Die idealtheoretischen Grundlagen. By W. Gröbner. Vienna, Springer, 1949. 12+212 pp. $5.70.

Dr. Gröbner's book is a textbook giving the fundamentals of the ideal theory needed in algebraic geometry. The exposition is clear, elegant, and easy to read.

About one-half of the material—basic field theory, the ideal theory of polynomial rings, and the more general commutative ideal theory—can be found in van der Waerden's textbook on algebra; the remainder—material on Hilbert's function, the ring of formal power series, integral algebraic quantities, and the syzygy theory of homogeneous polynomial ideals—can be found without too much difficulty in the literature, but appears here for the first time in textbook form.

Dr. Gröbner has organized all this material excellently, occasionally improving known proofs—especially in the resultant theory and the syzygy theory—and has at all times kept the geometry in the foreground; the motivation of the development of each subject is thus at all times quite clear.

As is obvious, different ideals in a polynomial ring can have the same locus of zeros—in Dr. Gröbner's terminology, the same "Nullstellengebilde" (NG). The author attempts to establish a 1-1 correspondence between ideals and geometric objects, associating with every polynomial ideal its "algebraische Mannigfaltigkeit" (AM), a term used here in a sense different from the customary. The AM of an ideal is to consist, namely, of its NG together with certain "infinitely near" loci. But the author does not go beyond a few suggestive remarks, and the main work on the idea remains to be carried out, if it can be at all.

The chapter on syzygy theory is an original contribution of Dr. Gröbner, and has appeared in the Monatshefte für Mathematik vol. 53 (1949) pp. 1–16. A part of this paper has been criticized by P. Dubreil in the Comptes Rendus, Académie des Sciences, Paris vol. 229 (1949) pp. 11–12, and the criticism applies also to some extent to the book. (Dubreil's and Gröbner's papers are reviewed in Mathematical Reviews vol. 11 (1950) p. 489.)

While illustrative examples are occasionally given in the footnotes, Dr. Gröbner could increase the usefulness of his textbook by including exercises and also further references to the literature.

A. SEIDENBERG

The variational principles of mechanics. By Cornelius Lanczos. (Mathematical Expositions, no. 4.) Toronto, University of Toronto Press, 1949. 25+307 pp. $5.75.

The most outstanding feature of this book is the enthusiastic style in which it is written. The enthusiasm is contagious to the extent that even the most iconoclastic reader can not but be intrigued by "Lagrange's ingenious idea" (on p. 39), by d'Alembert's stroke of genius (p. 88), by Gauss's "ingenious reinterpretation" thereof (p. 106), by the "amazing result" of Hamilton (p. 220), or by the biblical quotation at the head of the eighth Chapter: "Put off thy shoes from off thy feet, for the place whereon thou standest is holy ground."

The author gives on the whole an able exposition of the following topics: The Euler-Lagrange equations of the calculus of variations, d'Alembert's principle, the principle of least constraint, the Lagrangian equations of motion, principle of Hamilton, principle of least action, integration by ignoration of coordinates, the Legendre's transformation, the canonical equations of motion, integral invariants, canonical transformations, the brackets of Lagrange and Poisson, infinitesimal canonical transformations, the partial differential equation of Hamilton and Jacobi, solution by separation of variables, Delaunay's treatment of separable periodic systems, the significance of all this material in the development of both the older and more recent quantum mechanics.

In addition to the final chapter devoted to a brief historical survey, the entire treatment is more or less historical. To give an adequate idea of the general spirit and purpose of this mode of presentation, the reviewer can hardly do better than to quote from the preface:

"There is a tremendous treasure of philosophical meaning behind the great theories of Euler and Lagrange, and of Hamilton and Jacobi, which is completely smothered in a purely formalistic treatment, although it cannot fail to be a source of the greatest intellectual enjoyment, to every mathematically-minded person. To give the student a chance to discover for himself the hidden beauty of these theories was one of the foremost intentions of the author. For this purpose he had to lead the reader through the entire historical development, starting from the very beginning, and felt compelled to include problems which familiarize the student with the new concepts. These problems, of a simple character, were chosen in order to exhibit the general principles involved."

In following this historical approach, the author gives, for instance, Euler's original intuitive derivation of the differential equations of the calculus of variations as well as the more usual rigorous treatment of Lagrange. For having done this in an attractive and readable style, the reader will perhaps forgive the author for having fumbled slightly in his treatment of the Lagrange derivation. For in his discussion of the fundamental lemma of the calculus of variations he comes to an approximate equation (p. 59),

$$\frac{\delta I}{\epsilon} = E(\xi)\int_{\xi-\rho}^{\xi+\rho} \phi(x)dx \qquad \left(E(x) = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)$$

with the remark that the error tends to zero as ρ tends to zero. But he fails to remark that the whole integral on the right also apparently tends to zero, and nothing is said to show why the error might not be numerically equal to the whole right-hand side of the equation if $E(\xi) \neq 0$. The author fails even to state explicitly the essential hypothesis that $E(x)$ be continuous, merely stating rather vaguely that it is "practically constant" over the range of integration when ρ is small. Of course, these difficulties can be easily remedied without sacrifice of brevity and, for this reason, it is the greater pity that the remedies were not included.

It is possible to criticize other details of the book. Why, for instance, does the author drag in a half baked discussion of variation on page 38, where the subject under discussion is merely the problem of minimizing a function $f(u)$ of one or more variables? Here, of all places, whatever distinction can, in general, be made between a differential, du, and a "variation," δu, completely disappears; for u is merely an independent variable. Further on, in connection with the minimization of the definite integral $\int_a^b F(x, y, y')dx$, the author again (and this time quite properly) emphasizes on page 55 the distinction between differential and variation. But it seems to the reviewer that he should also emphasize their similarity. In fact, the variation δy *can* be regarded as a differential just as truly as the quantity denoted by dy. For in forming the variation of a function $y(x)$, the latter might be imagined as imbedded in an arbitrary sufficiently smooth family of functions $y(x, \alpha)$, where α is the parameter of the family and where $y(x, 0) = y(x)$. Then the "differential" $dy = y_x(x, 0)dx$, while the "variation" $\delta y = y_\alpha(x, 0)d\alpha$, the subscripts denoting partial differentiations in the usual manner.

Incidentally the wholesale use of infinitesimals, such as occurs in this book, while somewhat traditional, is rather repulsive to the reviewer. Many, if not all, of the theorems and proofs can be formulated in terms of derivatives just as easily as in terms of infinitesimals, differentials, virtual displacements, and variations. Moreover this can probably be done with no loss of brevity and with considerable gain in clarity and rigor. Other things being equal, relations involving derivatives would seem to be preferable to those involving infinitesimals, because the former are simple equalities whereas the latter are encumbered, either explicitly or implicitly, with higher order infinitesimals which we are supposed to neglect. The spectre of an incorrect appraisal of these orders is ever present.

By failing to give such a treatment using derivatives, the author has missed a golden opportunity of filling a real gap in the literature. Perhaps there are still other and more golden opportunities (both missed and taken) which a more astute reader might have detected. But at any rate, much of the author's scintillating style seems to a certain extent wasted on just another text book, whose claims for originality, even in exposition, are strictly limited.

A number of other minor comments may be made:

The author sometimes is so carried away by his own enthusiasm that, seemingly for rhetorical effect, he transgresses the limits of accuracy. At the bottom of page 113, for example, Hamilton's principle is stated as if it applied to an arbitrary mechanical system and the statement is even italicized. That the author knows better than this is indicated in other passages, one of which occurs on the middle of the next page. Of course, the book, though it mentions non-holonomic systems and other so-called polygenic systems, for which Hamilton's principle does not apply, is not primarily concerned with such systems; and so it is possible to excuse the author for an occasional slip of this kind.

Again the continual use of the phrase *the necessary and sufficient condition* instead of *a necessary and sufficient condition* in the statement of theorems is rather irritating. After all, the definite article is out of place until after the particular necessary and sufficient condition under consideration has somehow been uniquely characterized, and this is almost never the case prior to the statement of such theorems. The theorems themselves perform this function.

The book is well equipped with clearly designated summaries at the end of most of the sections.

There is a short bibliography at the end. The reviewer regrets that one of his favorite treatises on the subject has been omitted from this bibliography, namely *Lezioni di meccanica razionale* by Levi-Civita and Amaldi.

D. C. Lewis

The theory of valuations. By O. F. G. Schilling. (Mathematical Surveys, no. 4.) New York, American Mathematical Society, 1950. 8+253 pp. $6.00.

In the words of the author, the theory of valuations may be viewed as a branch of topological algebra. In fact, historically speaking, it represents the first invasion of topology, more precisely, of early metric topology, into the domains of algebra. The introduction of metric methods into algebra has been so fruitful that today many of the deeper algebraic theories carry their mark. In this regard, one should distinguish between the classical use in algebra of the natural metric of the real or complex number fields, such as in proving the "fundamental theorem of algebra," and the much more recent use of the far less evident metrics which are derived from arithmetic notions of divisibility and which constitute the principal notion of valuation theory. Such a metric occurs for the first time in Hensel's construction of the p-adic numbers, dating from the beginning of this century.

The first abstract definition of a valuation was given by Kürschák in 1913. The systematic development of valuation theory is due chiefly to Ostrowski and Krull to the continuation of whose work the author of the present book has made considerable contributions. From about 1920 onwards, valuation theory has played an important part in the theory of algebraic numbers, for instance in the reformulation and completion by Artin, Chevalley, and Hasse of the class field theory, and in the classification of the simple algebras over algebraic number fields by Hasse and Albert. Valuation theory proper and closely related other topological methods have played a fundamental rôle in Krull's work on the structure of commutative rings, and in the building of rigorous foundations for algebraic geometry, notably in the work of Zariski.

It is evident that the ramifications of valuation theory today are so extensive that no single book could possibly do justice to them all. The present book concentrates on the general significance of valuation theory for the algebraic and arithmetic structure of fields, division rings, and simple algebras, including topics such as the local class field theory and non-commutative ideal theory. It does not enter, for instance, into the interplay between absolute values and p-adic values which rules the global aspects of algebraic number theory, and refrains entirely from entering into the applications of valuation theory to general algebraic geometry. In order to indicate the actual scope of the book it will be best to review, briefly and partially, the content of each chapter.

Chapter 1 gives the formal concepts of valuation theory for division rings with unrestricted value groups, and the elementary results concerning the characterization of valuation rings, the relationships between valuations and homomorphisms, and the prolongations of valuations to algebraic extensions.

Chapter 2 is concerned with various notions of completeness of valuated fields, such as ordinary sequential completeness, F. K. Schmidt's extrinsic definition of maximal completeness, and finally a slightly less restrictive notion of completeness which is so designed that the principal tool of valuation theory, Hensel's arithmetic reducibility criterion for polynomials, can be proved inductively. The prolongation theory which follows is based on Ostrowski's notion of relative completeness, that is, on the hypothesis that Hensel's criterion is valid.

Chapter 3 deals with the ramification theory of valuations which is a generalization of Hilbert's ramification theory for algebraic number fields. The problems of this theory concern certain arithmetically defined subgroups of the Galois group of a field extension, the arithmetic and algebraic properties of the corresponding subfields, and their relationships to the value groups and residue class fields. The theory is presented here in a very general form applying to infinite extensions of general relatively complete fields. Free use is made of Krull's Galois theory for infinite extensions. The chapter also contains existence theorems for field extensions with prescribed features such as Galois group, maximal inertial subfield, value group, and residue class field. There are also some results on the structure of the Galois groups of certain special classes of extensions.

Chapter 4 treats the classical ideal theory which governs the arithmetic of algebraic number fields and fields of algebraic functions of one variable. E. Noether's system of axioms for ideal theory is replaced here by a system of postulates concerning the existence of a set of discrete valuations of rank one with certain finiteness and independence properties. The system is so devised that omission of one of the postulates leads to Artin's theory of quasi-divisibility which has been used in algebraic geometry. The chapter culminates in a description of the totality of fields whose arithmetic is that of the classical ideal theory. Besides the fields mentioned above, this includes certain infinite algebraic extensions of them. These results embody earlier work by MacLane and the author.

Chapter 5 contains the generalization to simple algebras of the ideal theory of Chapter 4. The main object is to show that the arithmetic of simple algebras whose centers have the classical ideal theory conforms to Asano's axiomatic pattern for non-commutative ideal theory. Another important topic of this chapter is the analysis of the class group of simple algebras over a relatively complete field. The results are more complicated and less explicit than those for the special case of algebras over p-adic fields, partly because the class group of the simple algebras over the residue class field, which is trivial in the p-adic case, enters the description.

Chapter 6 gives a generalized version of the local class field theory which establishes a correspondence between the abelian extensions of a complete field F with subgroups of the multiplicative group of F. As in the work of Moriya and Nakayama, it is assumed that the base field F is complete with respect to a discrete valuation of rank one, and that the residue class field is perfect and admits for each positive integer n exactly one cyclic extension of degree n. Under these conditions, the division algebras over F can be enumerated in a very simple fashion, and the correspondence of the local class field theory is established by means of the theory of simple algebras as was done by Chevalley for the p-adic fields. The theory is generalized further to cover infinite extensions, and a number of partial generalizations in several directions are given.

Chapter 7 may be regarded as a continuation of Chapter 2. The main concern is with the algebraic and topological structure of complete fields. It contains the Pontrjagin-van Dantzig-Jacobson theory of locally compact and totally disconnected topological division rings, Kaplansky's theory of maximally complete fields which is a generalization of the theory of Hasse, Schmidt, Teichmüller, and Witt for the case in which the valuation is discrete and of rank one, various equivalence problems for complete valuated fields, and a description of the structure of fields which are complete for a valuation of rank one and have a given residue class field.

In addition, there are two auxiliary appendices, one on the Galois theory for infinite extensions, and one on the general theory of linear algebras. Each chapter has a separate bibliography which greatly facilitates the approach to the extensive literature on the various aspects of valuation theory.

Throughout the book, a great effort is made to present the theory with as much generality as is feasible today. Frequently this is accomplished at the expense of much of the intuitive content of the main results. The presentation is extremely concise, unfortunately to the degree of sometimes omitting the easier parts of a proof, and occasionally of suppressing badly needed explanations of terminology and notation. Although these features will cause some difficulty to the general reader, they will not detract from the great value of this book to the specialist for whom it is apparently designed. It fully accomplishes its main purpose, namely to give a systematic account of the present status of valuation theory without dwelling on its applications to other fields. Indeed, a very considerable amount of recent research, a good deal of which is due to the author himself, is here collected in the short space of 250 pages.

G. Hochschild

Pfaff's problem and its generalizations. By J. A. Schouten and W. van der Kulk. Oxford, Clarendon Press, 1949. 16+542 pp. $12.50.

As originally conceived, Pfaff's problem was to integrate a single equation obtained by equating to zero a linear homogeneous differential expression. Subsequently, the number of equations was increased to a finite arbitrary r and the left members made skew-symmetric forms of arbitrary degrees. Once the dimension of the integral variety sought has been specified, the equations on the differentials are equivalent to equations on the first derivatives of the variables with respect to the parameters on the variety. The most recent generalization, first made in the junior author's thesis written under the senior author's direction in 1945, replaces the skew-symmetric equations defining the derivatives implicitly by parametric equations defining them explicitly, subject to certain conditions on the rank of matrices in the first and second derivatives.

This book gives in one of the notations associated with the tensor calculus a unified account of the main results in the area just outlined. The classical theories of first order linear partial differential systems, reduction of a Pfaffian form or equation to canonical form, and contact transformations are natural applications and are worked into the treatment at the appropriate places. In the realm of the authors' own contributions, there is an extended discussion of the following topics: arithmetical invariants, reduction of systems to canonical form, van der Kulk's integration theory, Cartan's prolongation, and a comparison of inferences to be drawn for partial differential systems from the existence theorems of Cartan and Riquier. The authors' remarks on prolongation will be read with interest but must remain in the domain of speculation until Cartan's prolongation process has been made precise. In particular, to the reviewer, the authors' attribution of symmetry to Cartan's theory at present seems to express a hope rather than an actuality.

This treatise contains a wealth of material gathered together in a uniform treatment for the first time, although it is by no means exhaustive. Numerous exercises are provided together with suggestions for their solution. In some quarters, the notation will be regarded as forbidding. It is to be regretted that all concerned including societies and congresses do not give more attention to the improvement and standardization of notation and nomenclature.

J. M. THOMAS

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NOTES

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The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Committee, who have been consulted regarding papers submitted for publication: R. P. Agnew, A. A. Albert, Warren Ambrose, R. D. Anderson, Richard Arens, Reinhold Baer, R. W. Barnard, E. F. Beckenbach, A. C. Berry, Salomon Bochner, D. G. Bourgin, R. C. Buck, Henri Cartan, H. S. M. Coxeter, W. F. Eberlein, Arthur Erdélyi, William Feller, M. K. Fort, P. R. Garabedian, A. M. Gleason, Oscar Goldman, W. H. Gottschalk, Philip Hall, P. R. Halmos, O. G. Harrold, Philip Hartman, G. A. Hedlund, M. H. Heins, M. R. Hestenes, W. V. D. Hodge, L. K. Hua, Florence D. Jacobson, Mark Kac, L. M. Kelly, N. H. McCoy, E. E. Moise, F. J. Murray, Leopoldo Nachbin, John von Neumann, Harry Pollard, J. F. Randolph, P. C. Rosenbloom, A. C. Schaeffer, M. M. Schiffer, Wladimir Seidel, Atle Selberg, Max Shiffman, M. F. Smiley, L. J. Snell, R. H. Sorgenfrey, George Springer, Otto Szasz, W. J. Thron, H. E. Vaughan, S. E. Warschawski, Antoni Zygmund.

A mimeographed list of chairmen of the various departments of mathematics in the United States for the academic year 1950–1951 is available and will be sent to any member of the Society who requests it.

The First National Congress of Applied Mechanics will be held June 11–16, 1951, at Chicago. The American Mathematical Society will be one of the sponsors of the Congress.

The Institute for Teachers of Mathematics, sponsored by the Association of Teachers of Mathematics in New England, will be held at Connecticut College, August 23–30. Information may be obtained from Mr. C. H. Mergendahl, Newton High School, Newton, Massachusetts, or Miss Katharine E. O'Brien, Deering High School, Portland, Maine.

Professor Marston Morse of the Institute of Advanced Study has been appointed a member of the National Science Board.

Professor H. B. Curry of Pennsylvania State College, Dr. H. W. Kuhn of Princeton University, and Professor S. S. Wilks of Princeton University have been awarded Fulbright research scholarships. Professor Samuel Eilenberg of Columbia University and Professor Norbert Wiener of Massachusetts Institute of Technology have been appointed Fulbright visiting lecturers for 1950–1951.

Professor E. P. Wigner, of Princeton University has received the Franklin Medal from the Franklin Institute.

Mr. Z. A. Melzak of McGill University has been awarded the William Lowell Putnam scholarship at Harvard University for the tenth annual competition.

Professor H. R. Pitt of Queen's University, Belfast, Ireland, has been appointed to a professorship at the University of Nottingham, Nottingham, England.

Dr. M. R. Siddiqi of Osmania University, Hyderabad, India, has been appointed director of research at Peshawar University, Peshawar, Pakistan.

Professor Thirukkannapuram Vijayaraghavan of Andhra University, Waltair, India, has accepted a position as director of the Ramanujan Institute of Mathematics, Madras, India.

Mr. R. F. Benton of the California Institute of Technology has accepted a position as mathematician with the U. S. Naval Air Missile Test Center, Point Mugu, California.

Associate Professor A. W. Boldyreff of the University of New Mexico has accepted a position at Sandia Corporation, Albuquerque, New Mexico.

Miss Evelyn Boyd of New York University has been appointed to an associate professorship at Fisk University.

Mr. R. G. Brown of the Naval Operations Evaluation Group, Washington, D. C., has accepted a position as research engineer at the Willow Run Research Center of the University of Michigan.

Professor G. R. Clements of the U. S. Naval Academy has retired.

Mr. J. B. Crabtree has been appointed to an assistant professorship at the University of New Hampshire.

Dr. M. D. Donsker of Cornell University has been appointed to an assistant professorship at the University of Minnesota.

Dr. Yael N. Dowker has been appointed assistant lecturer at the University of Manchester, Manchester, England.

Mr. E. J. Delate has accepted a position as statistician in the Quality Control Department of "Cel-O-Seal," E. I. DuPont de Nemours and Company, Buffalo, New York.

Mr. E. A. Fay has accepted a position as statistician at the U. S. Naval Ordnance Test Station, Inyokern, California.

Mr. Martin Goland has been appointed editor of Applied Mechanics Reviews. The editorial office of Applied Mechanics Reviews has recently been moved to the Midwest Research Institute, Kansas City, Missouri.

Assistant Professor Leonard Greenstone of the University of California, Los Angeles, has accepted a position as mathematician at the Stanford Research Institute.

Dr. I. N. Herstein of the University of Kansas has been appointed visiting lecturer at Ohio State University.

Mr. L. G. Jones has accepted a position as mathematician with the Boeing Airplane Company, Seattle, Washington.

Mr. C. R. Morris has accepted a position as mathematician with the U. S. Navy Hydrographic Office, Suitland, Maryland.

Dr. Leo Moser of the University of North Carolina has been appointed to an associate professorship at Texas Technological College.

Sister Charles Anne Mulligan of Albertus Magnus College has been appointed dean of studies at the College of St. Mary of the Springs, Columbus, Ohio.

Mr. C. H. Murphy has accepted a position as mathematician in the Ballistic Research Laboratory at Aberdeen Proving Ground, Maryland.

Mr. D. S. Park has accepted a position as mathematician with the National Bureau of Standards, Washington, D. C.

Assistant Professor L. G. Peck of Johns Hopkins University has accepted a position at the Los Alamos Scientific Laboratory.

Assistant Professor C. L. Perry of the University of Arkansas is on leave of absence and has accepted a position as mathematician at Oak Ridge National Laboratory.

Mr. M. L. Polk has accepted a position as engineer with Squire Laboratories, New York, New York.

Dr. Judson Sanderson of the University of Illinois has been appointed to an assistant professorship at Newcomb Memorial College, Tulane University of Louisiana.

Mr. J. W. Sawyer of the University of Missouri has been appointed to an assistant professorship at the Atlanta Division of the University of Georgia.

Dr. M. D. Springer has accepted a position as mathematical statistician with U. S. Naval Ordnance, Indianapolis, Indiana.

Mr. Bethuhne Vanderburg of Extension University has been appointed to an assistant professorship at the University of Alabama.

Assistant Professor A. R. Williams of the University of California has retired with the title assistant professor emeritus.

Mr. L. E. Wood of Cornell University has accepted a position as mathematician at Aberdeen Proving Ground, Maryland.

The following promotions have been announced:

K. J. Arrow, Stanford University, to an associate professorship.

F. B. Hildebrand, Massachusetts Institute of Technology, to an associate professorship.

Dr. H. D. Huskey, Institute for Numerical Analysis, National Bureau of Standards, to assistant director.

Simon Mowshowitz, University of Bridgeport, to a professorship.

James Singer, Brooklyn College, to an associate professorship.

The following appointments to instructorships are announced:

University of Colorado: Mr. H. C. McKenzie; Indiana University: Dr. R. E. MacKenzie; Iowa State College of Agriculture and Mechanic Arts: Mr. D. C. Benson; University of Kentucky: Mr. L. F. Boron; Montana State University: Mr. A. L. Duquette; University of Oregon: Mr. O. S. Rothaus; University of Vermont: Mr. J. E. Vollmer.

Mrs. Eliakim Hastings Moore died on October 23, 1950.

Professor M. M. Culver of the University of Pittsburgh died on November 2, 1950 at the age of fifty-nine years. He had been a member of the Society for nineteen years.

Professor Emeritus J. C. Morehead died on August 28, 1950 at the age of seventy-three years. He had been a member of the Society for thirty-eight years.

Professor Emeritus S. E. Rasor of Ohio State University died on October 17, 1950 at the age of seventy-seven years. He had been a member of the Society for nineteen years.

Professor Emeritus E. W. Sheldon of Acadia University died on June 14, 1950. He had been a member of the Society for thirty-seven years.

Professor Emeritus C. C. Spooner of Northern Michigan College of Education died on October 7, 1950. He had been a member of the Society for twenty-eight years.

Associate Professor Arthur Tilley of New York University died on October 4, 1950, at the age of fifty-nine years. He had been a member of the Society for twenty-one years.

Professor G. R. Trott of the University of Mississippi died on July 8, 1950. He had been a member of the Society for twenty years.

BULLETIN

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PREFACE

The following pages contain a list of the additions to the Society's Library during the year 1950.

Users of this fifth Supplement are referred to the Catalogue published in 1946 (Bulletin, vol. 52, No. 5, part 2) and to the preface to Supplement No. 1 (Bulletin, vol. 53, No. 1, part 2) for an explanation of the abbreviations and for a statement of the plan followed in reporting additions to the serial publications.

ARNOLD DRESDEN
Librarian

January 1951

SECTION I
SERIAL PUBLICATIONS

ACTA physica austriaca. Vienna
1+, 1947+
CHRISTIAAN Huygens
1–18, 1921–40||
CRANFIELD, England
College of aeronautics
Report
1+, 1946+
LENINGRAD
Universitet
Uchenye Zapiski
Serii͡a matemati cheskikh nauk
1–11, 1936–44
MATHEMATICAL and physical society of Egypt, Cairo
Proceedings
1+, 1937+
PHILIPS research reports. (Philips' gloeilampen fabrieken) Eindhoven, Holland
1+, 1945+
PRAGUE
Universita Karlova
Prirodovedecka fakulta
Spisy
3, 5–6, 32, 34, 40, 47, 117, 131, 134, 145, 181, 182, 1923–48
QUARTERLY journal of mechanics and applied mathematics. Oxford
1+, 1948+
RIVEON lematematika. Jerusalem
1+, 1946+
SIMON STEVIN; wis- en natuurkundig tijdschrift, Groningen
25+, 1946+

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NONSERIAL PUBLICATIONS

ADVANCES in applied mechanics. Ed. by R. v. Mises and T. von Kármán. v1, New York, 1948.
AKADEMJA umiejętnosci, Krakow. Wykaz prac z działu nauk matematycznoprzyrodniczych wykonanych w polsce w okresie okupacji niemieckiej w latach 1939–1945. [List of the works achieved in the field of mathematics and the sciences in Poland during the German occupation, 1939–1945] (Text in English and French) Krakow, 1947.
ALBERT, A. A. Solid analytic geometry. New York, 1949.
ARLEY, N. On the theory of stochastic processes and their application to the theory of cosmic radiation. New York, 1948.
ARTIN, E. *See* Waerden, B. L. van der.
ATHEN, H. Ebene und sphärische trigonometrie. Wolfenbüttel, 1948.
Vektorrechnung. Wolfenbüttel, 1948.
BEHNKE, H. Infinitesimalrechnung. Prepared for publication by P. K. Thullen. Part 1 3d ed., part 2 2d ed., Münster, 1948.
BERS, L. *See* KNOPP, K.
BERKELEY symposium on mathematical statistics and probability, University of California, 1945–1946. Ed. by Jerzy Neyman. Berkeley, 1949.
BERNSTEIN, S. Ed. by. A centenary of Marxism. New York, 1948.
BEYER, R. T. *See* WILLERS, F. A.
BIRKHOFF, G. Lattice theory. Rev. ed., New York, 1949. (Amer. math. soc. colloquium publ. v25)
BLASCHKE, W. Analytische geometrie. Wolfenbüttel, 1948.
Projektive geometrie. 2d ed., Wolfenbüttel, 1948.
BLUM, F. *See* WAERDEN, B. L. VAN DER.

Bolzano, B. Geometrické práce. Ed. by J. Vojtěch. Prague, 1948.

Brockmeyer, E., Halstrom, H. L., and Jensen, A. The life and works of A. K. Erlang. Copenhagen, 1948.

Chiarulli, P. Stability of two-dimensional velocity distributions of the half-jet type. Dayton, 1949.

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Chicago, university. Theses on the calculus of variations, 1933. Chicago, 1933.

Christianovich, S. A. Approximate integration of the equations of supersonic gas flow. Dayton, 1949.

Columbia university, statistical research group. Sampling inspection; principles, procedures, and tables for single, double and sequential sampling in acceptance inspection and quality control based on percent defective. New York, 1948.

Crank, J. The differential analyser. London, 1947.

Denjoy, A. Leçons sur le calcul des coefficients d'une série trigonométrique. Paris, 1941.

Erlang, A. K. *See* Brockmeyer, E.

Falkovich, S. On a class of Laval nozzles. Dayton, 1949.

  On the theory of a wing of finite aspect ratio in supersonic flow. Dayton, 1949.

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Gröbner, W. Moderne algebraische geometrie; die ideal-theoretischen grundlagen. Vienna, 1949.

  See also Severi, F.

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Haskind, M. D. An oscillating wing of finite aspect ratio in supersonic flow. Dayton, 1949.

Henderson, Archibald, The new crichton. Ed. by S. S. Hood. New York, 1949.

Hofreiter, N. *See* Gröbner, W.

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Hopf, L. Introduction to the differential equations of physics. Trans. by W. Nef. New York, 1948.

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Johnson, M. C. Time, knowledge, and the nebulae; an introduction to the meanings of time in physics, astronomy, and philosophy, and the relativities of Einstein and of Milne. New York, 1947.

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Karp, S. N. The oscillating airfoil in compressible flow. Part 2, Dayton, 1948.

Kattsoff, L. O. A philosophy of mathematics. Ames, Iowa, 1948.

Kelley, T. L. The Kelley statistical tables. Rev. ed., Cambridge, Mass., 1948.

Knopp, K. Theorie und anwendung der unendlichen reihen. 4th ed., Berlin, 1947.

  Problem book in the theory of functions. Trans by L. Bers. New York, 1948.

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Kruppa, E. *See* Müller, E. A.

Kudriashov, L. K. Plane parallel gas flow past an ellipse. Dayton, 1949.

Leja, F. Rachunek różniczkowy i całkowy ze wstępem do równań różniczkowych. Krakow, 1947.

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Luneburg, R. K. Mathematical analysis of binocular vision. Princeton, 1947.

MacRobert, T. M. Spherical harmonics; an elementary treatise on harmonic functions, with applications. 2d rev. ed., New York, 1948.

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Neyman, J. *See* Berkeley symposium.

Noether, E. *See* Waerden, B. L. van der.

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  Outflow of a jet of compressed air into moving air. Dayton, 1949.

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Symposia in applied mathematics.

  1. Providence, 1947. Non-linear problems in mechanics of continua. New York, 1949.

Thullen, P. K. *See* Behnke, H.

Unione matematica Italiana. Atti del secondo Congresso dell'unione matematica italiana tenuto in Bologna nei Giorni 4-5-6 Aprile 1940—xviiii sotto gli auspice della R. Universita de Bologna e del consiglio nazionale delle ricerche. Rome, 1942.

Vojtěch, J. *See* Bolzano, B.

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Waerden, B. L. van der. Modern algebra; in part a development from lectures by E. Artin and E. Noether. Trans. from 2d rev. German ed. by F. Blum, v1, New York, 1949.

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Wentzel, G. Quantum theory of fields. Trans. by C. Houtermans and J. M. Jauch. New York, 1949.

Wilder, R. L. Topology of manifolds. New York, 1949. (Amer. Math. Soc. colloquium publ. v32)

Willers, F. A. Practical analysis, graphical and numerical methods. Trans. by R. T. Beyer. New York, 1948.

CORRECTIONS TO BE MADE IN CATALOGUE OF THE LIBRARY[1]

p. 21. Lines 12–14 should read "Nos. 1–10 cover the period 1922–1930 and together are vol. 1
1+, 1922+"
p. 44. Add to entry for Bernoulli, James, before London, "Publ. by F. Maseres."
p. 67. Replace entry for Durfee, W. F. with "Durfee, W. P. Factor Table Sixteenth million, 15,000,000–16,002,309 (manuscript, 500 sheets) 1923–1929. Unique copy."
p. 120. Insert new entry "Maseres, F. *See* Bernoulli, James."
p. 123. Entry for Millsaps. Replace "California" by "California Inst. of Tech."

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[1] Bull. Amer. Math. Soc. v. 52 (1946) n5, part 2.

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MARCH, 1951
(Whole No. 611)

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The initiation fee and annual dues of members of the Society (see this BULLETIN, November, 1950, pp. 567–568) are payable to the Treasurer of the Society, A. E. Meder, Jr., 531 West 116th St., New York 27, N.Y.

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CHARACTERISTIC ROOTS AND FIELD OF VALUES OF A MATRIX

W. V. PARKER

1. **Introduction.** Let $A=(a_{ij})$ be a square matrix of order n whose elements are in the field of complex numbers. The complex number λ is a *characteristic root* of the matrix A if the determinant of the matrix $\lambda I-A$ is zero. It follows that λ is a characteristic root of A if, and only if, there exists a vector x such that $xx^*=1$ and

$$Ax^* = \lambda x^*, \tag{1}$$

where * is used to denote transposed conjugate. By taking transposed conjugates on both sides in (1) we obtain

$$xA^* = \bar{\lambda} x. \tag{2}$$

From (1) it follows that $\lambda=xAx^*$. The set of all complex numbers zAz^* where $zz^*=1$ is called the *field of values* [25][1] of the matrix A. It follows that *the characteristic roots of A belong to the field of values of A.*

Beginning with Bendixson [3] in 1900, many writers have obtained limits for the characteristic roots of a matrix. In many cases these were actually limits for the field of values of the matrix [14]. In an address delivered before the Mathematical Association of America in 1938, Browne [10] gave a summary of these results up to that time. It is the purpose here to discuss some of the results obtained since the time of Browne's paper.

2. **Some well known results.** If x and y are two vectors such that $xx^*=yy^*=1$ and X and Y are unitary matrices with leading vectors x and y respectively, then $xX^*=yY^*$ and hence $y=xX^*Y$, or $y=xU$ where U is a unitary matrix. Also, if $xx^*=1$ and $y=xU$ where U is a unitary matrix, then $yy^*=xUU^*x^*=xx^*=1$. It follows, therefore, that A and UAU^* have the *same field of values and the same characteristic roots* for every unitary matrix U. It may be readily shown [21] that *the field of values of A is identical with the set of all diagonal elements of the matrices UAU^* where $UU^*=I$.* If A is *Hermitian* there exists a unitary matrix U such that $UAU^*=\text{diag.}\ \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$

An addressed delivered before the Oak Ridge meeting of the Society, April 21, 1950, by invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings; received by the editors June 28, 1950.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

where $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ are the (real) characteristic roots of A. It follows immediately that the *field of values of an Hermitian matrix is the shortest segment of the real axis containing all the characteristic roots of A.*

From (1) and (2) the following relations are obtained

$$\begin{aligned} &\lambda = xAx^*, \qquad \bar{\lambda} = xA^*x^*, \qquad \bar{\lambda}\lambda = xA^*Ax^*, \\ (3) \qquad &\frac{\lambda + \bar{\lambda}}{2} = x\left(\frac{A + A^*}{2}\right)x^*, \qquad \frac{\lambda - \bar{\lambda}}{2i} = x\left(\frac{A - A^*}{2i}\right)x^*. \end{aligned}$$

From the first three of these relations we obtain two well known theorems. If $A = A^*$, then $\lambda = \bar{\lambda}$ and hence *the characteristic roots of an Hermitian matrix are all real.* If $A^*A = I$, then $\bar{\lambda}\lambda = 1$, and hence *the characteristic roots of a unitary matrix are all in absolute value one.* Again from the third of these relations we obtain at once Browne's theorem [8] *that $|\lambda|$ is bounded by the least and greatest of the characteristic roots of the positive semi-definite Hermitian matrix B defined by* $B^2 = A^*A$. It may also be shown [20] that for a given B there exists a matrix A which has as a characteristic root any number in this region. From the last two relations in (3) it follows that the characteristic roots of A lie in a rectangular region of the complex plane determined by the characteristic roots of $(A + A^*)/2$ and $(A - A^*)/2$ [16]. In fact the entire field of values of A lies in this rectangle.

3. **Limits for the characteristic roots of a matrix.** In 1930 Browne [9] obtained a limit for the characteristic roots of the matrix A in terms of the sums of the absolute values of the elements in the rows and the columns of A. Write

$$\begin{aligned} &R_i = \sum_j |a_{ij}|, \qquad T_j = \sum_i |a_{ij}|, \qquad 2S_i = R_i + T_i, \\ (4) \qquad & \\ &R_i' = R_i - |a_{ii}|, \qquad T_j' = T_j - |a_{jj}| \end{aligned}$$

and let R, T, and S denote the maximum R_i, T_i, and S_i, respectively. Browne showed that $|\lambda| \leqq (R+T)/2$. This result was improved by this author in 1937 [18] when it was shown that $|\lambda| \leqq S \leqq (R+T)/2$. Another improvement of Browne's result was given by Farnell in 1944 when he showed that $|\lambda| \leqq (RT)^{1/2} \leqq (R+T)/2$. In 1945 Barankin [1] further improved Farnell's result by showing $|\lambda| \leqq \max\,(R_iT_i)^{1/2}$.

The first two of these are actually limits for the field of values of A. Suppose that $\mu = xAx^*$ is any number in the field of values of A. Write $x = (x_1, x_2, \cdots, x_n)$, then $\mu = \sum_{i,j} a_{ij}x_i\bar{x}_j$. If $|x_i| = \xi_i$ it follows that

$$
\begin{aligned}
|\mu| &\leqq \sum_{i,j} |a_{ij}|\, \xi_i \xi_j \leqq \frac{1}{2} \sum_{i,j} |a_{ij}|\, (\xi_i^2 + \xi_j^2) \\
&= \frac{1}{2} \sum_i R_i \xi_i^2 + \frac{1}{2} \sum_j T_j \xi_j^2 = \sum_i S_i \xi_i^2 \leqq S \sum_i \xi_i^2 = S.
\end{aligned}
$$

Relation (1) may be written as a system of linear equations in the form

$$
\lambda \bar{x}_i = \sum_j a_{ij} \bar{x}_j \qquad (i = 1, 2, \cdots, n). \tag{5}
$$

If ξ_k is the greatest of the ξ_i, then since $\lambda \bar{x}_k = \sum_j a_{kj} \bar{x}_j$

$$
|\lambda \bar{x}_k| = |\lambda|\, \xi_k \leqq \sum_j |a_{kj}|\, \xi_j \leqq R_k \xi_k
$$

and hence $|\lambda| \leqq R_k$. In a similar manner $|\lambda| \leqq T_m$ for some $j = m$. This establishes the following theorem.

THEOREM 1. *If λ is a characteristic root of the matrix A and R is the greatest sum obtained for the absolute values of the elements of a row and T is the greatest sum obtained for the absolute values of the elements of a column, then $|\lambda| \leqq \min (R, T)$.*

This theorem was proved by the author in 1943. It was subsequently given by Barankin [1] and later again by Brauer [4]. In an added note Brauer pointed out that the theorem was proved by Perron in the second edition of his *Algebra* in 1933 and also by Specht in 1938. In this same paper Brauer obtained a better limit given by the following theorem.

THEOREM 2. *Let M_r be the maximum of the sums of the absolute values of the elements in each row of the matrix A^{2^r}. Then each characteristic root λ of A satisfies $|\lambda| \leqq (M_r)^{1/2^r}$.*

From (5) it follows that

$$
(\lambda - a_{kk}) \bar{x}_k = \sum_{j \neq k} a_{kj} \bar{x}_j
$$

so that

$$
|\lambda - a_{kk}|\, \xi_k \leqq \sum_{j \neq k} |a_{kj}|\, \xi_j \leqq \xi_k \sum_{j \neq k} |a_{kj}| = R_k' \xi_k
$$

and hence

$$
|\lambda - a_{kk}| \leqq \sum_{j \neq k} |a_{kj}| = R_k - |a_{kk}| = R_k'.
$$

In a similar fashion

$$|\lambda - a_{mm}| \leqq T_m - |a_{mm}| = T'_m.$$

This establishes the following theorem.

THEOREM 3 (BRAUER [4]). *Let* $A=(a_{ij})$ *be an arbitrary matrix and* R'_k *the sum of the absolute values of the non-diagonal elements of the kth row and* T'_m *the sum of the absolute values of the non-diagonal elements of the mth column of* A. *Each characteristic root of* A *lies in at least one of the circles* $|z-a_{kk}| \leqq R'_k$ *and in at least one of the circles* $|z-a_{mm}| \leqq T'_m$.

The equations (5) may also be written as

$$(\lambda - a_{ii})\bar{x}_i = \sum_{j\neq i} a_{ij}\bar{x}_j \qquad (i = 1, 2, \cdots, n). \tag{6}$$

If $\xi_k \geqq \xi_l \geqq \xi_j$, $j\neq k, l$, then from (6)

$$|\lambda - a_{kk}|\,\xi_k \leqq R'_k\xi_l \quad \text{and} \quad |\lambda - a_{ll}|\,\xi_l \leqq R'_l\xi_k.$$

It is assumed now that the characteristic root λ is not the diagonal element a_{kk}, so that $\xi_l \neq 0$. Then since

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}|\,\xi_k\xi_l \leqq R'_kR'_l\xi_k\xi_l,$$

it follows that

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}| \leqq R'_kR'_l.$$

The inequality is obviously true of $\lambda = a_{kk}$. In a similar manner

$$|\lambda - a_{mm}| \cdot |\lambda - a_{qq}| \leqq T'_mT'_q$$

for some m and q. This establishes the following theorem.

THEOREM 4 (BRAUER II [5]). *Each characteristic root* λ *of the matrix* A *lies in at least one of the* $n(n-1)/2$ *ovals of Cassini*

$$|z - a_{kk}| \cdot |z - a_{ll}| \leqq R'_kR'_l$$

and in at least one of the ovals

$$|z - a_{kk}| \cdot |z - a_{ll}| \leqq T'_kT'_l.$$

In his third paper Brauer [6] shows how the above theorems may be applied to specially selected polynomials in A to obtain better limits for the characteristic roots. He proved the following theorem.

THEOREM 5 (BRAUER III [6]). *Let* $A=(a_{ij})$ *be a square matrix of order* n *and* $f_1(y), f_2(y), \cdots, f_n(y)$ *be arbitrary polynomials. Denote the elements of the matrix* $f_r(A)$ *by* $a_{ij}^{(f_r)}$ *and set*

$$\sum_{j=1,\, j\neq i}^{n} \left| a_{ij}^{(f_r)} \right| = P_i^{(f_r)} \qquad (i, r = 1, 2, \cdots, n)$$

Each characteristic root λ *of* A *satisfies at least one of the* n *inequalities*

$$\left| f_s(\lambda) - a_{ss}^{(f_s)} \right| \leqq P_s^{(f_s)} \qquad (s = 1, 2, \cdots, n).$$

and at least one of the $n(n-1)/2$ *inequalities*

$$\left| f_s(\lambda) - a_{ss}^{(f_s)} \right| \cdot \left| f_t(\lambda) - a_{tt}^{(f_t)} \right| \leqq P_s^{(f_s)} P_t^{(f_t)}$$
$$(s, t = 1, 2, \cdots, n;\ s \neq t).$$

4. **Limits for the field of values of a matrix.** As pointed out above, several of the earlier theorems actually gave upper bounds for the field of values of A. Farnell [14] gives two limits for the field of values of A. *Let* $d_{rs}=(|a_{rs}|+|a_{sr}|)/2$, *and let* $S_r=\sum_s d_{rs}^2$. *Then if* λ *lies in the field of values of* A, $|\lambda| \leqq \max (S_r)$ *and* $|\lambda| \leqq (\sum_{r,s} d_{rs}^2)^{1/2}$.

In most instances the above bounds are obtained in terms of the distance from zero. However zero may lie entirely outside the field of values of A. Since the field of values of A is convex and contains all the elements of the diagonal of A, it also contains the centroid of these elements. That is, $c=(1/n)\sum_i a_{ii}$ belongs to the field of values of A. Since the trace, $T(A)=\sum_i a_{ii}$, is invariant under unitary transformations, the same c is obtained for every UAU^* where U is unitary. In fact it is possible to select U so that every diagonal element of UAU^* will be c [22].

If $B=A-cI$, the field of values of B is obtained by shifting the field of values of A the distance $|c|$ in the direction of $-c$. Applying the above theorems to B now gives bounds for the characteristic roots of A and the field values of A in terms of the distance from c. This was used by the author [22] to obtain a theorem on the *spread* of the characteristic roots of an Hermitian matrix.

THEOREM 6. *If* $A=(a_{ij})$ *is an Hermiitan matrix with characteristic roots* $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, *then*

$$(\lambda_n - \lambda_1)/2 \geqq \max |a_{ij}|, \qquad i \neq j.$$

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ALABAMA POLYTECHNIC INSTITUTE

THE ANNUAL MEETING OF THE SOCIETY

The fifty-seventh Annual Meeting of the American Mathematical Society was held at the University of Florida, Gainesville, Florida, Wednesday through Friday, December 27–29, 1950, in conjunction with the Annual Meeting of the Mathematical Association of America, and the Christmas Conference of the National Council of Teachers of Mathematics.

The attendance at the meeting was about 350, including the following 225 members of the Society:

C. R. Adams, R. B. Adams, O. R. Ainsworth, C. B. Allendoerfer, B. M. Armstrong, Nachman Aronszajn, W. L. Ayres, E. A. Bailey, R. H. Bardell, H. P. Beard, E. G. Begle, R. G. Blake, Laura Blakeley, A. L. Blakers, L. M. Blumenthal, M. I. Blyth, H. W. Bode, F. F. Bonsall, T. A. Botts, Samuel Bourne, G. F. Bradfield, O. E. Brown, C. C. Camp, R. Crowley Campbell, A. B. Carson, E. D. Cashwell, K. T. Chen, Sarvadaman Chowla, K. L. Chung, E. T. Church, C. E. Clark, R. F. Clippinger, Nathaniel Coburn, Eckford Cohen, J. B. Coleman, W. J. Conner, G. M. Conwell, L. P. Copeland, F. E. Cothran, J. C. Currie, P. H. Daus, J. J. Dennis, A. H. Diamond, V. E. Dietrich, L. L. Dines, W. L. Duren, Patrick Du Val, J. C. Eaves, Myrtle Edwards, D. O. Ellis, D. H. Erkiletian, J. L. Evans, W. B. Evans, R. M. Exner, H. F. Fehr, W. E. Ferguson, C. H. Fischer, Edward Fleisher, E. E. Floyd, Tomlinson Fort, J. S. Frame, Orrin Frink, J. W. Gaddum, H. M. Gehman, H. H. Germond, Wallace Givens, A. M. Gleason, H. E. Goheen, V. D. Gokhale, D. B. Goodner, E. H. Hadlock, Franklin Haimo, P. R. Halmos, P. C. Hammer, Frank Harary, F. S. Harper, E. A. Hedberg, Nicholas Heerema, M. H. Heins, Edwin Hewitt, E. H. C. Hildebrandt, A. J. Hoffman, T. C. Holyoke, A. S. Householder, J. A. Hratz, G. B. Huff, Ralph Hull, M. G. Humphreys, W. R. Hutcherson, Ernest Ikenberry, S. L. Jamison, B. W. Jones, F. B. Jones, P. S. Jones, R. V. Kadison, Shizuo Kakutani, Joseph Kampé de Fériet, H. T. Karnes, M. E. Kellar, J. L. Kelley, L. M. Kelly, A. J. Kempner, J. R. F. Kent, M. S. Klamkin, J. R. Kline, R. H. Knox, R. J. Koch, F. W. Kokomoor, Jacob Korevaar, G. B. Lang, R. E. Langer, Nathan Lazar, B. A. Lengyel, R. J. Levit, F. A. Lewis, Z. L. Loflin, Eugene Lukacs, E. J. McShane, C. C. MacDuffee, H. M. MacNeille, H. F. MacNeish, J. R. Mayor, Herman Meyer, H. A. Meyer, D. D. Miller, W. L. Miser, Benjamin Ernest Mitchell, J. M. Mitchell, C. N. Moore, L. J. Mordell, D. R. Morrison, G. D. Mostow, W. B. Moye, Zeev Nehari, John von Neumann, J. D. Newburgh, C. V. Newsom, O. M. Nikodým, A. J. Owens, O. O. Pardee, W. V. Parker, P. B. Patterson, L. G. Peck, W. D. Peeples, S. E. Pence, F. W. Perkins, I. E. Perlin, C. G. Phipps, Z. M. Pirenian, Harry Polachek, F. M. Pulliam, E. S. Quade, J. F. Randolph, L. T. Ratner, C. B. Read, L. M. Reagan, C. J. Rees, M. S. Rees, P. K. Rees, B. P. Reinsch, J. H. Roberts, F. V. Rohde, J. B. Rosser, W. A. Rutledge, Arthur Saastad, R. G. Sanger, J. W. Sawyer, M. A. Scheier, E. R. Schneckenburger, C. S. Scott, I. E. Segal, Esther Seiden, W. T. Sharp, C. N. Shuster, T. M. Simpson, C. B. Smith, W. S. Snyder, Andrew Sobczyk, J. J. Sopka, D. E. South, T. H. Southard, W. H. Spragens, H. D. Sprinkle, M. E. Stark, R. W. Stokes, P. M. Swingle, Gabor Szegö, T. T. Tanimoto, Olga Taussky, H. E. Taylor, J. M. Thomas, G. L. Tiller, John Todd, C. B. Tompkins, M. M. Torrey, E. A. Trabant, H. C. Trimble, C. A. Truesdell, A. W. Tucker, S. M. Ulam, Gilbert Ulmer, Eugene Usdin, L. A. Van Dyke, Henry Van Engen, A. H. Van Tuyl, T. L. Wade,

R. J. Walker, A. D. Wallace, J. L. Walsh, L. E. Ward. Jr., H. F. Weinberger, Alexander Weinstein, M. J. Weiss, F. P. Welch, J. G. Wendel, G. T. Whyburn, L. S. Whyburn, W. M. Whyburn, W. L. Williams, R. L. Wilson, W. H. Wilson, G. N. Wollan, F. M. Wright, Arthur Zeichner, A. D. Ziebur.

The twenty-fourth Josiah Willard Gibbs Lecture entitled *Some basic problems of statistical mechanics* was delivered by Professor G. E. Uhlenbeck of the University of Michigan on Wednesday, December 27. President-elect John von Neumann presided at this lecture.

Professor J. L. Walsh of Harvard University delivered his Retiring Presidential Address entitled *The location of critical points* on Thursday, December 28. Professor John von Neumann, President-elect, was the presiding officer.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited two speakers. On Thursday, December 28, Professor M. H. Heins of Brown University addressed the Society on *Regularity of growth of subharmonic functions*. President J. L. Walsh presided at this lecture. On Friday, December 29, Professor L. J. Mordell of St. John's College, Cambridge University spoke on *The product of n homogeneous linear forms*. Professor Patrick Du Val was the presiding officer.

The annual Business Meeting and Election of Officers was held at 10:15 A.M. on Friday, December 29. Details of proceedings are reported in the sequel.

A reception for members of the mathematical organizations and their guests was given by the University of Florida following the Gibbs lecture on Wednesday evening. This was in the Social Room of the Florida Union, headquarters of the Society for the meetings.

The dinner on Friday evening in the University Cafeteria was attended by approximately 250 members and guests. Professor T. M. Simpson, Dean of the Graduate School of the University of Florida, was toastmaster. President J. Hillis Miller of the University of Florida extended greetings and Professor R. E. Langer responded on behalf of the assembled guests. President Walsh presented a sterling silver tray to Secretary Kline as a token of appreciation by Society members of Professor Kline's ten years of service as Secretary. Professor Kline followed his expression of thanks by brief mention of some of the more significant activities of the Society during his period in office. Professor P. H. Daus of the University of California, Los Angeles, offered a resolution of thanks and appreciation to the University of Florida, the Committee on Arrangements, and all who assisted in making the meetings successful and enjoyable. This resolution was adopted by a rising vote.

The Annual Business Meeting of the Society was held on Friday morning, December 29. The Secretary reported on the affairs of the Society.

Upon recommendation of the Council, it was voted to amend the By-Laws of the Society so that the President and Treasurer will be ex officio members of the Board of Trustees.

It was voted that the members of the Society present vote to disapprove the action of the Council on the question of meetings at the University of California.

It was voted that the Secretary of the Society be instructed not to release the action of the Council concerning meetings at the University of California beyond the membership of the Society and that the Council be requested to reconsider the matter at its meeting in Minneapolis.

The Council met on Thursday evening, December 28, 1950.

The Secretary announced the election of the following thirty persons to ordinary membership in the Society:

Mr. Sholom Arzt, New York University;
Mr. Hans Boris Belck, General Electric Company, Pittsfield, Mass.;
Mr. Nicholas Bilotta, Department of Public Works, New York, N. Y.;
Visiting Professor Frank Featherstone Bonsall, Oklahoma Agricultural and Mechanical College;
Mr. Allen Virgil Butterworth, University of Chicago;
Mr. Robert Kent Butz, Alabama Polytechnic Institute;
Professor Madeleine Forró, Barat College, Lake Forest, Ill.;
Mr. Robert S. Freeman, New York University;
Mr. Thom Raymond Greene, Carnegie Institute of Technology;
Mr. Frederick Alexander Hiersch, Detroit, Mich.;
Mr. James Grady Horne, Jr., Tulane University of Louisiana;
Mr. Henry Paul Kramer, Radiation Laboratory, University of California;
Dr. Paolo Lanzano, St. Louis University;
Mr. Thomas James McNamara, Park College, Parkville, Mo.;
Professor James Horatio Means, Tillotson College, Austin, Tex.;
Mr. Gene Woodard Medlin, University of North Carolina;
Mr. Edward Peter Merkes, DePaul University;
Mr. Richard Allan Moore, Washington University;
Professor Elisha Netanyahu, Hebrew Technical College, Haifa, Israel;
Mr. John Adolph Nohel, Massachusetts Institute of Technology;
Mr. Richard Eugene Priest, University of Illinois;
Mr. Savatore John Rapisarda, Lawrence, Mass.;
Professor Arthur Saastad, DePaul University;
Mr. Federico Mariano Sioson, University of the Philippines;
Mr. H. D. Sprinkle, University of Florida;
Dr. Marie D. Wann, George Washington University;
Mr. William Hamer Warner, Carnegie Institute of Technology;
Mr. Gerard Paul Weeg, St. Ambrose College, Davenport, Iowa;

Sister Teresa Dolores Wolf, College of Mount St. Vincent, New York, N. Y.; Mr. Ti Yen, University of Illinois.

It was reported that the following ninety-six persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Professor Oscar Richard Ainsworth;
Brown University: Dr. George Walter Morgan;
Bryn Mawr College: Miss Padmini Rangaswamy Iyengar;
University of California: Messrs. Woodrow Wilson Bledsoe, Solomon Feferman, and Joseph William Weihe;
Cornell University: Messrs. Austin Arthur Howard and Gerson Bernhard Robison;
Duke University: Messrs. Gordon Cleaves Byers and Robert Taylor Herbst;
Equitable Life Insurance Company of Iowa: Messrs. Floyd Thomas Beasley and Tilmar Archibald Moilien;
University of Illinois: Messrs. Charles Glenn Carlyle, Alvin Vincent Fend, Joseph Hashisaki, John Edward Heisler, Jr., John William Rolley, and Robert Jerome Silverman;
Indiana University: Messrs. Jerald Laverne Ericksen, Lowell Keith Frazer, and David Eliza van Tijn;
Institute for Advanced Study: Professors Aldo Andreotti, Heinrich A. L. Behnke, Fabio Conforto, Vittorio Dalla Volta, Mr. Freeman John Dyson, Dr. István Sándor Gál, Professors Kenkichi Iwasawa, Jean Leray, Hidegorô Nakano, Giovanni Ricci, Dr. Leo Reino Sario, Professors Beniamino Segre and Kentaro Yano;
Iowa State College of Agriculture and Mechanic Arts: Mr. Arnold Marion Wedel;
Johns Hopkins University: Messrs. James Richardson Belford, Harry Wilson Clarke, Enrico Thomas Federighi, Irving Elwell Gaskill, Vernon Ray Hancock, Benjamin Fulton Howard, Jr., Harold Shulman, and Charles Junior Standish;
University of Kansas: Messrs. Kuo-Chih Hsu and Arthur Herman Kruse;
Lehigh University: Mr. Jacob Samoloff;
University of Maryland: Messrs. Russell Martin Brown, Jr., Daniel Richard Hughes, and Miss Edith Moss;
Massachusetts Institute of Technology: Messrs. Richard Horace Battin and Donald William Dunn;
University of Michigan: Messrs. Richard Blaine Barrar, John Gilbert Hocking, Robert William MacDowell, and Donald Ferdinand Mela;
Ohio State University: Messrs. Richard Albert Dean, Miles Sanford Edwards, Edward Richard Fadell, and Harold M. Horwitz;
University of Oregon: Mr. Arthur Eugene Livingston;
Pennsylvania State College: Messrs. Edward Jack Cogan and Albert Russel Erskine;
University of Pennsylvania: Messrs. Robert Ellis, Walter Hess Leser, and Philip Rabinowitz;
Princeton University: Messrs. Donald Wayne Bushaw, Herbert Amasa Forrester, George Allen Hufford, Roger Halliday Kyle, Marvin Lee Minsky, Peter Bjørn Nemenyi, Onorato Timothy O'Meara, Basil Andrew Rattray, Yomei Sawanobori, and David Lee Wallace;
Purdue University: Messrs. William Richard Fuller, Louis Eugene Grosh, Ladis Daniel Kovach, Maurice Dean Lamoree, and Walter Davidson Wood;
Rice Institute: Messrs. Jim Douglas, Jr., and John Delbert Rice;

Stanford University: Messrs. Gordon Bruce Hedrick, John Weston Hardy, Jr., Joel Franklin, and Donald Charles Benson;
Swarthmore College: Mr. Arthur Paul Mattuck;
University of Tennessee: Mr. Nickolas Heerema;
Wellesley College: Professor James Walter Warwick;
Western and Southern Life Insurance Company: Messrs. Eugene Warren Bates, Edward Bathurst Fackler, and Edwin Herbert Vonderbrink;
Yale University: Messrs. Austin Robert Brown, Jr., Stephen Peter Hoffman, Jr., Brian Eugene Owens, and Morris Weisfeld.

The Secretary announced that the person indicated below had been admitted to membership in the Society, under reciprocity agreements with various mathematical organizations: London Mathematical Society: Professor Charles Fox, McGill University.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 4411, including 353 nominees of institutional members and 47 life members. There are also 107 institutional members. The total attendance at all meetings in 1950 was 1413; the number of papers read was 436; there were 14 hour addresses, 3 half-hour addresses, 1 Gibbs Lecture, and 1 Retiring Presidential Address; the number of members attending at least one meeting was 1068.

The following appointments by the President were reported: As a Committee on Arrangements for the Summer Meeting at the University of Minnesota, September 4–7, 1951: Professors John M. H. Olmsted, Chairman, R. H. Cameron, H. M. Gehman, W. L. Hart, G. K. Kalisch, Fulton Koehler, W. S. Loud, S. E. Warschawski, and J. W. T. Youngs; as Chairman of the Committee on Printing Contracts for the period 1951–1953: Professor C. J. Rees; as Chairman of the Committee on Visiting Lectureships for the period 1951–1953: Professor R. P. Boas, Jr. (committee now consists of Professors R. P. Boas, Jr., Richard Courant, and H. S. Wall); as a member of the Committee on Places of Meetings for the period 1951–1953: Professor Orrin Frink (committee now consists of Professors R. W. Brink, Chairman, Orrin Frink, and J. M. Thomas); as members of the Committee on Applied Mathematics for the period 1951–1953: Professors R. V. Churchill and M. H. Martin (committee now consists of Professors M. H. Martin, Chairman, R. V. Churchill, K. O. Friedrichs, John von Neumann, William Prager, and Eric Reissner); as tellers for the 1950 annual election: Professors L. W. Cohen, F. W. Kokomoor, and E. R. Kolchin; as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for the period 1951–1952: Professor Deane Montgomery (committee now consists of Professors E. G. Begle, Chairman, A. A. Albert, and Deane Montgomery); as members of the Committee to Select Hour Speakers for Eastern Sectional Meetings: Professor N. E. Steenrod (to replace Professor E. G. Begle, term to expire December 31, 1951) and Professor M. H. Heins for the period 1951–1952 (committee now consists of Professors L. W. Cohen, Chairman, M. H. Heins, and N. E. Steenrod); as a member of the Committee to Select Hour Speakers for Southeastern Sectional Meetings for the period 1951–1952: Professor B. J. Pettis (committee now consists of Professors W. M. Whyburn, Chairman, Tomlinson Fort, and B. J. Pettis); as a member of the Committee to Select Hour Speakers for Western Sectional Meetings for the period 1951–1952: Professor P. R. Halmos (committee now consists of Professors J. W. T. Youngs, Chairman, P. R. Halmos, and Walter Leighton); as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for the period 1951–1952: Professor A. P. Morse (committee now consists of Professors J. W. Green, Chairman, M. R. Hestenes, and A. P. Morse); as Editor of Volume IV of the Proceedings of Symposia on Applied Mathematics: Professor M. H. Martin; as Chairman of the Committee to Consider Non-editorial Problems of the Memoirs: Professor A. W. Tucker.

It was reported that Professor W. T. Reid's name was substituted for Professor E. G. Begle as a nominee for membership at large of the Council on the ballot for the annual election.

At the annual election which closed on December 28, 1950, and at which 575 votes were cast, the following officers were elected:

Vice President, Professor G. A. Hedlund.

Secretary, Professor E. G. Begle.

Associate Secretary, Professor L. W. Cohen.

Treasurer, Dean A. E. Meder, Jr.

Librarian, Professor Jekuthiel Ginsburg.

Member of the Editorial Committee of the Bulletin, Professor W. T. Martin.

Member of the Editorial Committee of the Proceedings, Professor Nathan Jacobson.

Member of the Editorial Committee of the Transactions, Professor G. T. Whyburn.

Member of the Editorial Committee of the Colloquium Publications, Professor A. A. Albert.

Member of the Editorial Committee of Mathematical Reviews, Professor William Feller.

Member of the Editorial Committee of Mathematical Surveys, Professor Leo Zippin.

Representative on the Editorial Board of the American Journal of Mathematics, Professor Reinhold Baer.

Members at large of the Council, Professors D. H. Blackwell, H. F. Bohnenblust, H. S. M. Coxeter, Irving Kaplansky, and W. T. Reid.

Members of the Board of Trustees, Professors B. P. Gill, T. H. Hildebrandt, and J. R. Kline.

The Council voted unanimously to adopt the following three resolutions:

Whereas Professor John Robert Kline assumed the office of Secretary of the American Mathematical Society in 1941 and is now relinguishing that office after ten years of faithful and devoted service, and

Whereas Professor Kline has performed the duties of his office with such distinction that the Society has functioned in an outstanding manner and has enjoyed most wholesome growth, and

Whereas in discharging the heavy and important work of the Secretaryship Professor Kline has always given his time and physical energy generously and without stint, and

Whereas it is the wish of the Council of the American Mathematical Society to record its deep gratitude to Professor Kline:

Be it therefore resolved that the Council hereby express its most profound thanks to Professor Kline for his contributions to the development of mathematics in America through his work in the office of Secretary of the American Mathematical Society and otherwise, and

Be it further resolved that copies of this resolution be sent to Professor Kline, spread upon the minutes of the Council, and appropriately published by the Society.

The Council of the American Mathematical Society records its deep gratitude to Professor Arnold Dresden for the outstanding service which he has rendered through the office of Librarian for the Society. Under his able and careful handling, the Society library has grown steadily to a point where it now represents a highly valuable asset, both as to content and as to financial value. Through carefully selected and arranged exchanges, Professor Dresden has acquired much valuable material without drain on the funds of the Society. Through cataloging, arrangement, and numerous other services, he has increased the usefulness of the library to the Society membership.

The Council hereby expresses its appreciation to Professor Dresden and conveys to him its most profound thanks for the unselfish devotion to the service of American mathematics which he has ever shown in the performance of his duties as Librarian of the Society.

The Council of the American Mathematical Society records its deep gratitude to Professor Temple Rice Hollcroft for contributions which he has made to American mathematics through his many years of devoted and efficient service as Associate Secretary of the Society. Under his careful handling of the duties of his office, meetings of the Society for which he carried responsibility have been conducted with the smoothness that reflects careful thought and good planning. Through his programming, conducting, and reporting Society meetings, and through the multitude of other services encumbent upon his office, he has contributed greatly to the success of activities of the Society.

The Council hereby expresses its appreciation to Professor Hollcroft and conveys to him its most profound thanks for the fine service which he has rendered the American Mathematical Society through his work as Associate Secretary.

The President was authorized to appoint a committee to consider the request of Mrs. Charles C. Torrance that wives of members of the Society who are mathematicians be given the same privileges as emeritus members.

The Council voted that the following resolution be adopted:

> Owing to the expressed reluctance of a large number of mathematicians to attend meetings of the American Mathematical Society at the University of California as long as the conditions previously condemned prevail, the Society will not hold meetings on a campus of the University of California until these conditions are alleviated.

It was voted that a copy of the resolution be sent to the editors of Science.

The Council voted to approve the following dates of meetings of the Society in 1951: November 23–24, at the University of Oklahoma; December 26–28 in Providence, Rhode Island.

In an appendix to this report are excerpts from the report of the Treasurer for the fiscal year 1950 as verified by the auditors. A copy of the complete report will be sent, on request, to any member of the Society.

The Librarian reported the following additions to the Library: 163 bound volumes of periodicals, 80 books, and 46 pamphlets.

Certain invitations to give addresses in 1951 were announced: Professors Zeev Nehari and Tadasi Nakayama for the February meeting in Cincinnati.

Upon recommendation of the Committee to Nominate a Representative of the Society on the Policy Committee, Professor J. R. Kline was elected to serve on the Policy Committee for a four-year term beginning January 1, 1951.

The Bulletin Editorial Committee reported that the journal is more nearly current in its publication schedule than has been true for some years. A recommendation that a total of 600 pages be authorized for the 1951 Bulletin was accepted by the Council and referred to the Trustees. Professor R. P. Boas was reported as a new Assistant Editor for the Bulletin.

The Transactions and Memoirs Editorial Committee reported that the Transactions will have published almost exactly the 1100 pages authorized for 1950. The interval between receipt of the manuscripts and publication usually falls between six and nine months. The Council voted to recommend to the Board of Trustees that 1100 pages be published in the Transactions in 1951. In the Memoirs series, Number 1 appeared in July and Number 2 is now being printed. Four additional numbers are ready for the printer and it is hoped that they will be available early in 1951.

The Mathematical Reviews Editorial Committee reported that 766 pages of reviews had been published, the same as reported for 1949. The subscription list as of November, 1950, was 2,061. Beginning September 1, Dr. R. P. Boas, Jr. was replaced by Dr. J. V. Wehausen as Executive Editor.

The Proceedings Editorial Committee reported that the backlog of the journal was such that a period of approximately fourteen to sixteen months must now elapse between acceptance and publication of a particular paper. A recommendation that 1000 pages be authorized for the 1951 Proceedings was accepted by the Council and referred to the Trustees. This represents an increase of 150 pages over the 1950 volume of the Proceedings. Professors P. T. Bateman, N. J. Fine, A. W. Goodman, and E. H. Spanier were reported as new Assistant Editors for the Proceedings.

The Organizing Committee of the International Congress reported that the International Congress held at Harvard University from August 30 through September 6, 1950, was a complete success. The next International Congress of Mathematicians will be held in Amsterdam in 1954. Over 2,300 persons registered for the Congress, of whom about 300 came from outside of the continental United States and Canada. The Proceedings of the Congress, which represent the only unfinished business of the Congress, are expected to appear in 1951; they will comprise two volumes of about 850 pages each. In addition, the Conference on Algebraic Tendencies in Analysis will be published as a separate volume in the Mathematical Surveys Series, provided the manuscripts are received before March 1, 1951.

The Committee on Reciprocity Agreements reported that, "While it is felt that the American Mathematical Society should not enter into reciporicity agreements of the usual sort with Societies in the American area, it is strongly recommended that, as occasion permits, mathematicians residing in contiguous territory to the United States be extended cordial invitations to become members of the A.M.S. and that these invitations be supplemented by offering a favorable exchange rate where possible." The Council voted to accept the report and to discharge the Committee.

Upon recommendation of the Policy Committee, the Council voted that the National Council of Teachers of Mathematics be asked to designate one representative to serve on the Policy Committee.

The Colloquium Committee recommended that Professor Alfred Tarski be invited to deliver the Colloquium Lectures at the 1952 Summer Meeting. The Council voted to accept this recommendation.

The Council voted to adopt the following resolution:

> Since its founding sixteen years ago, the Duke Mathematical Journal has formed an important integral part in the American Mathematical Society's program for the publication of mathematical research. The value of the aid thus furnished by the Duke University has been greatly enhanced by the fact that the Society has not been called upon to share the financial burden. This assistance is especially significant in this period of heavy publication needs and strained Society resources.
>
> The officers and members of the Society deeply appreciate the nature and the extent of Duke's contribution. The Council conveys the gratitude of the Society and expresses its sincere hope that Duke University will maintain the Journal at the prewar levels of size and quality.

The Council voted to recommend to the Board of Trustees that a joint committee of the Council and the Board of Trustees be appointed to be available for consultation on matters relating to the Duke Mathematical Journal.

Abstracts of papers read follow. Presiding officers at the sessions for contributed papers were Professors P. H. Daus, C. R. Adams, W. V. Parker, J. B. Rosser, Tomlinson Fort, W. L. Duren, A. W. Tucker, Dr. Mina S. Rees, and Dr. A. S. Householder.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 101 was presented by Mr. Zeichner, 103 by Professor Bonsall, 112 by Professor Sobczyk, 115 by Professor Hewitt, and 138 by Dr. Weinberger. Dr. Amitsur was introduced by Dr. Jakob Levitzky, Dr. Dekker by Professor P. C. Rosenbloom, Mr. Peters by Professor Tomlinson Fort, and Professor Arf by Professor Alexander Weinstein.

ALGEBRA AND THEORY OF NUMBERS

80*t*. S. A. Amitsur: *Radical-like ideals.*

In the present paper an axiomatic study of the various radical-like ideals is initiated. This is achieved by considering properties E of rings which satisfy the following conditions: (1) The zero ring is an E-ring, that is, a ring which possesses the property E. (2) Each homomorphic image of an E-ring is also an E-ring. (3) If S is not an E-ring, then S contains a proper radical E-ideal, that is, an ideal Q such that the nonzero quotient ring S/Q contains no ideals with the property E (E-ideals), except the zero ideal. (4) Every ideal in an E-ring is an E-ideal. (5) If R is a right E-ideal in S, then SR is also an E-ideal in S. Some of the results obtained are: (I) If E satisfies (1) and (2), then the intersection of all the radical E-ideals of S is also a radical E-ideal (the lower E-radical). (II) If E satisfies the three conditions (1)–(3), then the lower E-radical of S is the maximal E-ideal in S (the E-radical of S). (III) if E satisfies (1)–(4), then the E-radical of an ideal A in S is the intersection of A and the E-radical of S. (IV) If E satisfies all conditions (1)–(5), then the radical N of S contains all right E-ideals of S and S/N contains no right E-ideals except the zero. (Received November 17, 1950.)

81. Eckford Cohen: *Sums of products in $GF[p^n, x]$.*

In a previous paper (*An extension of Ramanujan's sum*, Duke Math. J. vol. 15 (1948) pp. 85–90), the author deduced the number of representations of a polynomial F in $GF[p^n, x]$ as a sum of products, $F=\sum_1^r \alpha_i X_i^s Y_i$, with constant coefficients α_i, the X_i, Y_i satisfying certain conditions on degree. In the present paper, it is shown that certain types of polynomial coefficients can be introduced without materially altering the previous results. The methods used involve the evaluation of arithmetic sums which arise in connection with other problems as well. (Received November 17, 1950.)

82t. Harvey Cohn: *On stable maxima for the product of homogeneous forms.* II.

Continuing from a previous abstract (Bull. Amer. Math. Soc. Abstract 57-1-15), the author applies to his definitions of stable extrema the forms $|N(\theta)/D|$ where $N(\theta)$ and D are the norm and determinant in an algebraic module of degree n. Let $q=n(n-1)+1$ algebraic numbers $\Phi^{(k)}$ of minimum norm be given, with the q vectors, $\Phi_i^{(k)}/\Phi_j^{(k)}$ (where the $n(n-1)$ components i, j with $i\neq j$ refer to conjugates). Then consider sets of q integers A_k such that $\sum A_k\Phi_i^{(k)}/\Phi_j^{(k)}=0$. If such A_k are all of the same sign, or all zero, then the form has a stable extremum. Thus a differential property is decidable within the normal field. It is then seen that for the absolute extrema (critical lattices) solved by Davenport ($n=3$, $D^2=49$ and $D^2=-23$), the critical lattices for the product provide a stable minimum in the sense of the author's definitions. (Received November 3, 1950.)

83. J. C. E. Dekker: *The constructivity of maximal sum ideals in a certain Boolean algebra.*

A collection of positive integers will be called a set; a collection of sets will be called a class. If V is the class of all sets, V forms a Boolean algebra with respect to the operations of union and intersection. One classifies the maximal sum ideals in a Boolean algebra as atomic or nonatomic. It is generally suspected that while the atomic maximal sum ideals are constructive, the nonatomic maximal sum ideals are not constructive. The nature of this conjecture is however not clear, since no definition of a constructive sum ideal in a Boolean algebra is generally accepted. In this paper the concept of a recursively enumerable class is introduced. This concept enables one to propose two definitions for a constructive sum ideal in the Boolean algebra V. It is shown that according to each of these definitions, a maximal sum ideal in V is constructive if and only if it is atomic. The constructivity of several types of sum ideals in V is investigated. It is shown that the two definitions which are proposed for a constructive sum ideal in V are not equivalent. (Received November 6, 1950.)

84. J. C. Eaves: *A set of matrices possessing the "p" property.* Preliminary report.

In his paper *On k-commutative matrices* [Trans. Amer. Math. Soc. vol. 39 (1936) pp. 483–495] W. E. Roth introduced k-commutative matrices and obtained, among other properties, the skeleton of a matrix $B=(B_{ij})$ such that $A=(A_{ij})$ is k-commutative with respect to B, where A is taken to be a Jordan canonical matrix. Later

[*On the characteristic values of the matrix* $f(A, B)$, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 234–243] Roth discussed the skeletons of a pair of matrices, A and B, which have property "*p*." These papers suggest a set of block matrices $R_\lambda = (R_{\lambda ij})$ $(\lambda = 1, 2, \cdots, k)$ where $R_{\lambda ij}$ is an $n_i \times n_j$ matrix ($n_i = n_j$ for $i = j$; $n_s \neq n_t$ where $s \neq t$ for R_{ss}, R_{tt}) with the first $\{n_i, n_j\} - 1$ diagonals of each $R_{\lambda ij}$ containing only zero elements. There exists a product, P, of elementary permutative matrices such that for any R_λ then $P^{-1}R_\lambda P = T_\lambda$ where T_λ is a triangle matrix. This result is a generalization of Roth's second paper mentioned above. The matrices R_λ $(\lambda = 1, 2, \cdots, k)$ possess the triangle property and the "*p*" property and are k-commutative in pairs. This result is not necessarily true in case $n_s = n_t$, $s \neq t$ for R_{ss}, R_{tt}, even though several other restrictions are placed upon the R_λ. (Received November 13, 1950.)

85*t*. D. O. Ellis: *Notes on the foundations of lattice theory*

A *lattice* may be defined as a system closed under two binary single-valued operations, $a \vee b$ and $a \wedge b$, which satisfies: 1. *Double commutativity* ($a \vee b = b \vee a$ and $a \wedge b = b \wedge a$); 2. *Double associativity* ($a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$); 3. *Double idempotency* ($a \vee a = a \wedge a = a$); and 4. *Alternation* ($a \vee b = a$ if and only if $a \wedge b = b$). Systems satisfying 1, 2, and 4 are called *pseudo-lattices*. Systems satisfying 1, 2, 3, 4, and 5. *Single distributivity* (either $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all a, b, c or $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all a, b, c) are called *distributive lattices*. For pseudo-lattices, it does not follow that each identity of 5 implies the other so that a pseudo-lattice satisfying 6. *Double distributivity* (both identities of 5) is called a *distributive pseudo-lattice*. An element a of a pseudo-lattice is called *finitely potent* if it satisfies 7. $\exists n$ (a natural integer) $\ni \bigwedge_{n+1} a = a$ or $\bigvee_{n+1} a = a$; where $\bigwedge_{n+1} a$ and $\bigvee_{n+1} a$ indicate "powers" or "multiples" of a. In this note it is shown that elements satisfying 7 in a system (1, 2, 4, 6) satisfy 3 and, hence, that a *distributive pseudo-lattice each of whose elements is finitely potent is a distributive lattice*. In nondistributive pseudo-lattices, 7 does not imply 3. (Received July 24, 1950.)

86. J. S. Frame: *An irreducible representation extracted from two permutation groups.*

Let G be a finite group of order O_G, containing two subgroups A and B, of indices m and n respectively, such that the double coset AB contains pO_G elements, and all the remaining qO_G elements of G belong to a second double coset Ag_2B. The subgroups A and B induce representations of G by permutation matrices $R^A(g)$ and $R^B(g)$ of degrees m and n respectively, having as common irreducible components just two irreducible representations: F_1, the identity representation, and F_2, which is to be determined. An $m \times n$ incidence matrix $V = (v_{ij})$ is defined by setting $v_{ij} = 1$ or 0, according as the right cosets $A\rho_i$ and $B\sigma_j$ have or have not a common element, and an $m \times m$ correlation matrix K is formed from the rows of V, having k^2 as the mean square of its elements. It is proved that the degree f of F_2 is $1/k^2$, and the trace χ_λ of the matrices of F_2 for the h_λ elements of a class C_λ is $\chi_\lambda = f(1 - q_\lambda/q)$, where $q_\lambda h_\lambda$ denotes the number of elements of C_λ not in AB. Finally, a right ideal basis for F_2 is obtained from the ring elements $\bar{A}\bar{B}\sigma_j - \bar{G}$, where $\bar{A}$, $\bar{B}$, and $\bar{G}$ denote averages of the elements of A, B, or G in the group ring. (Received October 30, 1950.)

87. V. D. Gokhale: *Concerning the determinant of a Hermitian matrix of order 3 with elements in Cayley algebra.*

After defining the determinant for a Hermitian matrix, the paper proves that the determinant is unchanged by simultaneous interchanges of rows and columns, and that the determinant of the square of a matrix is equal to the square of the determinant of the matrix. An adjoint matrix is defined and it is shown that the product of a matrix and its adjoint is a diagonal matrix whose elements, in case the elements of the original matrix associate with each other, are the value of the determinant of the original matrix. (Received November 8, 1950.)

88. E. H. Hadlock: *A method of construction of an improperly primitive form.*

A method is derived and stated whereby, given the integral invariants Ω, Δ and possible sets of progressions involving 2 and the odd prime factors of Ω and Δ, the integral coefficients a, b, c, r, and s are determined in the improperly primitive form $f=ax^2+by^2+cz^2+2ryz+2sxz$ with determinant $d=\Omega^2\Delta\neq 0$ so that f will have associated with it the given invariants and sets of progressions. (Received November 6, 1950.)

89. Franklin Haimo: *Dense ideals in Boolean algebras.* Preliminary report.

Those ideals in Boolean algebras are studied which are the image spaces of endomorphisms α with the property $\alpha(x)\leqq x$. (Received November 13, 1950.)

90. T. C. Holyoke: *On the structure of multiply transitive permutation groups.*

Given a finite permutation group G fixing the letter 0 and transitive on the remaining letters $1, \cdots, n$, several sets of necessary and sufficient conditions are determined for the existence of a group H which is transitive on letters $0, 1, \cdots, n$ and contains G as the (largest) subgroup fixing the letter 0. Considering G and H to have an additional fixed letter, it is shown that the number of distinct groups H in which the given group G can be so embedded is greater than the number of distinct groups in which any one of the groups H can be similarly embedded. It is further shown that, once all groups which are exactly doubly transitive have been determined, the problem of constructing all groups with higher degree of transitivity is reduced to a very simple, straightforward process. New restrictions on the structure of k-fold transitive groups are found in terms of certain normalizers of the subgroups fixing $1, 2, \cdots, k$ given letters. (Received November 13, 1950.)

91*t*. L. K. Hua and Irving Reiner: *Automorphisms of the unimodular group.*

Let $\mathfrak{M}_n$ denote the group of n-rowed unimodular matrices, and let $\mathfrak{A}_n$ be the group of all automorphisms of $\mathfrak{M}_n$. For $n\neq 2$, it is shown that $\mathfrak{A}_n$ is generated by the set of all inner automorphisms of $\mathfrak{M}_n$, the automorphism $X\in\mathfrak{M}_n\rightarrow(X')^{-1}$, and (for even n only) the automorphism $X\in\mathfrak{M}_n\rightarrow(\det X)\cdot X$. It is also proved that $\mathfrak{A}_2$ is generated by the above and an additional automorphism τ given as follows: for $X\in\mathfrak{M}_2$ and $\det X=1$, set $\epsilon(X)=1$ or -1 according as X is or is not an element of the commutator subgroup of $\mathfrak{M}_2$, and let J be a fixed element of $\mathfrak{M}_2$ with $\det J=-1$; then define $X^\tau=\epsilon(X)\cdot X$ for $\det X=1$, $X^\tau=\epsilon(JX)\cdot X$ for $\det X=-1$. The proof is based on the establishment of canonical forms for involutions under similarity transformations, and normalizer subgroups of such canonical forms. (Received November 17, 1950.)

92. B. W. Jones: *On automorphs of symmetric matrices of odd order.*

Let A be a symmetric matrix of odd order and S an arbitrary skew matrix of the same order. Automorphs of A are expressed as polynomials in SA whose coefficients are explicitly given functions of the coefficients of the characteristic function of SA. The latter coefficients are shown to be forms in the elements of S whose matrices are concomitant matrices of A. These results have some advantages over Cayley's form. The method is partially based upon that of Cantor and Bachman for ternary forms. (Received November 9, 1950.)

93*t*. Jakob Levitzki: *On rings with bounded index.*

Let R be a ring satisfying the following conditions: (1) Every non-nil right ideal contains a nonzero idempotent. (2) the indices of the nilpotent elements of R are bounded (with bound $=n$). It is shown that if R is semi-simple (in the sense of Jacobson) and $A\supset 0$ is an ideal in R, then there exists an ideal B with unit such that $A\supseteq B$, $B\cong C_m$, $m\leqq n$, where C_m is a ring of all m by m matrices over a strongly regular ring C (that is, C satisfies condition (1) and has no nilpotent elements). If R is semi-simple it follows that R is isomorphic with a subdirect sum of rings of type C_m. If in particular R is simple, then $R\cong C_m$, where C is a field. This generalizes a theorem of Jacobson (Ann. of Math. vol. 46 (1945) Theorem 4). If R is semi-simple and the minimal (maximal) condition for two-sided ideals does not hold, then R contains an infinite number of central idempotents. This generalizes a result of Kaplansky (Trans. Amer. Math. Soc. vol. 68 (1950) p. 70). A characteristic minimal (maximal) condition for two-sided ideals is derived for a ring satisfying (1) and (2) to be semi-primary in the Wedderburn-Artin sense. In particular: If R is a simple nilring with bounded index, then R is a nilpotent null-ring. This answers a question raised by Jacobson (loc. cit. p. 700). (Received November 17, 1950.)

94*t*. L. H. Loomis: *Free σ-complete Boolean alegbras.*

Let m be any cardinal number. It is shown that there exists a σ-complete Boolean algebra B with a generating set of cardinal number m such that any other σ-complete Boolean algebra with a generating set of cardinal number m is a σ-homomorphic image of B. B is necessarily unique, and can be realized as a σ-algebra. The crux of the proof consists of showing that if b is a nonzero element of any generated Boolean σ-algebra than there exists a sequence $\{b_n\}$, where each element b_n is either a generator or the complement of a generator, such that $\bigcap_1^\infty b_n \subset b$. (Received November 22, 1950.)

95*t*. G. D. Mostow: *On an assertion of A. Weil.*

A locally compact group is said to have *property* (1) if there is some compact neighborhood of the identity invariant under all inner automorphisms; *property* (2) if there are arbitrarily small neighborhoods of the identity invariant under inner automorphisms. In his book, *Integration des les groupes topologique*, p. 129, Weil asserts that (2) is a consequence of (1). This assertion is proved false, even for Lie groups, by a counter example. However, the theorem is almost true for Lie groups in the following sense. *Theorem.* Let G be a Lie group with property (1). Then there is a compact connected central subgroup C such that G/C is the direct product of compact and abelian subgroups and thus satisfies property (2). Thus (a) the universal covering group of G is the direct product of a compact subgroup and a nilpotent subgroup, (b) the commutator subgroup of the radical of G is in C, (c) if G is simply connected, property (1) implies property (2), (d) if a neighborhood of the identity of G is separated by finite-dimensional representations of G, then property (1) implies property (2). (Received September 18, 1950.)

96. W. A. Rutledge: *The Hurwitz integral quaternions as a principal ideal domain.*

Let J represent the class of Hurwitz integral quaternions $a=a_1\rho+a_2i+a_3j+a_4k$ in which a_1, a_2, a_3, and a_4 are rational integers and $\rho=(1+i+j+k)/2$. Then J is a principal ideal domain in which the modules $J-aJ$, $J-bJ$ are J-isomorphic if and only if a and b are similar in the sense of Ore. Writing a, and likewise b, in a unique standard form $a=2^f r(1+i)^e c$ where r is a rational odd integer, e is zero or one, c is a quaternion without rational integral factor and of odd norm, the general theory of principal ideal domains is applied to J to get the following. (1) If $JaJ\subseteq Jc\cap cJ$ we say c is a total divisor of a. Then a and b have the same total divisors if and only if they have the same rational factors and same exponents of $(1+i)$. (2) Every ideal aJ is bounded and the bound is generated by $a^*=2^f rN(c)\cdot(1+i)^e$ where $a=2^f r(1+i)^e c$, $N(c)$ being the norm of c. (3) The module $J-aJ$ is indecomposable if and only if either a has no rational integral factor and $N(a)$ is a prime power or else $N(a)=2^m$. (4) Two integral quaternions a and b are similar if and only if they have the same norm and bound. (Received November 13, 1950.)

97t. Olga Taussky and John Todd: *The characteristic roots of products of matrices with rational integral elements.*

Let A and B be square matrices with rational integral elements and irreducible characteristic equation. The algebraic number field in which the characteristic roots of AB lie is investigated, in particular in the case that $B=A'$, the transpose of A. In exceptional cases the field can be identical with the field of the roots of A. Not every field can, of course, occur. Quadratic fields are studied in detail. (Received November 13, 1950.)

98. T. L. Wade: *Conjugate tensors.*

Let $T_{i_1i_2\cdots i_p}$ be an arbitrary covariant tensor (that is, one in which the p indices satisfy no particular type of symmetry), and let $T_{i_1i_2\cdots i_p}=\sum_{[\alpha]} T_{i_1i_2\cdots i_p}$ represent the decomposition of $T_{i_1i_2\cdots i_p}$ into tensors of various symmetry types. For each invariant type of symmetry $[\alpha]$ there is an absolute numerical tensor ${}_{[\alpha]}I^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_p} \equiv {}_{[\alpha]}I^{(i)}_{(j)}$, called an *immanant tensor*, and each immanant tensor has a *rank tensor* ${}_{[\alpha]}I^{(i_1)(i_2)\cdots(i_r)}_{(j_1)(j_2)\cdots(j_r)}$. For any $[\alpha]$ the rank tensor factors into two *split tensors* ${}_{[\alpha]}I^{(i_1)(i_2)\cdots(i_r)}$ and ${}_{[\alpha]}I_{(j_1)(j_2)\cdots(j_r)}$. With the aid of these split tensors the familiar concept of the conjugate tensor of a skew symmetric tensor (see Maria Pastori, *Tensori emisimmetrici coniugati*, Atti della Accademia Nazionale dei Lincei. Rendiconti (6) vol. 16 (1932) pp. 216–220) is generalized to give the conjugate tensor of a tensor ${}_{[\alpha]}T_{i_1i_2\cdots i_p}$ of any invariant type of symmetry, and these general conjugate tensors are studied. (Received November 13, 1950.)

ANALYSIS

99t. H. A. Antosiewicz: *On characteristic exponents and reduction theorems for linear differential systems.*

Let $dx/dt=A(t)x$ be a linear system of differential equations where the elements

$a_{ij}(t)$ $(i, j=1, 2, \cdots, n)$ of the matrix $A(t)$ are real, continuous functions for all $t \geqq 0$ for which there exist constants α_{ij} such that (I) $\lim_{t\to\infty} t^{-1}\int_0^t a_{ij}(\tau)d\tau=\alpha_{ij}$, (II) $\int_0^t (a_{ij}(t)-\alpha_{ij})d\tau=O(1)$ as $t \to \infty$. The following special cases are considered: (i) the integrals $\int_0^\infty (a_{ij}(\tau)-\alpha_{ij})d\tau$ converge absolutely, (ii) the functions $a_{ij}(t)$, in addition to satisfying (I) and (II), are monotonic increasing for sufficiently large t. In each case it is shown that the characteristic exponents of the system considered are uniformly bounded. If the characteristic roots $\rho_1, \rho_2, \cdots, \rho_n$ of the matrix (α_{ij}) are real and distinct, there exists a base $x^1, x^2, \cdots, x^n$ of solutions for which $\lim_{t\to\infty} t^{-1} \log \|x^i\| =\rho_i$ and $\log \|x^i\|-\rho_i t=O(1)$ as $t\to\infty$. This base is used to construct a matrix $C(t)$, $C(t)$ and $C^{-1}(t)$ having bounded elements, such that if $y=C^{-1}(t)x$, the system considered is transformed into $dy/dt=B(t)y$ where the matrix $B(t)$ is diagonal. This verifies in part a conjecture by S. P. Diliberto (*On systems of ordinary differential equations*, Annals of Mathematics Studies, no. 20, Princeton University Press, 1950), and represents a first generalization of the classical representation theorems for linear differential systems with constant or periodic coefficients. (Received November 10, 1950.)

100. Nachman Aronszajn: *Construction of reproducing kernels and Green's functions.*

A general method for the construction of reproducing kernels is described which is based on the classical C. Neumann's series for the inverse of an operator. The method is then applied to the construction of Green's functions. It allows the construction of the Green's function for a differential problem if the Green's function for another suitable problem is known explicitly. (Received November 13, 1950.)

101. Nachman Aronszajn and Arthur Zeichner: *A new type of auxiliary problem for approximation of eigenvalues by Weinstein's method.*

A new type of auxiliary problem for the approximation of eigenvalues by Weinstein's method is introduced. This new type allows the application of the method in many cases when the two previously known types (see N. Aronszajn, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 474–480, 594–601) cannot be applied. The new auxiliary problem is obtained by relaxation of continuity conditions inside the given domain. This procedure is particularly useful for those domains which are composed of a finite number of domains (with only boundary points in common), in each of which the problem is solvable. (Received November 13, 1950.)

102*t*. Lipman Bers: *Partial differential equations and generalized analytic functions.* II. Preliminary report.

Let $F(z)$, $G(z)$, $z=x+iy$, be a pair of functions such that (i) $\text{Im}\,(\bar{F}G)>0$, (ii) F_x, F_y, G_x, G_y exist and satisfy a Holder condition. Every complex-valued function $w(z)$ admits the unique representation $w=\phi F+\psi G$; ϕ, ψ real. w is called (F, G)-pseudo-analytic in D if at all points z_0 of D there exists the (F, G)-derivative $w(z_0) = \{d[w(z)-\phi(z_0)F(z)-\psi(z_0)G(z)]/dz\}_{z=z_0}$. This definition leads to a theory which (1) reduces to classical function theory for $(F, G)=(1, i)$, (2) generalizes most of the fundamental concepts and results of classical function theory, (3) bears the same relationship to the general theory of second order linear homogeneous partial differential equations of elliptic type as classical function theory does to that of the Laplace equation, (4) contains previous results on generalized analytic functions (L. Bers and A.

Gelbart, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 67–93; L. Bers, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 130–136) as special cases. (Received September 21, 1950.)

103. F. F. Bonsall and Morris Marden: *Zeros of self-inversive polynomials.*

Given the polynomial $f(z)=\sum_0^n a_k z^k$, let $f^*(z)=z^n\bar{f}(1/z)=\sum_0^n \bar{a}_k z^{n-k}$. The Schur-Cohn method of enumerating the zeros of $f(z)$ within the unit circle $|z|=1$ depends, in the case $|a_0|\neq|a_n|$, upon applying Rouché's Theorem to the polynomial $f_1(z)=\bar{a}_0 f(z)-a_n f^*(z)$ or to $f_1^*(z)$. In the case, however, that $f(z)$ is self-inversive, that is, if $f(z)=uf^*(z)$, $|u|=1$, the method requires the use of the Cohn theorem that $f(z)$ has the same number of zeros in the unit circle as does $[f'(z)]^*$. As proved by Cohn, this theorem appears to be unrelated to the theorem involving $f_1(z)$. In the present note, the authors give a simpler and more natural proof of the Cohn theorem based upon their observation that for any self-inversive polynomial $f(z)$, $[f'(z)]^*=\lim\{g_1(z)/[a_n(1-r^2)]\}$ as $r\uparrow 1$, with $g(z)=f(rz)$. An added remark by the authors is that, if $f(z)$ is self-inversive, no zero of $f'(z)$ can lie on the unit circle except at the multiple zeros of $f(z)$. (Received November 10, 1950.)

104*t*. Jean M. Boyer: *A note on Fourier constants.*

Let $b_n=(2/M)\sum_{k=1}^{M-1} f(k/M)\sin(nk\pi/M)$ and let $a_n=2\int_0^1 f(x)\sin(n\pi x)dx$. The author shows that if $f(x)$ is integrable and of bounded variation, $\lim_{M\to\infty} b_n(M)=a_n$ is uniform in n for $1\leqq n\leqq M$. (Received November 1, 1950.)

105. Sarvadaman Chowla: *Periodic simple continued fractions.*

Let $N(R)$ denote the number of terms in the period of the simple continued fraction for $R^{1/2}$. It is proved that $N(p)$, where p denotes primes, can be arbitrarily large. (Received November 15, 1950.)

106*t*. A. H. Copeland and Frank Harary: *Some embedding properties of rings.*

Our object is to find the minimal ring in which the complex numbers and an arbitrary Boolean algebra B can be embedded as a subring and the set of all idempotents respectively. In order to do this, we consider the ring F of all functions which map the points (as defined by the representation theorem) of the Boolean algebra into the complex numbers. This is a ring whose elements are random variables in the sense of Kolmogoroff. F is a ring containing both of these systems but is not minimal. In fact, the idempotents of F constitute the complete extension of B. There is a subring of F whose idempotents are the Borel extension of B and another subring whose idempotents are exactly B. The latter subring is the desired minimal ring and consists of discrete functions, that is, those which take on only a finite number of values. Both Hilbert spaces and Banach spaces are subsets of the function ring F. (Received November 13, 1950.)

107*t*. D. O. Ellis: *Notes on abstract distance geometry.* I. *The algebraic description of ground spaces.*

The terminology of this paper and its abstract will appear very shortly in Publicationes Mathematicae (David Ellis, *Geometry in abstract distance spaces*, Publicationes Mathematicae Debrecen vol. 2 (1951)). The purpose of the note is to derive from certain familiar algebraic properties of the metroid of a ground space the corresponding distance-theoretic properties of the ground space. The methods are very elementary and, for the most part, consist merely of translation between algebraic and distance-theoretic terminology. An interesting example of the results is: *If the metroid of a ground space is a nilpotent Abelian group (that is, every element is of period* 2) *then the ground space is normal; every point forms a complete metric base; and the ground space has the property of triangular fixity.* Another result (trivial to show directly) which arises in an unexpected fashion from our results is: *The set of nonnegative real numbers forms a groupoid under* $|x-y|$ *which is neither a quasigroup (hence not a loop) nor a semigroup.* (Received October 16, 1950.)

108*t*. Charles Fox: *Iterated transforms.*

This paper obtains generalizations of the iterated transforms $f(x)=\int_0^\infty e^{-xt}g(t)dt$; $g(x)=\int_0^\infty e^{-xt}h(t)dt=f(x)=\int_0^\infty (h(t)/(x+t))dt$, discussed by Widder in his work on Laplace transforms. The Laplace transform is first generalized and it is then proved that the factors e^{-xt} in the above formulae can be replaced by general Laplace transforms of Fourier kernels. Integral operators are then defined by means of Mellin transforms. In most cases these operators, T_1 and T_2, are expressible in the form $T_1p\{x\}=\int_0^\infty p(xu)m(u)du$ and $T_2q\{x\}=\int_0^\infty q(xu)n(u)du$, where $m(x)$ and $n(x)$ are a pair of Fourier kernels. Iterated transforms of a very general character are then obtained as follows: $g(x)=\int_0^\infty T_1p\{xt\}h(t)dt$; $f(x)=\int_0^\infty T_2q\{xt\}g(t)dt$; $f(x)=\int_0^\infty (p, q)\{t\}h(xt)dt$, where $(p, q)\{t\}=\int_0^\infty p(ut)q(t)dt$. Further it is shown that when $p(x)=q(x)$ and $m(x)=n(x)$, then there exist Parseval theorems of the type $\int_0^\infty f(ut)h(t)dt=\int_0^\infty g(ut)g(t)dt$. The conditions required for the validity of these theorems are mainly that $h(x)$, $p(x)$, and $q(x)$ should belong to $L^2(0, \infty)$ and that $M(s)$ and $N(s)$, the respective Mellin transforms of $m(x)$ and $n(x)$, should be bounded on the line $s=1/2+i\tau$ $(-\infty<\tau<+\infty)$ and that $M(s)N(1-s)=1$. The investigations are based largely upon the properties of Mellin transforms. (Received November 14, 1950.)

109*t*. B. R. Gelbaum: *A nonabsolute basis for Hilbert space.*

By the use of a construction of M. S. Al'tman (Doklady Akademii Nauk SSSR (N.S.) vol. 69 (1949) pp. 483–485 (Russian)) and an earlier result of the author (*Expansions in Banach spaces*, Duke Math. J. vol. 17 (1950) pp. 187–196) the existence of a nonabsolute basis in Hilbert space is demonstrated. (Received October 13, 1950.)

110*t*. A. W. Goodman: *Typically-real functions with assigned zeros.*

A function $f(z)=z^q+\sum_{n=q+1}^{\infty} b_nz^n$ is said to be typically-real of order p if it is regular in $|z|<1$, if all the coefficients b_n are real, and if there is a ρ such that for $\rho<r<1$, $\Im f(z)$ changes sign $2p$ times on the circle $z=re^{i\theta}$. Let $f(z)$ have zeros $\beta_1, \beta_2, \cdots, \beta_s$, $0<|\beta_j|<1$, and let $t\geqq 0$ be defined by $q+s+t=p\geqq 1$. An upper bound is obtained for $|b_n|$, $n=q+1, q+2, \cdots$ in terms of $|\beta_j|$. If $t=0$ or if t is odd this bound is sharp for every q, s, n, and $|\beta_j|$, $j=1, 2, \cdots, s$. (Received June 21, 1950.)

111*t*. Israel Halperin and George Lorentz: *Multiply subadditive functions.*

Let S be a set with operation $e_1\dotplus e_2$, S defined for some pairs $e_1, e_2\in S$. Interesting special cases arise when e are sets and $e_1\dotplus e_2$ is the union of disjoint e_1, e_2, or when S

is a complemented modular lattice with transitive perspectivity and $e_1 \dotplus e_2$ the union of independent elements. The authors discuss functions $W(e)$ on S admitting representations (a) $W(e) = \sup |\phi(e)|$, or (b) $W(e) = \sup \phi(e)$, where sup is taken for a certain family $\{\phi\}$ of functions additive on S (nonnegative in case (b)). The necessary and sufficient condition for (a) is the multiple subadditivity of W (in addition, W must be increasing in case (b)). W is multiply subadditive on a Boolean ring S if $pW(e) \leqq \sum_{r=1}^{n} W(e_r)$ whenever the e_r cover e exactly p times, that is, there are disjoint decompositions $e_r = \bigcup_{k=1}^{p} e_{rk}$, $r = 1, \cdots, n$, such that $e = \bigcup_{r=1}^{n} e_{rk}$, $k = 1, \cdots, p$, again as disjoint decompositions. (This definition is properly changed for an arbitrary S.) For the proof, S is imbedded into the Abelian group G of all formal sums $\sum \pm e_r$, $e_r \in S$ (with some identifications); the Hahn-Banach theorem in G is used. (Received October 18, 1950.)

112. P. C. Hammer and Andrew Sobczyk: *Critical points of a convex body.*

In real n-dimensional linear space L, let any positive-homogeneous, subadditive functional $p(x)$, positive for $x \neq \theta$, be called a *sub-norm*. Any bounded convex region C in L with θ interior to C is the set of x for which a sub-norm $p(x)$ is less than or equal to 1; and conversely. Let $y = (y-x)+x$, $\beta(x-y)+x$, with $\beta \leqq 1$, be boundary points of C, and let $r(x) = \max\ [1/(\beta+1)]$ for boundary points y. Then if r_0 is the minimum value of $r(x)$, another non-negative function associated with C is $(r(x)-r_0)/(1-r_0) = u(x)$. A point where $r(x)$ is minimum is called a *critical point* of C. For a plane trapezoid, $u(x)$ is not a sub-norm. For a plane triangle, $u(x)$ is the characteristic sub-norm with origin at the critical point, and likewise for an n-dimensional simplex; for the latter the critical point also is unique. On the other hand, the critical set for a triangular prism is an interval; in general for a region C which is the product of a symmetric n_1-dimensional region C_1, and an n_2-dimensional region C_2 with unique critical point and ratio $r_0 > 1/2$, $n_1+n_2 = n$, the critical ratio is r_0 and the critical set is the set of (x_1, θ_2) where $\|x_1\| \leqq (2r_0-1)$. Conjugate regions and relations with convex linear topological spaces are discussed. (Received November 8, 1950.)

113*t*. M. H. Heins: *Riemann surfaces of infinite genus.* I.

Riemann surfaces of infinite genus with one ideal boundary element which is null in the sense of harmonic measure are considered. By an end of such a surface is meant a subregion whose complement is compact. For the purposes of the present investigation it suffices to consider ends with simple analytic boundary and all further reference to ends is taken in this restrictive sense. Use is made of R. Nevanlinna's recent investigations concerning Riemann surfaces with null boundary. Results: (1) There exist admitted surfaces with the property that for each end every nonconstant function meromorphic on the end takes all values infinitely often with the exception of at most two (extension of a result of Myrberg). (2) There exist admitted surfaces tolerating nontrivial analytic functions omitting a countable infinity of values. (3) There exist ends tolerating nontrivial bounded harmonic functions not possessing a limit at the ideal boundary. (4) A function analytic and bounded on an end possesses a limit at the ideal boundary, and if the function is not identically constant, it is $(1, m)$ on some subend $(n \geqq 2)$. (Received November 6, 1950.)

114*t*. M. H. Heins: *Riemann surfaces of infinite genus.* II.

This is a continuation of Part I. (5) Let the family of non-negative harmonic functions on an end which vanish continuously on the relative boundary be introduced. If it is generated by a finite subset of its members (the family being considered as an additive semimodule with operator), the least number of generators is termed the dimension of the end. Otherwise the dimension is taken as ∞. Dimension is independent of ends of an admitted surface and is a conformal invariant of the surface. (6) For each end of an admitted surface the bounded harmonic functions whose domain is the given end all possess limits at the ideal boundary if and only if the dimension is one. (7) If n of (4) is taken as least possible, then the dimension does not exceed n. (8) There exist two-sheeted transcendental hyperelliptic Riemann surfaces tolerating harmonic functions which are nontrivial, singularity-free, and $O(\log r)$. (Received November 6, 1950.)

115. Edwin Hewitt and H. S. Zuckerman: *Equivalence of two notions of integral.*

Let X be a locally compact Hausdorff space, and let $\mathfrak{C}_+$ be the set of all continuous, real-valued, non-negative functions on X each of which vanishes outside of some compact set. Let M be a real-valued, non-negative functional defined on $\mathfrak{C}_+$ such that $M(f+g)=M(f)+M(g)$ for $f, g\in\mathfrak{C}_+$ and $M(\alpha f)=\alpha M(f)$ for α real and greater than or equal to 0 and $f\in\mathfrak{C}_+$. For all compact subsets A of X, define $\mu(A)$ as $\sup_{0\leqq f\leqq\phi_A} M(f)$. For all open subsets G of X, define $\mu^*(G)$ as $\sup \mu(A)$, taken over all compact $A\subset G$. For all subsets P of X, define $\mu_*(P)$ as $\inf \mu_*(G)$, taken over all open $G\supset P$. It is proved that μ^* is an outer measure on X in the sense of Carathéodory with respect to which every compact set and every open set of finite μ_*-measure is measurable. Following Daniell, Bourbaki, Stone, and McShane, one defines a class of summable functions on X with respect to M by using the classes of upper and lower semi-continuous functions on X, in the usual fashion. It is proved that a function is summable if and only if it is measurable (μ^*) and has a finite integral (μ^*). In this case, the Daniell sum is equal to the μ^*-integral. (Received November 17, 1950.)

116t. R. C. James: *A nonreflexive separable Banach space isometric with its second conjugate space.*

For a sequence $x=(x_1, x_2, \cdots)$, let $\|x\|=\text{l.u.b.}\ [\sum_{i=1}^{n} (x_{p_i}-x_{p_{i+1}})^2+(x_{p_{n+1}}-x_{p_1})^2]^{1/2}$, where the l.u.b. is over all positive integers n and finite increasing sequences of positive integers $p_1, p_2, \cdots, p_{n+1}$. Let B be the Banach space of all x for which $\lim_{n\to\infty} x_n=0$ and $\|x\|$ is finite. This space B is isomorphic with a nonreflexive Banach space which is known to be isomorphic with its second conjugate space and which is isometric under the natural mapping with a maximal closed linear subspace of its second conjugate space. B^{**} is the space of all $F=(F_1, F_2, \cdots)$ for which $\|F\|=\lim_{n\to\infty} \|(F_1, \cdots, F_n, 0, 0, 0, \cdots)\|$ is finite. The mapping $x=(x_1, x_2, \cdots)\leftrightarrow F_x=(x_2-x_1, x_3-x_1, \cdots, x_n-x_1, \cdots)$ is an isometry of B with B^{**}. (Received November 13, 1950.)

117. S. L. Jamison: *Perturbation of normal operators.* Preliminary report.

A normal operator N on a Hilbert space is a closed, densely defined linear transformation of the space into itself which commutes with its adjoint, that is, $N^*N=NN^*$. If a bounded operator which depends on a real parameter ϵ is expressible as a convergent power series in ϵ: $T(\epsilon)=T_0+\epsilon T_1+\epsilon^2 T_2+\cdots$ where $T_0, T_1, T_2, \cdots$ are bounded operators, then $T(\epsilon)$ is said to depend regularly on ϵ. An isolated portion of the spectrum of a regular normal operator $N(\epsilon)$ not only moves continuously with ϵ but a measure of how fast it moves is obtained. In particular, if the unperturbed operator $N(0)=N_0$ possesses an isolated eigenvalue λ_0 of finite multiplicity m, then there are m numerical functions $\lambda^{(i)}(\epsilon)$ and m elements $\phi^{(i)}(\epsilon)$ (depending regularly on ϵ) such that the $\lambda^{(i)}(\epsilon)$ are eigenvalues of the perturbed operator $N(\epsilon)$, $\lambda^{(i)}(0)=\lambda_0$, and the $\phi^{(i)}(\epsilon)$ are corresponding mutually orthogonal normed eigenelements. These results generalize those of Franz Hellich (Math. Ann. vol. 113 (1936)) and Bela v. Sz. Nagy (Comment. Math. Helv. vol. 19 (1947)) who proved them for self-adjoint operators. A first method of proof is a reduction to the self-adjoint case, and a second method is a generalization of Nagy's techniques. (Received November 9, 1950.)

118. R. V. Kadison: *Isometries in operator algebras.*

An isometry (norm preserving linear isomorphism) of $C'(X)$, the set of all continuous complex-valued functions on the compact-Hausdorff space X, is a ring isomorphism of $C'(X)$ followed by multiplication by a unitary function. The noncommutative extension of this result to arbitrary C^*-algebras (uniformly closed, self-adjoint algebras of operators on a Hilbert space containing an identity) is proved. Specifically, if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-algebras and ρ is an isometry between them, then ρ is the composition of a linear isomorphism which preserves the adjoint operation and is multiplicative on powers (C^* or quantum isomorphism) with left multiplication by a unitary operator of $\mathfrak{A}_2$. The converse, that any such map is an isometry, is also proved. As a preliminary to these results all extreme points on the unit sphere of a C^*-algebra $\mathfrak{A}$ are determined. They are those partially isometric operators U in $\mathfrak{A}$ such that $(I-F)\mathfrak{A}(I-E)=(0)$ where $U^*U=E$ and $UU^*=F$. In the case of weakly closed C^*-algebras (rings of operators), the form of C^*-isomorphisms is studied. Applications are made to factors. It is proved for example that factors that are isometric (equivalent) as Banach spaces are * isomorphic as algebras. (Received November 6, 1950.)

119. Joseph Kampé de Fériet: *Autocorrelation function of a truncated function.*

In a previous paper (to be published in the Proceedings of the International Congress of Mathematicians) the author has given a new way to obtain the energy spectrum (defined by Norbert Wiener, Acta Math. vol. 55 (1930) pp. 117–258) of a function $u(t)$ which belongs to L^2 for every finite interval and has a continuous autocorrelation function; his method is based on the definition of an autocorrelation function $\rho_T(h)$ for the truncated function $u_T(t)$ (equal to $u(t)$ in $|t|\leqq T$ and to zero outside). The present paper establishes that $\rho_T(h)$, which is zero for $|h|\geqq 2T$, is represented in $|h|\leqq 2T$ by the absolutely and uniformly convergent Fourier series: $\rho_T(h) = \sum_{-\infty}^{+\infty} |a_n|^2 \exp (n\pi i h:2T)$, where a_{2n} and a_{2n+1} are respectively the nth Fourier constant of u_T and of $u_T \exp (-\pi i t:2T)$. From this it follows that not every convergent trigonometrical series with positive coefficients can represent an autocorrelation function of a truncated function: one can choose arbitrarily only the even (odd) coefficients (provided that their sum converges). (Received November 13, 1950.)

120. M. S. Klamkin: *Sums and asymptotic sums of series related to the harmonic series.*

Various elementary identities are derived and are applied to determine sums and asymptotic sums of series related to the harmonic series. Among the series summed are the following whose pth terms are: $S_{1,p}/p$; $S_{1,p}/p^2$; $S^2_{1,p}/p$; $S^3_{1,p}/p$; $S^2_{1,p}$; $S_{2,p}/p$; $S_{2,p}/p^2$; $S_{3,p}/p$; $S_{1,p}/p^3$ $S_{1,p}S_{2,p}/p$; $S_{n,p}$ where $S_{m,p}=\sum_{r=1}^{p}(1/r^m)$. (Received November 13, 1950.)

121. R. J. Levit: *A generalization of the Jacobi polynomials orthogonal on a finite set of points.*

The Legendre, Hermite, and Laguerre polynomials are limiting cases of certain finite systems of polynomials which satisfy orthogonality conditions expressed in terms of finite sums (references in G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, §2.8, 2.9). In the present paper a system of polynomials $p_n^{(h,\alpha,\beta)}(x)$ of degrees $n=0, 1, \cdots, N-1$ is defined which is a similar sense constitutes a finite generalization of the Jacobi polynomials. More precisely, the polynomials $p_n^{(h,\alpha,\beta)}(x)$ satisfy orthogonality conditions of the form $\sum_{i=0}^{N-1} w^{(h,\alpha,\beta)}(x_i) \cdot p_m^{(h,\alpha,\beta)}(x_i)p_n^{(h,\alpha,\beta)}(x_i)=0$ for $m\neq n$, where $w^{(h,\alpha,\beta)}(x)$ is a certain positive function on the finite set $x_0<x_1<\cdots<x_{N-1}$; and, whenever $\alpha>-1$, $\beta>-1$, $p_n^{(h,\alpha,\beta)}(x)$ approaches the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ while $w^{(h,\alpha,\beta)}(x)$ approaches the Jacobi weight function $(1-x)^\alpha(1+x)^\beta$ as $h\to 0$. The usual properties of orthogonal systems, such as a recurrence relation, formula of Rodrigues' type, and so forth, are developed. For certain values of h, α, β the polynomials $p_n^{(h,\alpha,\beta)}(x)$ continue to form an orthogonal system even though the inequalities $\alpha>-1$, $\beta>-1$ do not hold. In some of these cases they exhibit the peculiarity of being orthogonal on two distinct sets of points with respect to $w^{(h,\alpha,\beta)}(x)$. (Received November 13, 1950.)

122*t*. A. J. Lohwater: *Boundary values of functions analytic in the unit circle.*

The author shows that, if $f(z)$ is analytic with no zeros in $|z|<1$, and if $f(z)$ possesses radial limit values of modulus 1 for almost all $e^{i\theta}$ of $|z|=1$, then, in order that $f(z)$ not be identically constant, there must exist a path interior to $|z|<1$ and terminating at a point of $|z|=1$ along which either $f(z)\to 0$ or $f(z)\to\infty$. If there is no path along which $f(z)\to\infty$, then $|f(z)|<1$ in $|z|<1$. If, in addition to the above hypotheses, $f(z)$ is of bounded characteristic in $|z|<1$, the path mentioned above becomes radial. A uniqueness theorem for harmonic functions of bounded mean modulus follows as a corollary. (Received October 26, 1950.)

123*t*. A. N. Milgram and P. C. Rosenbloom: *Heat conduction on Riemannian manifolds.* II. *Properties of solutions.*

In their previous abstract (Bull. Amer. Math. Soc. Abstract 57-1-4) the authors discussed existence and certain properties of the solutions of the equations (1) $\Delta\alpha=\partial\alpha/\partial t$ where α is an exterior differential form defined on a closed orientable Riemannian manifold of class C^2. Let $T_t\alpha$ denote the solution at time t having initial value α. A spectral analysis of the complete semigroup of transformations T_t is made by elementary means. As a corollary, the decomposition theorem of deRham and the theories of the equations $\Delta\alpha=\lambda\alpha$ and $\Delta\alpha=\beta$, where β is a given form, are obtained. $T_t\alpha$ can be represented as the scalar product of α with the double form $J_p(x, y, t)$ which is the so-called fundamental solution of the equation (1). The integral equation for $T_t\alpha$ leads immediately to the series representation of J_p and makes it possible to deduce readily that the solutions of (1) have the same degree of smoothness as the manifold. This integral representation of T_t shows that this operator is completely continuous. J_p behaves like the euclidean fundamental solution as $t\to 0$. (Received November 13, 1950.)

124t. K. S. Miller: *Self-adjoint factorizations of differential operators.*

In this short note, the following theorem is proved: Let L be an ordinary linear differential operator, $Lu=p_0(x)u^{(n)}+p_1(x)u^{(n-1)}+\cdots+p_n(x)u$ of even order $n=2r$. $p_i(x)\in C^{n-i}$ and $p_0(x)>0$ in some closed finite interval $[a, b]$. Then there exists a subinterval of $[a, b]$ in which L has a representation $L=f(x)P_1P_2\cdots P_r$ where each P_α, $\alpha=1, 2, \cdots, r$, is a linear differential operator of the second order and formally self-adjoint. (Received October 30, 1950.)

125. Josephine M. Mitchell: *On the spherical summability of multiple orthogonal series.*

Let $\{\phi_m\}$ $(m=m_1\cdots m_q;\ m_k=0, 1, 2, \cdots;\ k=1, \cdots, q)$ be a complete orthonormal system of real Lebesgue square integrable functions defined on a measurable set $E\subset q$-dimensional Euclidean space. Let $\sum a_m\phi_m$ be a multiple orthogonal series, where $\{a_m\}$ is a set of constants satisfying $\sum a_m^2<\infty$. We consider the (spherical) convergence and (C, α) summability of the orthogonal series, that is, respectively, the existence of $\lim_{N\to\infty} S_N$, where $S_N=\sum_{\nu=0}^{N} 0_\nu$, $0_\nu=\sum a_m\phi_m$ if $\nu=m_1^2+\cdots+m_q^2$ or 0 if $\nu\neq m_1^2+\cdots+m_q^2$, and of $\lim_{N\to\infty} \sigma_N^\alpha$, where $\sigma_N^\alpha=(p_N^\alpha)^{-1}\sum_{\nu=0}^{N} p_{N-\nu}^\alpha\, 0_\nu$ $(\alpha>0)$, $p_N^\alpha=(\alpha+1)\cdots(\alpha+N)/N!$. By means of methods similar to those used in the case $q=1$, it is possible to extend all the well known sufficient conditions for simple orthogonal series to the convergence and (C, α) summability of the multiple series $\sum a_m\phi_m$. In addition we prove that the (C, α) summability methods are all equivalent for $\alpha>0$ and equivalent to the Riesz mean summability of order α and type ν. These results represent a considerable improvement over known results of the same type on spherical convergence and summability of multiple Fourier series. (Received November 17, 1950.)

126. C. N. Moore: *On convergence factors for summable double series whose Cesàro means do not remain bounded.*

It is usual in dealing with convergence factors in convergent or summable double series to make the assumption that the partial sums or the corresponding means of partial sums remain bounded. This is the case of greatest interest and the one which yields the simplest results. However, for the sake of completeness, it seems desirable to investigate the modifications in the theory which result from dropping this assumption. The present paper deals with the situation in question for the case of Cesàro summable double series. (Received November 14, 1950.)

127t. G. D. Mostow: *On the L^2-system of a Lie group.*

Theorem. Suppose G is a connected Lie group. Then there is a compact subgroup $K\subset G$ and an analytic submanifold E of G such that (1) the mapping $(k, e)\to k\cdot e$ of the analytic manifold $K\times E$ onto G is an analytic isomorphism (that is, analytic mapping with nonvanishing Jacobian); (2) coordinates can be introduced on E in such a way that Haar measure on G becomes the product measure of Haar measure on K and ordinary *euclidean* measure on E. From (1) we conclude that a Lie group can be embedded analytically in euclidean space. From (2) we conclude that the L^2-system of G (with respect to Haar measure) admits a base of functions which are analytic on G. (Received November 10, 1950.)

128t. Zeev Nehari: *Some extremal maps.*

Let D be a multiply-connected domain in the complex z-plane and let S be the class of univalent functions $w=f(z)$ in D which have a simple pole of residue 1 at a given point ζ of D. If a, b are two distinct points of D, the region of variability Δ of the expression $\log \{f(a)-f(b)\}$ $(f(z)\in S)$ is found and a geometric characterization of the conformal maps associated with the boundary points of Δ is given. In particular, it is shown that the function $f_0(z)$ solving the problem $|f(a)-f(b)|=\max$ maps D onto a slit domain whose slits are confocal elliptic arcs with the foci $f_0(a)$, $f_0(b)$. The problem $|f(a)-f(b)|=\min$ leads to a corresponding hyperbolic slit domain. Results of the same type are also obtained for the domain of variability of $\log \{f(a)f(b)\}$. (Received November 13, 1950.)

129. G. O. Peters: *Note on Sterling's numbers*. Preliminary report.

The author uses the definition $x^{(n)}=x(x-1)\cdots(x-[n-1])$, then $x^{(n)}$ can be expressed as a polynomial in powers of x, and x^n as a polynomial in factorials of x, using Sterling's numbers of the first and second kind respectively. The author then uses the definition $x^{(-n)}=1/(x+1)(x+2)\cdots(x+n)$ and expresses $x^{(-n)}$ as a power series, and expresses x^{-n} as a factorial series, and calls the coefficients in these series Sterling's numbers of the first and second kind of negative degree. The Sterling's numbers of the first and second kind may thus be defined by $x^{(n)}=\sum_{\nu=n}^{0} S_n^\nu x^\nu$ and $x^n=\sum_{\nu=n}^{0} \zeta_n^\nu x^{(\nu)}$. And the Sterling's numbers of the first and second kind of negative degree may be defined by $x^{(-n)}=\sum_{\nu=n}^{\infty} S_{-n}^{-\nu} x^{-\nu}$ and $x^{-n}=\sum_{\nu=n}^{\infty} \zeta_{-n}^{-\nu} x^{(-n)}$. The author finds a simple relation between the well known Sterling's numbers of positive degree and the Sterling's numbers of negative degree, given by $S_{-k}^{-l}=(-1)^{l+k}\zeta_l^k$ and $\zeta_{-k}^{-l}=(-1)^{l+k}S_l^k$. (Received November 14, 1950.)

130t. M. O. Reade: *On Gustin's bilinear integral operator*. Preliminary report.

Gustin's bilinear integral identity for harmonic functions [Amer. J. Math. vol. 70 (1948) pp. 212–220] is used to obtain information concerning subharmonic functions. A typical result is the following one. Let $f(x, y)$ be subharmonic for $x^2+y^2<1$, and let $\mu(e)$ be the non-negative mass distribution associated with $f(x, y)$. Then for each triple of positive constants λ_k, $k=1, 2, 3$, $\lambda_1\lambda_2=\lambda_3^2$, $\lim_{r\to 0}\{(2/\pi r^2)\int_0^{2\pi}[f(x_0+\lambda_1 r\cos\phi, y_0+\lambda_1 r\sin\phi)f(x_0+\lambda_2 r\cos\phi, y_0+\lambda_2 r\sin\phi)-f^2(x_0+\lambda_3\cos\phi, y_0+\lambda_3 r\sin\phi)]d\phi=(\lambda_1-\lambda_2)^2 f(x_0, y_0)D_s\mu(x_0, y_0)$ holds almost everywhere in $x^2+y^2<1$. (Received November 13, 1950.)

131. P. K. Rees: *Rotation of isometric circles in transforms of Fuchsian groups.*

This paper gives several theorems concerning the angles $gog_t=\theta_t$ and $gog_s=\theta^s$ where g, g_t, and g_s are the centers of the isometric circles of the Fuchsian transformations $G(z)=(\alpha z+\bar{\nu})/(\nu z+\bar{\alpha})$, $\alpha\bar{\alpha}-\nu\bar{\nu}=1$; $T(z)=(az+\bar{c})/(cz+\bar{a})$, $a\bar{a}-c\bar{c}=1$; and $S(z)=GTG^{-1}=(Az+\overline{C})/(Cz+\overline{A})$, $A\overline{A}-C\overline{C}=1$. The transformation G is considered as variable, and T is any fixed transformation of a group. The coefficients α and ν are restricted to real values so as to lead to a simple geometric situation, but a, c, A, and C are complex. The following is typical of the theorems: If the imaginary part of ac is different from zero, then the necessary and sufficient condition that $\sin\theta_s=0$ is that the center $g=-\bar{\alpha}/\nu=-\alpha/\nu$ of the isometric circle of G be at an intersection of the real axis and the fixed circle Q_8 of T with center at $[(a+\bar{a})(a-\bar{a})+(c+\bar{c})(c-\bar{c})]/2(ac-\overline{ac})$ and radius $[(a+\bar{a})(a-\bar{a})+(c+\bar{c})(c-\bar{c})-4(ac-\overline{ac})^2]^{1/2}/2(ac-\overline{ac})$. (Received November 13, 1950.)

132. J. J. Sopka: *Reynolds operators on the algebra of all continuous functions on a compact Hausdorff space.*

A Reynolds operator T on an algebra A with a unit is an endomorphism of A which takes the unit into itself and which has the property: $T(fT(g))=T(f)T(g)$ for all f and g in A. If an order relation $\geqq$ is defined in A, T is called order-preserving if $f\geqq 0$ implies $T(f)\geqq 0$. In the case where $A=C(K)$, the family of all continuous functions on a compact Hausdorff space K, then to each order-preserving Reynolds operator T on $C(K)$ there corresponds a partition $\mathcal{K}$ of K into closed subsets K^α of K, and a family of Radon measures $\{\mu_\alpha\}$, one μ_α defined in each K^α, such that for each f in $C(K)$ the value of $T(f)$ is constant on each K^α and equal to the integral of f over K^α with respect to the measure function μ_α. A necessary and sufficient condition is given for a system $[K, \mathcal{K}, \{\mu_\alpha\}]$ to admit a Reynolds operator. (Received November 10, 1950.)

133*t*. Otto Szász: *Identities and inequalities concerning orthogonal polynomials and Bessel functions.*

Several papers have been written recently generalizing a nonlinear inequality concerning Legendre polynomials, due to P. Turán. In these papers analogous inequalities were derived for ultraspherical polynomials, Laguerre and Hermite polynomials, and for Bessel functions. In the present paper some new proofs and further improvements of these results are given. The methods are based again partly on recursion formulas and partly on linear differential equations, satisfied by the functions under consideration. (Received November 3, 1950.)

134. T. T. Tanimoto: *Manifolds in Hilbert space.*

Let X be an abstract real Hilbert space with inner product $x_1 \times x_2$, and $u \in X$ a vector function of vectors $x_1, x_2, \cdots, x_r \in X$ linear in all x's whose domains are X. Call u an r-hyperlinear transformation $\alpha^{[r]}$. Continuity, boundedness, and their equivalence are developed in the usual way. Let $\sigma_{(q)}$ be a transformation of X into X, not necessarily linear, by a function of a point q. (Proper distinctions are made between vectors and points.) The derivatives of $\sigma_{(q)}$ with respect to q are defined conventionally and are linear or hyperlinear operators. A manifold is defined as the range of $\sigma_{(q)}$. The tangent linear manifold M_q at q is defined to be the range of $\alpha_{(q)}=d\sigma_{(q)}/dq$. Nonsingular points with usual properties are defined. If E, a function of q, projects any vector on M_q and if $\beta_{(q)}^{[r]}$ is an r-hyperlinear operator, a function of q, the intrinsic differential, if $d\beta_{(q)}^{[r]}$ exists, is defined by $d^*\beta_{(q)}^{[r]}=Ed\beta_{(q)}^{[r]}$. dE is not proved to exist but we define $d^*E=E(\alpha^{-1})_{-1}(d\alpha)_{-1}(1-E)$ where $(\alpha^{-1})_{-1}$, $(d\alpha)_{-1}$ are the adjoints of α^{-1} and $d\alpha$ and where $\alpha=d\sigma/\alpha q$. If d_1q and d_2q are distinct vectors at q in M_q we define the Riemann tensor by $R^{[3]}d_1qd_2qu=d_1^*d_2^*u-d_2^*d_1^*u$ for any vector u at q in M_q. The curvature is defined by $K=R^{[3]}b_1b_2b_1\times b_2/(b_1^2b_2^2-(b_1\times b_2)^2)$, where b_1 and b_2 are distinct unitary vectors at q in M_q. (Received November 13, 1950.)

135. H. E. Taylor: *On the type and form of the mapping function of certain Riemann surfaces.* Preliminary report.

Three classes of Riemann surfaces are defined, all open and with four first-order branch points in each sheet S_n on the real axis over $w=a_n, b_n, a_{n-1}, b_{n-1}$. For class I, $0<a_1<b_1$, $a_{2k\pm 1}>a_{2k}$, $b_{2k-1}>a_{2k-1}$, $a_{2k}>b_{2k}$, $k=1, 2, 3, \cdots$. For class II, $b_{-1}<a_{-1}<0<a_1<b_1$, $a_{2k\pm 1}>a_{2k}$, $b_{2k-1}>a_{2k-1}$, $a_{2k}>b_{2k}$, $a_{-2k}>a_{-2k\pm 1}$, $b_{-2k}>a_{-2k}$, $a_{-2k-1}>b_{-2k-1}$, $k=1, 2, 3, \cdots$. For class III, $0<a_1<b_1<b_2$, $b_{2k}<a_{2k}<a_{2k+1}<b_{2k+1}<b_{2k+2}$,

$k=1, 2, 3, \cdots$. Any surface of class I or III is parabolic and any surface of class II is the image of the z-sphere less two points. The meromorphic function $w=f(z)$ corresponding to surfaces of class I is such that $f'(z)=e^{\delta z}\prod_{k=1}^{\infty}(1-z/\alpha_k)(1-z/\beta_k)/(1-z/\gamma_k)$ with $\delta \leqq 0$; α_k, $-\beta_k$, $-\gamma_k$ all $\rightarrow\infty$ as $k\rightarrow\infty$; $\beta_k<\gamma_k<\beta_{k-1}<0<\alpha_k<\alpha_{k+1}$; and with the residues of $f'(z)$ at the poles all zero. A partial converse of this result is obtained. Analogous results are obtained for surfaces of class II. The methods employed involve approximation of the given surfaces by sequences of elliptic surfaces as developed by G. R. MacLane (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 99–113); the use of the theory of normal families of functions; and the results of Carathéodory (Math. Ann. vol. 72 (1912) pp. 107–144) on the mapping of a sequence of plain domains by a family of functions. (Received November 9, 1950.)

136t. F. G. Tricomi: *A class of nonorthogonal polynomials related to those of Laguerre.*

A systematic study is made of the polynomials $l_n(x)=(-1)^n L_n^{(x-n)}(x)$ and of the more general ones in which the upper index of the Laguerre polynomial is $x-y-n$, which appear as coefficients in some important expansions. These polynomials (for which the degree of l_n in x is the greatest integer in $n/2$) have a simple generating function and the recurrence relation $(n+1)l_{n+1}-(n+y)l_n+xl_{n-1}=0$ holds. Further its zeros are all real and the kth of them ($k=1, 2, \cdots$ fixed) is a decreasing function of n which approaches the limit $y+k-1$ as $n\rightarrow\infty$. An asymptotic expansion of l_n for $n\rightarrow\infty$, and x and y bounded is found and some further properties established. The generalized polynomial is a particular solution of the parabolic partial differential equation $(\xi+\eta)u_{\xi\xi}+(\eta-\xi)u_\xi+nu=0$ where $x=\xi+\eta$, $y+n-1=\xi-\eta$, which is closely related to the differential equation of the confluent hypergeometric functions. (Received November 9, 1950.)

137t. H. S. Wall: *J-matrices of interior order k*. Preliminary report.

Let $u_n=(u_n^{ij})$ be a k by k matrix of polynomials in x of degree $\leqq n$, $=n$, $<n$, according as $i<j$, $i=j$, $i>j$, respectively, and Φ a k by k symmetric matrix of functions in $BV[-\infty, +\infty]$ such that $\int u_m' d\Phi u_n=\delta_{mn}\cdot I$, $m, n=0, 1, 2, \cdots$. If $C_p=\int x^p d\Phi$, $p=0, 1, 2, \cdots$, $\{u_n\}_{n=0}^{\infty}$ exists only if the determinants of the segments of the matrix (C_{i+j}) are different from 0. There exist sequences $\{A_p\}_{p=1}^{\infty}$, $\{B_p\}_{p=1}^{\infty}$ of k by k matrices of numbers, A_p nonsingular with only 0's above the main diagonal, B_p symmetric, such that $u_{n-1}A_n+u_n(B_{n+1}+xI)+u_{n+1}A_{n+1}'=0$, $n=0, 1, 2, \cdots$, $u_{-1}=0$. The matrix J of the quadratic form $\sum(\xi_p' B_p\xi_p+\xi_p' A_p\xi_{p+1}+\xi_{p+1}' A_p\xi_p)$, ξ_p a k-dimensional column vector, is called a J-matrix of interior order k. A substantial part of the theory of continued fractions ($k=1$) is extended to J. A reciprocal $r(\rho)$ of $J+zI$ is determined by its kth segment $r(\rho)_k$, which is an arbitrary symmetric k by k matrix ρ. The kth segment f_m of the reciprocal of $(J+zI)_{mk}$ plays the role of mth approximant. If $\mathfrak{J}(J+zI)>0$ for $\mathfrak{J}(z)>0$, there exists for $\mathfrak{J}(z)>0$ a bounded $r(\rho)=\lim r(f_m)$ as $m\rightarrow\infty$ over suitable values, and $\rho=\rho(z)=\int d\Phi(x)/(z-x)$, the quadratic form $\xi'\Phi\xi$ a nondecreasing function of x. Application is made to the moment problem of Boas and Pólya. (Received October 13, 1950.)

138. H. F. Weinberger and Alexander Weinstein: *On the connection between the eigenvalues and eigenfunctions of some Sturm-Liouville problems.*

Using methods previously developed by one of the authors (A. Weinstein, *Reissner* *anniversary volume*, Ann. Arbor, Edwards, 1949, pp. 404–414), explicit expressions for the *eigenvalues* and *eigenfunctions* of the Sturm-Liouville problem $u''-r(x)u+\lambda u=0$ with periodic boundary conditions are given in terms of the eigenvalues and eigenfunctions of the same equation with the boundary conditions $u'=0$. A generalization of this process leads to similar results for the fourth order Sturm-Liouville problem $(k(x)u'')''-\lambda\rho(x)u=0$ with any self-adjoint boundary conditions. The eigenvalues and eigenfunctions of this problem with the added condition (*): $u=0$ at a given interior point are obtained in terms of the Green's function (and hence of the eigenvalues and eigenfunctions) of the same problem without the condition (*). This result has application to the theory of continuous beams. (Received October 25, 1950.)

139. J. G. Wendel: *On isomorphism of group algebras.*

The classes of real- and complex-valued integrable functions $x(g)$ on a locally compact group G form Banach algebras, $L(G)$. It is shown in both the real and complex cases that two such algebras $L(G)$ and $L(\Gamma)$ are isometrically and algebraically isomorphic if and only if the groups are topologically isomorphic. The relation between the algebra isomorphism T and the group isomorphism τ is given by the formula $(Tx)(\tau g)=c\chi(g)x(g)$, $x\in L(G)$, $g\in G$, $Tx\in L(\Gamma)$, $\tau g\in\Gamma$, where c is a constant, $\chi(g)$ is a continuous character on G. The proof uses the left translation operators on $L(G)$, and exploits methods due to Kawada (Mathematica Japonicae vol. 1 (1948) pp. 1–5). (Received November 8, 1950.)

140. A. D. Ziebur: *The asymptotic solutions of a certain type of ordinary differential equation of the second order.*

This paper is concerned with the solutions of the differential equation (1) $d^2u/dz^2+[\lambda^2P(z)^2/z+a_2/z^2+a_1/z+\sum_{i=0}^{\infty}q_i(z)/\lambda^i]u=0$, where λ is a large complex parameter, a_1 and a_2 are arbitrary constants, and $P(z)$ and the $q_i(z)$ are analytic in some bounded domain R_z which contains the origin. It is supposed that the above series converges for all λ greater than some fixed value and that $P(z)$ is nonvanishing in R_z. An algorithm for the determination of the coefficients of a formal series in inverse powers of λ, segments of which serve to represent solutions of equation (1) asymptotically, is developed, and the asymptotic forms of certain particular solutions are determined. The Stokes phenomenon is avoided by the admission of Bessel functions into the asymptotic representations. The methods used are essentially those of R. E. Langer (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 461–490). The confluent hypergeometric function $M_{k,m}(z)$ of Whittaker is a solution of a particular equation of the form (1), and by specializing the results obtained for the more general case, asymptotic expansions of $M_{k,m}(z)$ with respect to k are obtained. (Received October 30, 1950.)

141*t*. H. J. Zimmerberg: *On normalizable transformations in Hilbert space.*

The existence theorems for normalizable transformations in Hilbert space developed by Zaanen (Acta Math. vol. 83 (1950) pp. 197–248) are extended and applied to systems of definite Fredholm integral equations previously considered by the author (Duke Math. J. vol. 15 (1948) pp. 371–388). (Received November 9, 1950.)

APPLIED MATHEMATICS

142. O. R. Ainsworth: *Theory of waves from a point source in one of*

two contiguous semi-infinite isotropic elastic media. Preliminary report.

Two isotropic elastic media, one occupying $z>0$ and the other $z<0$, are considered. First a P-wave instantaneous point source with S waves occurring in the reflected and transmitted fields is considered. By symmetry the S waves are all oriented perpendicular to $z=0$. Making the stresses and displacements continuous across $z=0$, evaluate the undetermined functions, and the contour integrals for the displacements are reduced to definite integrals of algebraic quantities by applying a theorem of E. Pinney. The S-wave sources, oriented parallel and perpendicular to $z=0$, are also considered separately, and, again, the displacements are given as definite integrals. (Received November 15, 1950.)

143. Cahit Arf: *On a free boundary problem in elasticity.*

The author considers the free boundary problem stated in his paper (Revue de la Faculté des Sciences de Université d'Istanbul (A) vol. 12 (1947)) for doubly connected domains C. Using the formulas of Muschelisvili and the theory of uniformization he shows that all solutions of this problem can be obtained by means of Weierstrass' functions σ, ζ, θ, corresponding to the periods ω, $i\omega'$ with $\omega>0$, $\omega'>0$. Explicit formulas are given which furnish the totality of the solutions of the problem for the particular case when the stresses σ_x, σ_y, τ are finite everywhere in C and on its boundary. (Received December 14, 1950.)

144*t*. Raoul Bott and R. J. Duffin: *The Wang algebra of networks.*

According to K. T. Wang, the determinant of Kirchhoff's equations is the "product" of any complete set of meshes of the network evaluated by the rules $r_i+r_i=0$ and $r_i \cdot r_i=0$. Each factor is the sum of the resistances r_i in the mesh. More generally, let D be the determinant of the quadratic form $\sum_1^n r_i x_i^2$ where the vector $(x_1, x_2, \cdots, x_n)$ is constrained to lie in a given m-dimensional subspace K of a real vector space V. Then D is a homogeneous multilinear form of degree m with positive coefficients as a function of the parameters r_i (see Bull. Amer. Math. Soc. Abstract 55-3-173). It is shown that D is obtainable from the Grassmann product $(x \cdot y = -y \cdot x)$ for any complete set of m vectors of K. The coefficients of D are equal if and only if the vectors of K having coordinates ± 1 or 0 constitute an m-dimensional vector space under addition mod 2. For networks without transformers the constraint imposed by either one of Kirchhoff's laws has this property. The observation that the Wang algebra is simply the Grassmann algebra mod 2 then extends Wang's rule to its limit. (Received November 15, 1950.)

145. Nathaniel Coburn: *The independent scalars in isotropic turbulence.*

The independent scalars of the correlation tensors in isotropic turbulence are defined by the following three properties: (1) they must be the orthogonal components of the correlation tensors; (2) these scalars must be defined in such a manner that the correlation tensors satisfy the continuity relations identically; (3) the maximum number of such scalars must occur in each correlation tensor. In this paper, we furnish a method for constructing such scalars. The method is based upon the following procedure. First, one determines the maximum number of independent scalars associated with a given correlation tensor. Secondly, by a contraction process, a new tensor is constructed which possesses the maximum number of independent scalars.

By forming the curl of this tensor, one obtains the desired tensor. The method is applied to: (1) the case of isotropic turbulence; (2) the case of axial symmetric turbulence. Finally, it is shown that our method generates all possible correlation tensors which depend linearly upon a maximum number of independent scalars and their derivatives and whose divergence vanishes identically. (Received July 28, 1950.)

146*t*. David Gilbarg: *Unsteady flows with finite closed wake.*

The free boundary problem for unsteady flows differs from the classical Helmholtz free boundary problem in that the boundary condition, flow speed $= |\text{grad}\ \phi| = \text{const.}$ on free streamlines, for the Helmholtz problem, is replaced by the condition (obtained from the nonstationary form of Bernoulli's equation) $\partial\phi/\partial t + |\text{grad}\ \phi|^2 = c(t)$ on free streamlines, where ϕ equals the velocity potential. If unsteady plane flows past polygonal obstacles are considered, it is possible to derive in closed form the entire set of flows with finite closed wake (cavity) behind such obstacles. Standard methods of conformal mapping suffice. In the case of symmetric flow past a flat plate, von Kármán (Annali di Matematica Pura ed Applicata (4) vol. 29 (1949) pp. 247–249) has exhibited an example in which the wake is of constant shape. It is seen that this is a particular example of a class of flows with this property. (Received November 13, 1950.)

147. O. O. Pardee: *An improperly-set boundary value problem in potential theory.*

From the theory of electron flow the following problem is set. Given Cauchy conditions of potential and normal derivatives as continuous functions on an open surface, determine a solution in the large. No exact solution exists in general, but a solution satisfying the boundary conditions to an arbitrary degree of approximation is obtained for two dimensions, with suitable restrictions on the continuity of the boundary functions and on the smoothness of the boundary curve. The solution exists in any finite simply-connected region whose boundary is formed by the given (support) curve and a sufficiently smooth closing curve joining the end points of the support curve. A singularity is allowed at an arbitrarily chosen point on the closing curve. By solving a Neumann problem for this region, the normal derivative condition is satisfied exactly. To satisfy the remaining condition approximately, the region is mapped onto the upper half-plane, the support curve going into a finite portion of the x-axis and the singular point, to infinity. By means of an incomplete Fourier transform the approximate solution is obtained. Some special solutions are obtained for three dimensions. (Received November 13, 1950.)

148*t*. H. E. Salzer: *Equally weighted quadrature formulas over semi-infinite and infinite intervals.*

Equally weighted quadrature formulas of the type $\int_a^b F(x)f(x)dx \doteq K\sum_{i=1}^n f(x_i)$ have been studied by Chebyshev and others for only finite values of a and b. This present study is concerned with formulas of the type (1) $\int_0^\infty e^{-x}f(x)dx \doteq (1/n)\sum_{i=1}^n f(x_i)$ and (2) $\int_{-\infty}^\infty e^{-x^2}f(x)dx \doteq (\pi^{1/2}/n)\sum_{i=1}^n f(x_i)$ which are exact for $f(x)$ equal to I. any polynomial of degree n, or II. any polynomial of degree at most $m<n$ when the points x_i are specified to all lie within the real intervals of integration. For I., the coefficients of the polynomials of which the x_i are the zeros were obtained for $n \leqq 10$, in two different ways, and the values of x_i (not all real for $10 \geqq n > 2$ in (1) and for $10 \geqq n > 3$ in (2)) were calculated to 7D. For II., it was proven that, in (1) $m=3$ requires $n \geqq 7$ and m cannot exceed 3 even for n as high as 14, and in (2) $m=5$ requires $n \geqq 7$, with no possible increase in m even for n as high as 10. The points x_i for a formula in (1) where $n=7$, $m=3$, and a formula in (2) where $n=7$, $m=5$ were calculated to $12D$ and $7D$ respectively, together with checking factors to $10S$ and $7S$ respectively. (Received September 19, 1950.)

149 C. B. Smith: *An application of tensor methods to plates laminated of orthotropic material.*

Plates laminated from orthotropic layers, that is, layers that have two perpendicular axes of elastic symmetry in the plane of the layer, are usually constructed so that the adjacent layers have their axes of symmetry respectively parallel. This paper first discusses the elastic properties of a two-layer plate constructed with the axes of symmetry of the first layer inclined at any angle with the axes of symmetry of the second layer. It is shown that a plate of this type may be considered to be orthotropic. By using the formulas obtained for the two-layer case, plates with any even number of layers may be constructed (the adjacent layers will not necessarily have their axes of symmetry respectively parallel) which again may be considered to be orthotropic. Finally the results of this paper are applied to plates of plywood. (Received November 6, 1950.)

150. C. A. Truesdell: *A measure of vorticity.*

The local rotation in a continuous medium in motion is measured by the vorticity vector $w^{\rightarrow} \equiv \text{curl}\, v^{\rightarrow}$, $v^{\rightarrow}$ being the velocity. Since $w^{\rightarrow}$ is not dimensionless, its numerical magnitude in any given motion has no individual significance, since it may be assigned any magnitude whatever by mere choice of units. In this paper an absolute measure of vorticity, $W \equiv w/(2II_d)^{1/2}$, where II_d is the second principal invariant of the rate of deformation tensor, is introduced and studied. All motions (with the exception of a uniform translation) are thus assigned a non-negative numerical degree of rotationality, varying from $W=0$ for an irrotational motion to $W=\infty$ for a rigid rotation. The class of motions for which $W=1$ is characterized; sufficient, but not necessary, is that the motion be slow. On the basis of the range of W, the Couette flows may be classified into five types, and it turns out that Rayleigh's criterion of stability for these flows (at infinite Reynolds number) may be put into the following form: a necessary and sufficient condition for stability is $W>1$ at all points. It is shown that for the Gerstner waves $W \leqq 1$, the value 1 being attained at the extreme possible amplitude. If $W \leqq 0.1$ it is necessary that the amplitude nowhere exceed 1.5% of the wave length. Thus Gerstner's waves are strongly rotational except for very small amplitudes. (Received October 26, 1950.)

151. A. H. Van Tuyl: *On the coefficients of capacity of two neighboring spheres.*

Closed expressions are found for the coefficients of capacity of two neighboring spheres in the form of definite integrals whose integrands contain elliptic functions. Using these integral representations, series expansions are obtained which converge rapidly when the distance between the spheres is small. The expansions are used in the study of the charge density at the points of former contact when two charged tangent spheres are separated slightly. (Received November 13, 1950.)

152. Alexander Weinstein: *On dislocations in shafts under torsion.*

The superposition of a uniform flow and of a flow generated by a vortex ring in a space of five dimensions yields the solution of the problem of torsion of a nearly cylindrical shaft of revolution with an interior crack bounded by a circular line of radius b. The potential of the vortex line is given in terms of Bessel functions by the formula const. $\int_0^\infty \exp(-|x|t)J_1(yt)J_2(bt)dt$. This formula follows from the results given by A. Weinstein, *Proceedings of the seventh international congress for applied mechanics*, 1948, pp. 108–119. In this formula x and y denote cylindrical coordinates in the meridian plane. (Received October 19, 1950.)

GEOMETRY

153t. John DeCicco: *Commutators of turns, slides, and dilatations upon a surface.*

New proofs and extensions of certain theorems of Kasner concerning infinitesimal lineal element transformations are given. The commutators between the symbols U_1F, U_2F, U_3F of the three one-parameter groups of turns, slides, dilatations upon a surface Σ are $(U_1, U_2)F = U_3F$; $(U_2, U_3)F = KU_1F$; $(U_3, U_1)F = U_2F$, where K is the Gaussian curvature of Σ. The turns and slides form a group of a finite number of parameters if and only if Σ is of constant Gaussian curvature. If $K = \text{const.}$, then this is the whirl group of three parameters, which contains the dilatations as a subgroup. Two infinitesimal contact transformations, neither of which are point transformations, such that they possess the same transversality law and the commutator is a point transformation, are two infinitesimal dilatations of two conformally equivalent surfaces. Two infinitesimal contact transformations possess the same linear involutorial law of transversals if and only if they are infinitesimal dilatations of two conformally equivalent surfaces. (Received November 27, 1950.)

154t. John DeCicco: *Conservative physical systems of curves upon a surface.*

A system S_k of ∞^3 curves in a given field of force upon a surface Σ consists of curves along which a constrained motion is possible such that the pressure P is proportional to the normal component N of the force vector. Thus $P = kN$ where $k(\neq -1)$ is the constant factor of proportionality. A conservative S_k is a physical system in a conservative field of force. The special cases $k = 0, 1, -2, \infty$ define the ∞^3 dynamical trajectories, catenaries, brachistochrones, velocity curves, respectively. The Lagrangian and Hamiltonian equations for a conservative S_k are obtained. A conservative S_k, where $k \neq \infty$, can be separated into ∞^1 natural families $S_k(E)$, according to the value of the energy constant E. The natural family $S_k(E)$ consists of ∞^2 geodesics of a surface, and hence forms the trajectories of the one-parameter group of dilatations of this surface. The infinitesimal dilatation of this group is an extension of the infinitesimal contact transformation of mechanics, studied extensively by Lie, Vessiot, Kasner. Similar results are obtained for a velocity system S_∞. (Received November 27, 1950.)

155. Patrick Du Val: *Surfaces whose canonical systems are hyperelliptic.*

If the irreducible canonical system on a regular algebraic surface of genera $p_g = p_a = p$, $p^{(1)} = n+1$, consists of hyperelliptic curves, it has $n - 2p + 4$ simple base points on a model free from exceptional curves. The canonical model is a double normal rational ruled surface of order $p-2$ on which the base points appear as lines λ, which are constituents of the branch curve (for $p = 3$ a double plane, for $p = 6$, $n = 8$ alternatively a double Veronese surface). There is a residual branch curve f of order

$n+2p$, which has a $(k+2)$ple point at every point common to k of the lines λ, and further triple points at general points on these lines, sufficient on each to reduce its virtual grade to -2. These points are base points of the bicanonical system, so that on the bicanonical model the base points of the canonical system are isolated branch-points at conical nodes. Generally the lines λ are generators of the ruled surface (or belong to a pencil for $n=3$) and the residual branch curve meets each generator in six points and has two triple points in each line λ. There are a small number of exceptional cases for $p \leqq 6$, $n \leqq 9$. (Received November 13, 1950.)

156. A. J. Hoffman: *Cyclic affine planes.* Preliminary report.

Consider a finite affine plane that admits a collineation which leaves one point fixed and permutes the others cyclically. As R. C. Bose has indicated (J. Indian Math. Soc. (2) vol. 6 (1942)), such a "cyclic affine plane" with n points on a line can be constructed from a difference set of n integers (mod n^2-1; $n+1$) (in the notation of Chowla, J. Indian Math. Soc. (2) vol. 9 (1945)), and conversely. The author makes additional comments about cyclic affine planes, in analogy with M. Hall's discussion of cyclic projective planes (Duke Math J. vol. 14 (1947)). The most interesting is that $p|n$ implies p is a "multiplier" of the difference set. This extends a theorem of Chowla (loc. cit.). (Received November 14, 1950.)

157. Benjamin Ernest Mitchell: *Synthetic development of Laguerre's representation of the imaginary point.*

The complex complete quadrangle developed from four imaginary points conjugate in pairs has a real diagonal triangle. The fate of this diagonal triangle when one of the conjugate pairs become the circular points at infinity determines the Laguerre representation. Four of the six sides become isotropics. One becomes the line at infinity. The sixth, real and carrier of the second conjugate point-pair, remains finite. One of the diagonal points goes to infinity. The other two take positions symmetric to the line of the finite conjugate point-pair, and with them constitute a parti-real square. They, when ordered, are the real representation of the finite conjugate imaginary point-pair. (Received November 9, 1950.)

158*t*. Benjamin Ernest Mitchell: *Relation of two lines in the complex plane.*

Since a line in the complex plane is two-dimensional in points, two such lines may be expected to have relationship in excess of their common point. And such they have. Theorem: Any two lines in the complex plane share a common point and a conjugate point-pair. The "pictures" (Study) of the four points involved when the conjugates of the lines are also taken into consideration determine the type of lines involved. The special cases of Kreutzlage, Trapezlage, and Lage isometrich (Study) are considered. (Received November 9, 1950.)

159*t*. Samuel Verblunsky: *On the shortest path through a number of points.*

It is proved that, given ν points $P_i(x_i, y_i)$ in the strip $0 \leqq y \leqq 1$ with $x_1 \leqq x_2 \leqq \cdots \leqq x_\nu$, there is a path whose first point is P_1, whose last point is P_j say, which contains the ν points, and whose length does not exceed $x_j - x_1 + .7\nu$. Here, .7 cannot be replaced by a number less than $3^{-1/2}$. It is deduced that, given n points in a unit square, there is a path containing them of length less than $2+(2.8n)^{1/2}$. (Received November 6, 1950.)

Statistics and Probability

160t. K. L. Chung: *An extension of renewal theory.*

Let $\{X_n\}$, $n=1, 2, \cdots$, be a sequence of independent random variables having the common distribution function $F(x)$ and let $S_n=\sum_1^n X_k$. Suppose that the mean of $F(x)$ is a finite positive number m. If $F(x)$ is of the aperiodic lattice type, then $\lim \sum_1^\infty P(S_n=x)$ is equal to $1/m$ or 0 according as $x\rightarrow+\infty$ or $x\rightarrow-\infty$ on the lattice. If $F(x)$ contains a nonvanishing absolutely continuous part, then $\lim \sum_1^\infty P(x<S_n \leqq x+h)$ is equal to h/m or 0 according as $x\rightarrow+\infty$ or $x\rightarrow-\infty$. These theorems extend theorems of Erdös, Feller, and Pollard and of Doob and Blackwell on renewal theory. The method of proof is new. (Received November 14, 1950.)

161t. F. G. Tricomi: *The probable length of the resultant of n random unit vectors.*

In a problem of biology it was required to determine the average l_n of the length r indicated in the title, which is a continuous random variable in the range $(0, n)$. It was possible for the author to determine rigorously the probability density $\phi_n(r)$ of r, which is given by $(1/2)r\{\tau_{n-1}[(n-r)/2]-\tau_{n-1}[(n+r)/2]\}$ where $\tau_n(x)$ is another probability density already determined by the author in 1932. (Jber. Deutschen Math. Verein. vol. 42, pp. 174–179). Actually $\tau_n(x)$ is the probability density of the sum of n independent random variables, each with the constant probability density *one* in the range $(0, 1)$. Moreover, an asymptotic representation of $\phi_n(r)$ is found (a distribution of the type $Cr^2 \exp[-3r^2/2n]$) and thereafter the asymptotic expression $2(2n/3\pi)^{1/2}+O(n^{-1/2})$ of the average is obtained. (Received November 9, 1950.)

Topology

162t. W. R. Baum: *The group of null-spherods.*

A combinatorial null-spherod in a complex K is a simplicial mapping of an oriented n-sphere into K which has degree 0 in each n-simplex of K. Two null-spherods s_1, s_2 are ν-homotopic if there is a combinatorial deformation of s_1 into s_2 such that all spherods between them are also null-spherods. Using these notions instead of ordinary sphere images and homotopies, define, in analogy to the Hurewicz groups, new groups $\Psi^n(K)$. Similar groups can be defined for continuous mappings and continuous deformations; they are topological invariants of K. $\Psi^n(K)$ is isomorphic to the factor-group $\Gamma^n(K^n)/\mathrm{P}_0^n(K, K^n)$, where $\Gamma^n(K^n)$ and $\mathrm{P}_0^n(K, K^n)$ are subgroups of the Hurewicz homotopy group $\pi_n(K^n)$ (K^n=n-dimensional skeleton of K) introduced by H. Hopf (Comment. Math. Helv. vol. 17 (1945) pp. 307–326). $\Gamma^n(K^n)$ is that subgroup of $\pi_n(K^n)$ whose elements are represented by n-spherods in K^n which are 0 in the sense of homology. The elements of $\pi_n(K^n)$ which are null-homotopic in K form a group $\mathrm{P}^n(K, K^n)$. Then $\mathrm{P}_0^n(K, K^n)$ is the subgroup of $\mathrm{P}^n(K, K^n)$ which is generated by all differences $\rho-\rho'$ where the elements $\rho, \rho'\in\mathrm{P}^n(K, K^n)$ are free-homotopic on K^n. (Received November 13, 1950.)

163t. A. L. Blakers and W. S. Massey: *On the homotopy groups of spheres.*

In this paper the definition of the J-homomorphism of G. W. Whitehead (Ann. of Math. vol. 51 (1950) p. 214) is extended to give a homomorphism $J_1:\pi_p(R_{k-1}, R_{k-2})\rightarrow\pi_{p+k}(S^k; E_+^k, E_-^k)$, and a homomorphism of the homotopy sequence of the pair (R_{k-1}, R_{k-2}) into one of the homotopy sequences of the triad $(S^k; E_+^k, E_-^k)$. A "Hopf homomorphism" $H_1:\pi_n(S^r; E^r_+, E^r_-)\rightarrow\pi_{n+1}(S^{2r})$ is defined for all n, r. $\kappa_*:\pi_p(R_{k-1}, R_{k-2}) \approx \pi_p(S^k)$ is the isomorphism induced by the fiber map of R_{k-1} on S^k and $E^{(k+1)}$ denotes the $(k+1)$-fold suspension. The commutativity relation $H_1J_1(\alpha)=E^{(k+1)}\kappa_*(\alpha)$, $\alpha\in\pi_p(R_{k-1}, R_{k-2})$ is proved. Thus if $\pi_p(S^k)$ has an element whose successive suspensions are nontrivial, then $\pi_{p+k}(S^k; E^k_+, E^k_-)$ is nontrivial. Hence either $\pi_{p+k}(S^k)\neq 0$ or $\pi_{p+k-2}(S^{k-1})\neq 0$. This result overlaps with that of Everett Pitcher (Bull. Amer. Math. Soc. Abstract 56-1-38). (Received August 30, 1950.)

164*t*. A. L. Blakers and W. S. Massey: *Homotopy groups of triads.* III.

Let $(X; A, B)$ be a triad such that every point of X belongs to the interior of A or the interior of B, and A, B, and $A\cap B$ are all arcwise connected. Theorem: If $(A, A\cap B)$ is m-connected, $m\geqq 2$, and $(B, A\cap B)$ is n-connected, $n\geqq 2$, then the triad $(X; A, B)$ is $m+n$ connected. This theorem is still true if $m=1$, or $n=1$, provided $A\cap B$ is simply connected. Corollary 1: Let (X, A) be a triangulable pair such that (X, A) is m-connected, $m\geqq 1$, and A is n-connected, $n>1$. Let $\tilde{X}$ be the space obtained by identifying all of A to a single point, x_0. Let ϕ: $(X, A)\rightarrow(\tilde{X}, x_0)$ denote the identification map, and ϕ_p: $\pi_p(X, A)\rightarrow\pi_p(\tilde{X}, x_0)$ the homomorphisms induced by ϕ. Then ϕ_p is an isomorphism onto for $p\leqq m+n$, and is a homomorphism onto for $p=m+n+1$. Corollary 2: Let X be a simplicial complex and A and B subcomplexes such that $X=A\cup B$, and A, B, and $A\cap B$ are all connected. Assume that $(A, A\cap B)$ is m-connected, $m\geqq 1$, $(B, A\cap B)$ is n-connected, $n\geqq 1$, and $A\cap B$ is r-connected, $r>1$. Then the injections $\pi_p(A, A\cap B)\rightarrow\pi_p(X, B)$ and $\pi_q(B, A\cap B)\rightarrow\pi_q(X, A)$ are isomorphisms into for $p\leqq m+r$ and $q\leqq n+r$ respectively. (Received November 13, 1950.)

165*t*. A. L. Blakers and W. S. Massey: *Generalized Whitehead products.* II.

Let $(X; A, B)$ be a triad. A new product is defined which associates with elements $\alpha\in\pi_p(B, A\cap B)$ and $\beta\in\pi_q(A, A\cap B)$ an element $[\alpha, \beta]$ of $\pi_r(X; A, B)$, $r=p+q-1$. This product is bilinear if p and q are greater than 2, and satisfies the commutation rule $[\alpha, \beta]\ (-1)^{pq}[\beta, \alpha]$. If ∂_+: $\pi_r(X; A, B)\rightarrow\pi_{r-1}(A, A\cap B)$ and ∂_-: $\pi_r(X; A, B)\rightarrow\pi_{r-1}(B, A\cap B)$ denote the boundary operators of the triad $(X; A, B)$, and ∂_1: $\pi_q(A, A\cap B)\rightarrow\pi_{q-1}(A\cap B)$, ∂_2: $\pi_p(B, A\cap B)\rightarrow\pi_{p-1}(A\cap B)$ are boundary operators of the pairs $(A, A\cap B)$ and $(B, A\cap B)$, then $\partial_+[\alpha, \beta]=-[\partial_2\alpha, \beta]$, and $\partial_-[\alpha, \beta]=(-1)^{p-1}[\alpha, \partial_1\beta]$. Various other relations between this new product and the products previously defined by J. H. C. Whitehead and the authors are established. (Received November 13, 1950.)

166*t*. A. L. Blakers and W. S. Massey: *On the groups* $\pi_p(S^n; E^n_+, E^n_-)$.

Let S^n be an n-sphere, E^n_+ and E^n_- the upper and lower caps, and $S^{n-1}=E^n_+\cap E^n_-$ the equator. Choose a generator $\iota\in\pi_n(E^n_-, S^{n-1})$. Define a homomorphism θ_p: $\pi_r(E^n_-, S^{n-1})\rightarrow\pi_p(S^n; E^n_+, E^n_-)$, where $r=p-n+1$, by $\theta_p(\alpha)=[\alpha, \iota]$ for any $\alpha\in\pi_r(E^n_-, S^{n-1})$. Here $[\alpha, \iota]$ denotes the generalized Whitehead product defined in the previous abstract. Let ϕ: $(S^n; E^n_+, E^n_-)\rightarrow(S^n_1\vee S^n_2; S^n_1, S^n_2)$ be a map which identifies all of the equator S^{n-1} to a single point, $x_0\in S^n_1\vee S^n_2$. The space $S^n_1\vee S^n_2$ consists of two spheres with a single point in common. Let ϕ_p: $\pi_p(S^n; E^n_+, E^n_-)\rightarrow\pi_p(S^n_1\vee S^n_2; S^n_1, S^n_2)$ denote the homomorphism induced by ϕ. Theorem: If $p<3n-3$, then $\phi_p\theta_p$ is an isomorphism onto; if $p=3n-3$, then $\phi_p\theta_p$ is a homomorphism onto. Corollary 1: If $p<3n-3$, then $\pi_p(S^n; E^n_+, E^n_-)$ has a direct summand isomorphic to $\pi_p(S^{2n-1})$. Corollary 2: If $\pi_{3n-3}(S^{2n-1})\neq 0$, then $\pi_{3n-3}(S^n; E^n_+, E^n_-)\neq 0$. Corollary 3: The following groups are not zero: $\pi_{2n}(S^n; E^n_+, E^n_-)$, $n \geqq 3$; $\pi_{2n+1}(S^n; E^n_+, E^n_-)$, $n \geqq 4$; $\pi_{2n+2}(S^n; E^n_+, E^n_-)$, $n \geqq 5$. (Received November 13, 1950.)

167. A. L. Blakers: *Geometric proof of an algebraic theorem of Eilenberg-MacLane.*

S. Eilenberg and S. MacLane (Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 443) have proved the cohomology equivalent of the following algebraic theorem: *The suspension homomorphism* S: $H_q(K(\pi, n), G) \to H_{q+1}(K(\pi, n+1), G)$ *is an isomorphism onto for* $q < 2n$, *and is onto for* $q = 2n$. In the present paper this result is deduced as a corollary of a theorem about the homotopy groups of certain triads, and consequently the proof is rather geometric. Let K be a complex with $\pi_i(K) = 0$, $i < n$, $n < i < m$, $m > 2n$, $n > 1$, and with $\pi_n(K) = \pi$. (See J. H. C. Whitehead, Ann. of Math. vol. 50 (1949) p. 261). Let $\tilde{K}$ denote the join of K with a pair of points. Then it follows from a general theorem about triad homotopy groups (Bull. Amer. Math. Soc. Abstract 57-2-162) that $\pi_i(\tilde{K}) = 0$, $i < n+1$, $n+1 < i \leqq 2n$, while $\pi_{n+1}(\tilde{K}) \approx \pi$. It now follows (see Eilenberg-MacLane, Ann. of Math. vol. 46 (1945) p. 508) that the homomorphism k: $H_q(K, G) \to H_q(K(\pi, n), G)$ is an isomorphism onto for $q \leqq m$, while the corresponding homomorphism $\tilde{k}$: $H_q(\tilde{K}, G) \to H_q(K(\pi, m+1), G)$ is an isomorphism onto for $q \leqq 2n$ and is onto when $q = 2n+1$. Moreover, the homology suspension homomorphism h: $H_q(K, G) \to H_{q+1}(\tilde{K}, G)$ is an isomorphism onto for all dimensions, and the theorem now follows from the commutativity relation $\tilde{k}h = Sk$. (Received November 13, 1950.)

168t. Raoul Bott: *A set of new combinatorial invariants for cell complexes.* Preliminary report.

Let K be a finite n-dimensional cell complex, with the n cells $\sigma_i\{i=1, 2, \cdots, \lambda\}$. If S is a subset of the first λ integers, denote by K_s the complex derived from K by deleting all n cells σ_i with $i \notin S$. $\beta^\alpha(K_s)$ shall stand for the αth Betti number of K_s, $|s|$ shall be the number of elements in the set S. Set $p_\mu = \sum_{\beta^n(K_s)=\mu}(-)^{|s|}$; $q_\mu = \sum_{\beta^n(K_s)=\mu}((-1)^{1/2})^{|s|-\beta^{n-1}(K_s)}$. Then p_μ^2, q_μ^4 are invariants of K under subdivision; however they are not homotopy invariants. (Received November 16, 1950.)

169. K. T. Chen: *On isotopic invariants of links.*

An n-link L is the union of n disjoint simple closed curves in a euclidean 3-space E. Each of the closed curves is called a component of L. A link with polygonal components is said to be polygonal. An isotopy ϕ: $L \times I \to E$ is regular if each link $\phi_{t_0}(L)$, $0 \leqq t_0 \leqq 1$, is polygonal with respect to some coordinate system of E, and, given $\epsilon > 0$, there is $\delta > 0$ such that, for each t, $|t - t_0| < \delta$, there is a homeomorphism τ: $E \to E$ with $\tau\phi_t(L)$ being polygonal and $|\tau\phi_t(p) - \phi_{t_0}(p)| < \epsilon$, $p \in L$, with respect to the same coordinate system. Let G and G' be the fundamental groups of $E - \phi_0(L)$ and $E - \phi_1(L)$ respectively. Then G/G_d and G'/G'_d are isomorphic ($d = 1, 2, \cdots$), where G_d and G'_d are the dth lower central commutator subgroups of G and G' respectively. However groups G and G' are, in general, not isomorphic. Under a regular isotopy, each finitely generated abelian group G_d/G_{d+1} is therefore invariant and yields numerical invariants. (Received November 13, 1950.)

170t. D. O. Ellis: *On separable metric spaces.*

Let M be a separable metric space. Let M^* be a subset dense in M which is at most countable. We assume that M^* is fixed as to sequentialization and let its elements be $q_1, q_2, \cdots$. It is easily shown (and well known) that M^* forms a metric base for M. We attach to each point of M as *metric coördinates* the distances $pq_1, pq_2, \cdots$. The results of this note are: 1. *The mapping of M into Fréchet's space E_ω defined by corresponding coördinates is biuniform and continuous and is a homeomorphism when M is compact;* 2. *The coördinate convergence in M is equivalent to the original metric convergence;* 3. *If p and q are points of M with coördinates $\{x_i\}$ and $\{y_i\}$, respectively, then* $pq = \text{g.l.b.}_i(x_i + y_i) = \text{l.u.b.}_i |x_i - y_i|$. Two unsolved problems are also proposed in the note. (Received August 11, 1950.)

171. D. O. Ellis: *On strict metricity in metric spaces.*

A *γ-metric space* is a semimetric space satisfying (*) $xy + yz \geqq \gamma xz$ for $x \neq y \neq z \neq x$. If M is a γ-metric space for $\gamma > 1$, M is called a *more than metric space*. A metric space not satisfying (*) for any $\gamma > 1$ is called *strictly metric*. If M is a metric space, the *curvature ray*, $K(M)$, of M is the set of all $\gamma \in [0, \infty]$ satisfying $\gamma = \min ((xy+yz)/xz, (xy+xz)/yz, (xz+yx)/xy)$ for $x, y, z \in M$, $x \neq y \neq z \neq x$. The principal results of the present paper are: 1. *Every more than metric space is discrete in its metric topology;* 2. *For a metric space of more than two distinct points not containing any linear triples to be strictly metric, it is necessary and sufficient that $1 \in K'(M)$;* 3. *For a metric space to be strictly metric it is sufficient that either the space contain a linear triple or that the space contain at least one accumulation point in its metric topology.* (Received August 3, 1950.)

172t. E. E. Floyd: *Some retraction properties of the orbit decomposition spaces of periodic maps.*

Let X be a finite-dimensional locally compact metric space and let T be a periodic map on X. Let X^* denote the orbit decomposition space of X, T. Then if X is an absolute neighborhood retract (absolute retract), so also is X^*. (Received November 13, 1950.)

173t. E. E. Floyd: *Examples of fixed point sets of periodic maps.*

Examples are constructed to prove that if G is any nontrivial abelian group, there exists a finite complex X and a simplicial map T of X onto X of some prime period such that X is homologically trivial over G while the fixed point set L of T is not homologically trivial over X. Similarly there exists an X and a T such that X is a homological sphere over G while L is not. These results answer a question of P. A. Smith (see page 372 of his appendix in Lefschetz, *Algebraic topology*). (Received November 13, 1950.)

174. F. B. Jones: *Certain homogeneous, unicoherent, indecomposable continua.*

It is shown in a metric space that: *Every homogeneous, unicoherent, compact continuum is indecomposable.* It follows that in the plane: *Every homogeneous, bounded continuum which does not separate the plane is indecomposable.* Is every such continuum a pseudo-arc? (Received November 22, 1950.)

175t. Saunders MacLane and J. H. C. Whitehead: *The complete 3-homotopy type of a complex.*

In a recent note in Proc. Nat. Acad. Sci. U.S.A. (vol. 36 (1950) pp. 41–47) the authors have shown that the three-type of a CW complex can be described by the homotopy groups Π_1 and Π_2 of K, the operators of Π_1 on Π_2, and the homotopy obstruction cohomology class k in $H^3(\Pi_1, \Pi_2)$. We now consider the corresponding problem of the classification of maps of one such complex K into another. By making special choices in the construction of K, we obtain a specific cocycle l in $Z^3(\Pi_1, \Pi_2)$. The 3-homotopy classification of maps can then be formulated combinationally in terms of the actions of these maps on the cocycles l. (Received November 14, 1950.)

176t. Deane Montgomery and Leo Zippin: *Four-dimensional groups.*

This paper shows that every four-dimensional locally compact separable metric locally connected group is a Lie group. The proof makes use of the existence of a closed two-dimensional subgroup which has recently been shown by the authors. (Received November 13, 1950.)

177. G. D. Mostow: *On the topology of noncompact Klein spaces.* I.

A new representation theorem for positive definite symmetric matrices is proved and with its help a simplified model for certain Klein spaces is obtained. *Theorem.* Let S denote the space of all real symmetric $n\times n$ matrices, let E be a linear subspace of S, let $F=\{X \mid X\in S, \text{Tr } XY=0, \text{ all } Y\in E\}$, and let P, E, F denote exp S, exp E, exp F respectively. Assume that for all x, $y\in E$, $xyx\in E$. Then for any $p\in P$, $p=efe$ uniquely and continuously with $e\in E$ and $f\in F$. Let K be a compact Lie group, K' a closed subgroup of K, F a euclidean space, and λ a continuous homomorphism of K' into the orthogonal group on F. Let $f^k=\lambda(k)(f)$ if $k\in K'$ and $f\in F$. If $(x, f)\in K\times F$, define $C_{x,f}=\{(xk, f^k) \mid \text{all } k\in K'\}$. *Definition.* (K, K', F, λ) is the base space obtained upon taking the $C_{x,f}$ as fibers in $K\times F$. *Theorem.* Any Klein space whose isotropy subgroup is connected and semisimple is topologically (K, K', F, λ) where K, K', F, λ are suitably determined. If λ can be extended to a continuous mapping of K into the group of auto-homeomorphisms of F such that $\lambda(xk)=\lambda(x)\cdot k$, $x\in K$, $k\in K'$, then $(K, K', F, \lambda)\cong K/K'\times F$. This is the case if K/K' admits a continuous cross-section n K. (Received December 20, 1950.)

178t. R. L. Taylor: *The extendibility of a crossed module.*

J. H. C. Whitehead has defined the concept of a *crossed module* [Bull. Amer. Math. Soc. vol. 55 (1949) pp. 454–496], and has shown that crossed modules appear in the homotopy systems of complexes. MacLane and Whitehead have defined the *obstruction* of a crossed module [Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 41–48] and used it to characterize the 3-type of a complex. The present paper is the first in a series showing the role played by crossed modules in the theory of universal covering groups. The concept of an *extension* of a crossed module is defined, and it is proved that a crossed module is extendible if and only if its obstruction vanishes. Other necessary and sufficient conditions for the extendibility of special kinds of crossed modules are given, in terms of the cohomology theory of groups. (Received November 16, 1950.)

179t. R. L. Taylor: *Universal coverings of a nonconnected group.*

A *covering* of the topological group G is a topological group G^* together with a homomorphism p of G^* onto G such that for each arcwise component M of G, $p^{-1}(M)$ is arcwise connected, and $p/p^{-1}(M)$ is a fibre mapping with discrete fibre. Such a covering is *universal* if the arcwise components of G^* are simply connected. A nonconnected topological group G may fail to admit a universal covering, even if G is a compact 1-dimensional Lie group. Let G be a topological group such that N, the arcwise component of the identity, has sufficient local properties for the existence of a universal covering N^* of N. It is proved that N^* forms a crossed module over G, and that the universal coverings of G are in a natural 1-1 correspondence with the extensions of the crossed module (N^*, G). Hence G admits a universal covering if and only if the obstruction of the crossed module (N^*, G) vanishes (R. L. Taylor, *The extendibility of a crossed module*, Bull. Amer. Math. Soc. Abstract 57-2-178). This obstruction may be regarded as an element of $H^3[G/N, \pi_1(N)]$; it is an invariant of G. The results are generalized in a subsequent paper (R. L. Taylor, *Universal coverings of a ringoid*, Bull. Amer. Math. Soc. Abstract 57-2-180). (Received November 16, 1950.)

180t. R. L. Taylor: *Universal coverings of a ringoid.*

A *ringoid* is an additive group P (not necessarily abelian) together with an operation of multiplication in P, satisfying the axioms for a groupoid, such that if the products ab and cd are defined, then the product $(a+c)(b+d)$ is defined, and $(a+c)\cdot(b+d)=ab+cd$. An example is the set of all strict homotopy classes of paths in an additive topological group ("strict" meaning "end points fixed"). It is proved that the concept of a ringoid is equivalent to that of an additive group P together with a subgroup E and an ordered pair of retraction-homomorphisms L, R of P onto E, such that every element of $L^{-1}(0)$ commutes with every element of $R^{-1}(0)$. From every ringoid P a crossed module $M(P)$ is constructed in a natural way. The concepts of *covering* and *universal covering* of a ringoid are defined algebraically, and it is proved that the universal coverings of a ringoid P are in a natural 1-1 correspondence with the extensions of the crossed module $M(P)$. This permits a generalization of the results of a previous paper (R. L. Taylor, *Universal coverings of a nonconnected group*, Bull. Amer. Math. Soc. Abstract 57-2-179) to the case of a topological group G of which no local properties are assumed. (Received November 16, 1950.)

181t. R. L. Taylor: *The classification of fibre bundles.* I.

Let B be a topological space, and G a topological group. Let B be coverable by two closed sets M_1 and M_2, each contractible in B, whose intersection M_{12} is contractible in M_1 and in M_2, and is a neighborhood retract of B. Assume also that every homotopically trivial map defined on M_{12} is extendible to B (for example, let B be metric separable and M_{12} a compact ANR). Let N be the arcwise component of the identity in G, and let M'_{12} be an arbitrary arcwise component of M_{12}. Theorem: The abstract fibre bundles over B in G are in 1-1 correspondence with the conjugacy classes (under G/N) of all homotopy classes of maps of the pair (M_{12}, M'_{12}) into the pair (G, N). Feldbau's classification of bundles over a sphere [C. R. Acad. Sci. Paris vol. 208 (1939) pp. 1621–1623] and Steenrod's classification of bundles over a 1-complex [lectures, University of Michigan, 1947] are special cases of this theorem. (The suspension of any compactum may be covered in the manner described above.) Some corollaries on the homotopy classification of mappings are given. (Received November 16, 1950.)

182t. R. L. Taylor: *The classification of fibre bundles.* II.

This paper develops a method of classifying fibre bundles without using universal bundles. The method provides new results, which may be combined with the universal bundle theory to yield corollaries on the homotopy classification of mappings. Let B be a topological space, and G a topological group. A covering C of B is *admissible* if it satisfies certain conditions resembling those of the previous paper, but without restriction on the number of sets used. A *homotopy cocycle* on C in G consists of a homotopy class of maps of $M_i \cap M_j$ into G for each ordered pair M_i, M_j of sets in C;

homotopies must be stationary throughout the intersection of any 3 distinct sets, and an algebraic condition of the form $H_{ij}H_{jk}=H_{ik}$ is required. *Equivalence* of homotopy cocycles is defined by a kind of conjugation by classes of homotopically trivial maps. Theorem: If C is an admissible covering of B, the abstract bundles over B in G are in 1-1 correspondence with the equivalence classes of homotopy cocycles on C in G. The main theorem of the preceding paper is an easy corollary. (Received November 16, 1950.)

183*t*. R. L. Taylor: *The classification of fibre bundles.* III.

The technique developed in the previous paper is applied here to the classification of all fibre bundles over a compact 2-manifold B, in an arbitrary topological group G. Complete results are obtained in terms of the ringoid $P(G)$ of all strict homotopy classes of paths in G. If G admits an algebraic universal covering G^* (R. L. Taylor, *Universal coverings of a ringoid*, Bull. Amer. Math. Soc. Abstract 57-2-180), the results can be stated in terms of G^* instead of $P(G)$. Special cases: (1) If the center of G meets every arcwise component of G, the bundles are in 1-1 correspondence with the direct sum $\pi_1(N)+\mathrm{Hom}\ [\pi_1(B), G/N]$ or $\pi_1(N)/2\pi_1(N)+\mathrm{Hom}\ [\pi_1(B), G/N]$ according as B is orientable or not. (N is the arcwise component of the identity in G; a slight additional hypothesis is needed for the nonorientable case.) (2) Let O^n be the group of all orthogonal transformations of S^n; let p=genus of $B(p=2-\chi$ if B is nonorientable). Then the bundles fall into 2^{2p} or 2^p "types," according as B is orientable or not. If $n>1$, each type includes exactly two abstract bundles. If $n=1$, the same holds except for one type, whose members are in 1-1 correspondence with the non-negative integers. (Received November 16, 1950.)

184*t*. Mrs. B. R. Vanderslice and D. W. Hall: *Chromatic polynomials.* Preliminary report.

G. D. Birkhoff and D. C. Lewis, Jr. have made a conjecture (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 355–451, in particular pp. 411–413) which, if true, will establish the four color conjecture in the affirmative. This paper is a first step towards the experimental verification of the Birkhoff-Lewis conjecture for the special case of the truncated icosahedron. This particular map is of interest since it is the simplest map with triple vertices containing no region of fewer than five sides and having the property that its pentagons are mutually isolated. In the work towards the computation of the chromatic polynomial for the truncated isocahedron, the polynomials have already been obtained for several hundred new maps, quite a few of which contain more than twenty regions. (Received November 14, 1950.)

185. A. D. Wallace: *Rim compact spaces.*

The space X is rim compact at $A\subset X$ if each neighborhood of A contains a neighborhood of A with a compact boundary. We term X rim compact (RC) if it is RC at each of its points. Let now X be any RCH space. It is shown, among other things, that: (i) X is RC at any of its closed sets with a compact boundary; (ii) X is completely regular; (iii) any quasicomponent of X with a compact boundary is also a component. We remark that, under different names, RC spaces have been investigated by Freudenthal, Jones, Zippin, and others. (Received November 13, 1950.)

W. M. WHYBURN,
Associate Secretary

Appendix

Excerpts from Report of Treasurer

December 20, 1950

To the Board of Trustees of the
American Mathematical Society

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1950, with certain pertinent comments.

Investment Portfolio and Income

On November 30, 1950, the market value of securities held for Invested Funds exceeded book value by $36,000 while the market value of securities held for Current Funds was less than book value by just over $100. On the whole portfolio, the market value therefore exceeds book value by $35,900. Reserves held in accounts "Reserve for Investment Losses" ($4,386) and "Profit on Sales of Securities" ($18,168) may still be considered adequate protection against contingent depreciation in market value.

The following is a summary of the changes in the security holdings made during the year.

Acquired

$16,100	U.S. Treasury 2½s 1957–52
100	shares Consumers Power Co. common
50	shares Goodyear Tire and Rubber Co. cum. pref. $5.00
100	shares Niagara Mohawk Power Co. common
60	shares U. S. Steel Corp. 7% cum. pref.
10	shares U. S. Smelting, Refining and Mining Co. 7% cum. pref.

Sold

$ 7,000	U. S. Treasury 2s 1954–52
30	Aetna Insurance Co. common
100	rights Consumers Power Co. common
100	shares Crane Co. common
100	shares International Nickel Co. of Canada common
15	shares National Biscuit Co. 7% cum. pref.
200	shares National Dairy common
100	shares Owens Illinois Glass Co. common

53 shares Texas Company common
50 shares Union Oil Company of California $3.75 cum. pref.

Redeemed

$10,000 U. S. Savings Bonds

The investment portfolio, valued at market November 30, 1950, now includes Government bonds 33 per cent, other bonds 8 per cent, preferred stock 18 per cent, and common stock 41 per cent.

All savings accounts of the Society have been closed and the money used to meet current obligations. These accounts represented accumulated unspent income from earlier years.

Income received during the year from investment of Current Funds amounted to $1,837. This represents a return of 2.3 per cent computed on average book value of investments. Income on Invested Funds amounted to $10,859, representing a return of 5.6 per cent. Total investment income from all sources was $12,696, representing a return of over 4.3 per cent.

Income from the Henderson Estate was $6,025; in 1949 it was $5,200. It may be noted that the Society's share in the income of this estate has now increased by $500 annually, as one of the other beneficiaries died during the year.

During the year the Society received the principal of the Helen A. Merrill Bequest, securities valued at $650. By action of the Trustees, this has been set up as a special fund. There are probably a number of members of the Society who are in a position to include the Society among the beneficiaries of their estates, either by bequest of a specific sum or by ultimately turning over to the Society funds which have been used to provide a life trust for a relative. The importance of such bequests, even in small amounts, cannot be overestimated. The officers of the Society will be glad to furnish information as to the legal form for such bequests.

The General Financial Picture

A brief summary of the scope of the Society's program may be of interest. The annual general receipts of the Society are just short of $65,000. The expenses of maintaining the Society's offices, holding meetings, and generally conducting its affairs come to approximately $32,000. The *Bulletin*, *Proceedings*, and *Transactions* are regarded as general publications of the Society, whereas all other publications are charged against special publication funds set up for the purpose. The annual cost of the *Bulletin*, *Proceedings*, and *Transac-*

tions is about $48,000. Clearly, therefore, the Society is annually depleting its reserves to a serious extent. Either additional sources of income must be found, or the scope of the Society's activity must be curtailed.

The recent increase in dues will probably make it possible to support the general Society activities, including publication of the *Bulletin* through general receipts. The *Proceedings*, though sent to each member, are—like the *Transactions*—more specialized in character than the *Bulletin* and it would seem logical that special appropriations, which might in whole or in part come from general funds, rather than unspecialized income, be used for their support.

Apparently, the assets of the Society have increased by more than $17,000. The increase is fictitious. The International Congress account has increased by more than $27,000—money taken in at the Congress which is being held by the Society to meet the further expenses of the Congress, including the publication of the *Proceedings*. Furthermore, the Society has not yet been billed for all of the publications properly chargeable to the business of fiscal 1950. It is estimated that these bills will amount to $10,000.

Except for the use of $10,840 in its savings accounts and $10,000 from redeemed U. S. Savings Bonds the Society was not compelled this year to liquidate any of the investments of current funds. In effect we have borrowed approximately $12,000 from the International Congress Fund to obtain cash to meet the ordinary expenses of the Society.

The situation remains critical and the Treasurer cannot but urge the Trustees to give full and earnest consideration to the difficulties faced by the Society because of its lack of sufficient income to discharge the obligations placed upon it by the growth and vitality of American mathematics.

Respectfully submitted,
ALBERT E. MEDER, JR.,
Treasurer

BALANCE SHEET

Assets

	November 30, 1950	November 30, 1949
CURRENT FUNDS:		
Cash	$ 33,332.51	$ 12,057.26
Due from Invested Funds		31.58
Account Receivable from United States Government	5,842.66	3,440.49
Investments	67,328.66	76,292.42
	$106,503.83	$ 91,821.75
INVESTED FUNDS:		
Cash	$ 629.83	$
Investments	194,242.32	192,153.46
	$194,872.15	$192,153.46
TOTAL ASSETS	$301,375.98	$283,975.21

Liabilities

CURRENT FUNDS:		
Publications	$ 26,348.27	$ 23,928.48
International Congress	45,531.48	18,089.17
Policy Committee	(648.55)	(160.70)
Prize Funds and Other Special Funds Accumulated Income	3,524.71	9,553.82
Sinking Fund	54.04	1,244.97
Profit on Sales of Securities	2,049.85	2,032.98
Miscellaneous	1,644.02	986.90
Surplus	28,000.01	36,146.13
	$106,503.83	$ 91,821.75
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 71,000.00
Prize Funds and Other Special Funds	33,683.22	33,033.22
Life Membership and Subscription Reserve	2,635.08	2,851.70
Mathematical Reviews	65,000.00	65,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	18,167.96	15,851.07
Due to Current Funds		31.58
	$194,872.15	$192,153.46
TOTAL LIABILITIES	$301,375.98	$283,975.21

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1949–1950

	1950	1949
GENERAL RECEIPTS		
Dues—Ordinary Memberships	$ 33,010	$ 30,017
Dues—Contributing Memberships	788	746
Dues—Institutional Memberships	15,224	9,408
Initiation Fees	1,442	1,091
Investment Income	13,171	12,526
Miscellaneous	1,517	122
	$ 65,152	$ 53,910
PUBLICATIONS RECEIPTS	93,507[1]	73,775[2]
OTHER RECEIPTS	5,545	1,884
TOTAL RECEIPTS	$164,204	$129,569
GENERAL EXPENSE	$ 37,915	$ 25,507
COST OF PUBLICATIONS AND SALES	135,712	118,837
OTHER EXPENSE	100	533
TOTAL EXPENSES	$173,727	$144,877
EXCESS OF EXPENSES OVER RECEIPTS	$ 9,523	$ 15,308

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[1] This includes $21,500 received under a contract with the Office of Air Research for support of Mathematical Reviews.

[2] This includes $7,140 received under a contract with the Office of Naval Research for support of Mathematical Reviews.

BOOK REVIEWS

Leçons sur le calcul des coefficients d'une séries trigonométrique. Quatrieme Partie. *Les totalisations. Solution du problème de Fourier.* Premier Fascicule: *Les totalisations.* Deuxième Fascicule: *Appendices et tables générales.* By Arnaud Denjoy. Paris, Gauthier-Villars, 1949. Fasc. I, pp. 327–481, 1500 fr.; Fasc. II, pp. 483–715, 2200 fr.

These two books constitute the concluding part of a treatise, in four parts, devoted to the solution of one of the fundamental problems of trigonometrical series. The problem is to express trigonometrical series, that are everywhere convergent, in the Fourier form. This problem was successfully attacked in 1921 by Denjoy in five notes published in the Cômptes Rendus de l'Academie des Sciences de Paris. In 1938, he gave a course of lectures on this topic at Harvard University, and these lectures are written up now in the form of a book.

The background to the problem is furnished by the uniqueness theory of trigonometrical series on the one hand and by the theory of integration of functions of a real variable on the other. If two trigonometrical series converge everywhere to the same sum, the series are identical; that is, the corresponding coefficients in the two series are equal. This is a theorem of Cantor (1872) which guarantees that if a function $f(x)$ has a trigonometric development

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

which converges everywhere, then this development is unique. If, in addition, the function $f(x)$ is Lebesgue integrable, then it is a Fourier development; in other words, the coefficients a_n, b_n are Fourier coefficients, related to $f(x)$ by means of the classical formulas of Fourier. This is a theorem of de la Vallée-Poussin (1912). The problem is to uphold this in the general case, without the restrictive assumption of Lebesgue integrability on $f(x)$. Denjoy solved this problem by evolving an elaborate process of *totalisation symmetrique à deux degrés* for the calculation of the coefficients. This process may be looked upon as a generalization of the notion of integration, and termed "trigonometric integration." It is more powerful than what is commonly known as the ordinary Denjoy integral which Denjoy had invented for the analogous, but somewhat less complicated, problem of proving that a function which is differentiable everywhere is the "integral" of its derivative. Interesting contributions have been made to the theory of trigonometric integration, such as Verblunsky's approximate Denjoy integral and Burkill's Cesaro-Perron integral, in the intervening years between the author's original researches (1921) and the publication of this book, though there is no mention of them here.

Starting from de la Vallée-Poussin's result, the treatise covers all the ground that is necessary to reach the final result of the author. The fourth part, under review, comprises Chapters VII to IX, with a few Appendices at the end. In Chapter VII the author develops the theory of the Denjoy integral (*totalisation simple*) with the aid of a new notion of totalisation of series, presented here for the first time. In Chapter VIII he treats Stieltjes integrals relative to general measures. In Chapter IX he presents a complete solution of the main problem, explained above, and illustrates with examples the impossibility of relaxing any of the conditions formulated in his definition of the "trigonometric integral." The Appendices deal mainly with the special Denjoy integral using majorants and minorants, besides containing a rather severe criticism of Perron's definition of integral on the ground that it is nonconstructive.

In contrast with the earlier notes of the author which were brief, the present work is very elaborate and even diffuse. It bears witness to the highly ingenious and original mind of the author. To appreciate it, one has to read the book in full; no part of it can be detached from the rest. This, however, is not an unmixed blessing. Though the title of the book sounds very special, its content is not narrow; it is really a survey, in the singular fashion of the author, of the various sectors of the theory of functions of a real variable that surround the very difficult problem of the calculation of coefficients of trigonometrical series. The work that is embodied in this book has already had considerable influence on that of other mathematicians; in this sense, one regrets that the book did not appear sooner. Anyone interested in the theory of non-absolutely convergent integrals would consider the book valuable.

K. CHANDRASEKHARAN

Lectures on classical differential geometry. By D. J. Struik. Cambridge, Addison-Wesley, 1950. 8+221 pp. $6.00.

There is many a good reason to welcome this new book on differential geometry.

First of all, there is the very fact that it is devoted to *classical* differential geometry, that is, to the wealth of ideas from which all further developments have been derived. The comprehensive historical, biographical, and bibliographical information will convey to the student the enthusiasm of the pioneers in this field of research, put him in contact with the original ideas in their bare form (not overshadowed by symbolism, however powerful), and give him access to the more extensive treatises on this subject.

Another good reason is that the visual content of "geometry" is emphasized, also by a large use of appropriate illustrations. The geometric point of view, so helpful also in researches on more abstract spaces, pervades and illuminates these lectures.

The analytic apparatus (Gibbs vector notation) is appropriate to the subject and its use is always subordinate to the development of the geometric ideas.

A large collection of problems, some for class use and some serving as hints for advanced research, enriches the volume.

The presentation of ideas and of proofs and the typographical presentation are excellent.

E. Bompiani

Grundzüge der Galois'schen Theorie. By N. Tschebotarow. Trans. and ed. by H. Schwerdtfeger. Groningen, Noordhoff, 1950. 16 +432 pp. f 20.00.

The present (German) edition is a reworked and annotated version of the original (Russian) edition, the date of which is unknown to the reviewer but is certainly prior to 1940; it is based on a series of lectures given at the University of Kasan. The author believes that the modern or abstract recasting of algebra is responsible for increased insight and spectacular advances, but that the abstract approach is not suited to the young student, who should have a thorough foundation of concrete mathematics on which subsequently to lay the abstractions and generalizations. In this book, which is intended to meet the needs of such students, he endeavors to preserve the spirit of classical concrete galois theory and at the same time to introduce such notions as will facilitate the readers' ability to assimilate and appreciate the abstract theory, presumably at some later time.

Chapter I (110 pages) is devoted to group theory, with emphasis on finite groups and, especially, permutation groups, and includes an appendix on A. Loewy's "Mischgruppen" (abstract system of which a realization is the set of all isomorphisms of a field extension of a field K into an algebraic closure of K, with multiplication only sometimes defined), and an appendix containing some remarks on a theorem of Bertrand concerning the symmetric group. Chapter II (76 pages) treats polynomials and fields (with emphasis on number fields, but with occasional excursions into abstract fields), contains considerable material on irreducibility criteria and introduces the concepts of the root (or splitting) field of a polynomial and normal field extension. Chapter III (91 pages) concerns the galois group of a polynomial (defined as a permutation group), gives the fundamental theorem of galois theory, and contains an appendix on Loewy's treatment of non-normal field extensions. Chapter IV (90 pages) discusses the connection between solvable groups and solvability by radicals, with numerous classical applications, and contains an appendix giving a table of the cyclotomic polynomials for $2 < m \leqq 60$. Chapter V treats finite fields and the question of equations with prescribed galois group. A final Appendix is devoted to the elements of the theory of rational numbers.

The book proceeds at a leisurely pace and is readable. In the reviewer's opinion, the author has eschewed the modern approach with excessive zeal; this at times results in a loss of clarity. On the positive side one finds a great wealth of illustrative examples and exercises.

E. R. KOLCHIN

An introduction to probability theory and its applications. Vol. I. By William Feller. New York, Wiley, 1950. 12+419 pp. $6.00.

This is the first volume of a projected two-volume work. In order to avoid questions of measurability and analytic difficulties, this volume is restricted to consideration of discrete sample spaces. This does not prevent the inclusion of an enormous amount of material, all of it interesting, much of it not available in any existing books, and some of it original. The effect is to make the book highly readable even for that part of the mathematical public which has no prior knowledge of probability. Thus the book amply justifies the first part of its title in that it takes a reader with some mathematical maturity and no prior knowledge of probability, and gives him a considerable knowledge of probability, with the necessary background for going further. The proofs are in the spirit of probability theory and should help give the student a feeling for the subject.

Probability theory is now a rigorous and flourishing branch of analysis, distinguished from, say, measure theory, by the character and interest of its problems. It is true that probability theory, like geometry, had its origin in certain practical problems. However, like geometry, the theory now concerns itself with problems of interest per se, many of which are very idealized, and have only a remote connection or no presently visible connection, with practical problems.

At the same time the development of the science is continually stimulated by challenging problems arising in the various fields of application. This book contains a huge number of examples illustrating almost every aspect of the theory developed. These examples are very interesting and not at all of the ad hoc variety. It is no mean feat to present so many interesting examples between two covers. They enhance the interest of the theory even for the pure mathematician, except perhaps for the extreme diehard of the "God save mathematics from its applications" school.

The introductory chapter consists of a brief philosophical introduction. The author declares that the modern theory of probability, whose successes it is his intention to depict, is limited to what he says may roughly be called "physical or statistical probability." The intuitive notion of probability, connected with general inductive reasoning and the correctness of judgments, does not, he says, come within the scope of this theory. In the first chapter he introduces his axiom system, with the proviso that in the present book only denumerable spaces are to be considered. Chapter 2 treats the classical elements of combinatorial analysis. Chapter 3 is called *The simplest occupancy and ordering problems*, and Chapter 4 is called *Combination of events*. Both treat various combinatorial problems. In these chapters occur one or two problems on testing hypotheses, in which a hypothesis is rejected because under it the probability of the event observed is very small. This reviewer has misgivings about such arguments. What about other events of small probability which are not held to contradict the hypothesis being tested? It is only when one takes into account the admissible alternative hypotheses that this procedure will make sense. Naturally this question of admissible alternatives is not treated in the present book, since it is not within its scope and would take the author too far afield.

Chapters 5, 6, and 7 treat usual and classical topics. In Chapter 5 the author introduces the notion of conditional probability. In chapters 6 and 7 he discusses the binomial, multinomial, and Poisson distributions, and the normal approximation to the binomial distribution. It is amazing what a fresh and well-informed approach can do even to these standard topics. As an example we cite the discussion of large deviations (that is, the probability that the reduced number of successes will exceed x, where $x \to \infty$ in a suitable manner as $n \to \infty$, where n is the number of Bernoulli trials) on page 144, and problems 12 to 17 on pages 147 and 148, where results of Smirnov and Khintchine are developed.

Chapter 8 is entitled *Unlimited sequences of Bernoulli trials*. The proof by Doob (Ann. of Math. vol. 37 (1936) pp. 363–367) of the impossibility of a gambling system is given. The strong law of large numbers (for Bernoulli trials) is proved by using one of the above-mentioned results on large deviations. Then follows a proof of the law of the iterated logarithm. This is carried further in two exercises which introduce the reader to Feller's own results (Trans. Amer. Math. Soc. vol. 54 (1943) pp. 373–402). In Chapter 9 the author introduces the notions of chance variable and of expectation. In addition to the usual results (and many results not usual, such as introducing the reader to stratified sampling by means of two exercises), he proves Kolmogorov's inequality, which is used in the next chapter to prove that $\sum \sigma_i^2/i^2 < \infty$ is a sufficient condition for the strong law of large numbers to hold for independent chance variables. The author does not employ spaces whose elements are infinite sequences, because this would raise measure theoretic questions which he is determined to avoid in the present volume; strong convergence and similar properties are defined by means of passages to the limit. In Chapter 10 the author proves the (weak) law of large numbers for identically distributed, independent chance variables with a finite expected value (Khintchine's theorem). Since the author does not, in this volume, make use of theorems on the characteristic function (generating functions of integral-valued variables are used elsewhere), with the aid of which this theorem can be proved very expeditiously, he makes use of the device of truncating the chance variables appropriately so that the main parts have, of course, finite variances. In subsequent sections and numerous exercises he discusses the strong law of large numbers under varying conditions, various implications of the central limit theorem (proof deferred until the second volume), and his own theory of fair games (Ann. Math. Statist. vol. 16 (1945) pp. 301–304) with application to the Petersburg paradox. In Chapter 11 integral-valued variables are discussed. Generating functions and the notion of convolution are introduced. Numerous applications are made; a typically interesting one is to obtain the probability of the termination of a chain reaction.

The remaining chapters are, in many respects, the richest in the book, replete with recent results of the author and others, and it is useless to do more than name their topics. Two chapters are devoted to recurrent events (roughly, the stochastic process starts from scratch each time a certain pattern has occurred), and two chapters treat Markov chains. One chapter treats problems in gambler's ruin, random walk, and diffusion processes, the latter two in several dimensions. The final chapter is a brief introduction to the simplest time-dependent stochastic processes.

The outline given above of the book's contents is very bare and grossly inadequate. Since the book is not a unified treatment of just a few topics it is, however, difficult to do otherwise in a reasonable compass. To sum up, this is a superb book, and a delight to read. The gathering together of so much material in so brilliant a manner represents a prodigious amount of labor for which the mathematical public is greatly indebted. The reviewer congratulates the author; he has set a lofty standard for would-be writers of similar books to attain.

J. WOLFOWITZ

Theoretische Mechanik. Eine einheitliche Einführung in die gesamte Mechanik. By G. Hamel. (Die Grundlehren der mathematischen Wissenschaften, vol. 57.) Berlin, Springer, 1949. 16+796 pp. 161 figs.

In writing this book, the author had the following aims: to give a unified treatment avoiding the usual separation into mechanics of particles and mechanics of continua, and, following Lagrange, to present a deductive treatment based on the principle of virtual work, d'Alembert's principle, and Lagrange's "liberation principle" (Befreiungsprinzip). According to the author, Lagrange has used this last principle consistently even though he never formulated it explicitly. The author states this principle as follows: "if a constraint imposed on a mechanical system is relaxed, the corresponding reaction becomes an applied force which depends primarily on the deformation previously prevented by the constraint." For example, in the transition from an incompressible to a compressible perfect fluid, the Lagrangian multiplier of the condition of incompressibility becomes the pressure and this depends on the density variations which were originally excluded by the condition of incompressibility.

In this reviewer's opinion, the author has been entirely successful in carrying out his intentions. It goes almost without saying, that the resulting treatise is not suitable for beginners in spite of the fact that the subject is developed from first principles. In fact, the author's remark concerning Hertz' *Mechanics* ("geistreich, aber schwer zu lesen") applies equally well to his own book which is also rich in ideas but hard to read. To the reader who has already achieved mastery of the field along conventional lines, however, the work will open new horizons.

An unusual feature for a book written at this level is an extensive collection of Problems and Solutions (pp. 527–789).

The space available for this review does not permit detailed comments on the contents; the following list of chapter headings (with particularly significant section headings added in parentheses) will have to suffice. The concept of force and Newton's law (with an appendix on special relativity theory). Statics of constraint systems with a finite number of degrees of freedom (Lagrange's liberation principle). Statics of systems with infinitely many degrees of freedom (Theory of thin shells and plates. Foundations of the theory of elasticity. Viscous fluids and gases). Basic principles of kinetics (Vortex theorems of Lagrange and Helmholtz). Holonomic systems of a finite number of degrees of freedom—Lagrange's equations (Dirichlet's theorem on stability). Mathematical elaboration (Canonical equations. Canonical transformations). Minimum principles (Minimum principles of elasticity). The rigid body in space. Nonholonomic systems of a finite number of degrees of freedom.

W. PRAGER

Algebraic curves. By R. J. Walker. Princeton University Press, 1950. 10+201 pp. $4.00.

Modern algebraic geometry is one of the very active fields of mathematical research and there is a genuine need for a textbook on the elements of the subject. Up to now there have been available in English only sets of lecture notes, which while the work of the leaders in the field, themselves testify to the need for a more formal and elaborate publication. The volume under view is clearly an attempt to meet this need, and while this reviewer does not believe that it is wholly satisfactory, the book is a considerable contribution to the problem.

The first two chapters of the book are devoted to algebraic and geometric preliminaries. It is in the third chapter that the author begins the study of his subject matter, algebraic curves over an algebraically closed field of characteristic zero, starting with a discussion of multiple points of such curves. A weak form of Bezout's theorem is derived and used to relate the order of a curve with the multiplicities of its singular points. The chapter also contains a proof of the theorem on reduction of singularities, followed by a sketchy treatment of neighboring points. The first part of chapter four is devoted to formal power series, leading to the notion of a place of a curve. The basic algebraic result here, the algebraic closure of a certain fractional power series field, is handled in great detail. This material is then applied to a formulation and proof of Bezout's theorem and to the derivation of some of Plücker's formulas. The chapter ends with a proof of a simple case of Nöther's $AF+BG$ theorem. Chapter five opens with more algebraic material, this time on ideals and field extensions. The field of rational functions on a curve is discussed and used to obtain satisfactory formulations of the concepts of rational and birational correspondences. Lüroth's theorem and the remaining Plücker formulas are then derived. Finally valuations are defined and their connection with places is established. The last chapter is given over to linear series, applying them first to obtain a nonsingular birational transform of an irreducible curve. Study of the canonical series then leads to the genus of a curve and to the Riemann-Roch theorem. Two further topics bring the book to a close: partial classification of curves under birational equivalence leading to appropriate canonical forms, and treatment of the nonsingular cubic including Salmon's theorem on the cross-ratio.

A great deal of the book is given over to purely algebraic topics, making possible an exposition of the theory of curves which is completely rigorous. Indeed many of the curve-theoretic proofs are valid for the more general situations to which the reader might progress in his further study. On the other hand the exposition of the algebra makes no such provision for the reader's education. It is so extremely concise and so thoroughly tailored to the special purposes of the book that this reviewer believes its study to be a most uneconomical use of the reader's time.

The quantity of out and out algebraic geometry is relatively small, by comparison with older treatises, but its quality is very high. A question might be raised on the propriety of having the resolution of singularities take place before the student has studied parametrizations, for without them the proof is unnecessarily hard and the achievement of only ordinary singularities is not easily appreciated. With a few such exceptions the individual topics which are treated are treated well. Nevertheless, the total impact of the book is disappointing. The book might be very illuminating as a companion volume to one of the older works, but the beginner in the subject is not likely to benefit from it. He is given no awareness of the rich body of knowledge to which it relates, and, although he is told from time to time that a certain item is important, he sees no systematic working out of a basic problem which could independently justify his activity.

HOWARD LEVI

Weltsystem, Weltäther und die Relativitätstheorie. By Karl Jellinek. Basel, Wepf, 1949. 15+450 pp. 45 Swiss fr.

The subtitle of this book on special and general relativity and on relativistic cosmology reads: *An introduction for the experimental natural scientist.*

The author's approach to the subject is often laborious, and there seems to be some danger that a reader might get lost in the details of specific transformation equations and long-winded discussions of apparent paradoxes. Here are a few examples of the method of presentation: Minkowski's space-time geometry is introduced in the sixth and last section of the long chapter on special relativity, whereas the proof of the Lorentz invariance of Maxwell's equations is given in section 5. The chapter on tensors does not include any discussion of covariant differentiation or the proof of the tensor character of the curvature tensor. In the chapter on general relativity, the Schwarzschild line element is given without derivation. On the other hand, a detailed calculation is made to show that certain radial lines are geodesics; here a simple symmetry argument would have sufficed.

The author attaches some importance to the possibility of retaining a "world ether" consisting of "tiny ether atoms." This point of view is perhaps old fashioned at a time when most physicists would dispute the usefulness of introducing a mechanistic model for its own sake.

For purposes of reference the value of the book is impaired by the very large number of abbreviations. An index of these would have been useful.

A. SCHILD

The theory of algebraic numbers. By Harry Pollard. (Carus Mathematical Monographs, no. 9.) Buffalo, The Mathematical Association of America; New York, Wiley, 1950. 12+143 pp. $3.00.

For many years no suitable treatment of algebraic number theory has been available in English. The present monograph, presenting a concise and careful exposition of the elements of the classical theory, is thus most welcome.

The introductory material covers the unique factorization of the rational integers and of the Gaussian integers $m+ni$, together with the (little) Fermat theorem in both cases. The familiar example of nonunique factorization for the integers $m+n(-5)^{1/2}$ is then presented. Then algebraic numbers are defined in general, and the (constructive) existence of transcendental numbers is demonstrated. The elementary properties of algebraic number fields and of algebraic integers are then developed. For the fundamental theorem of ideal theory (every ideal can be represented uniquely as a product of prime ideals) both the classical proof and the Ore modification of the Noether-Krull axiomatic proof are presented. Among the consequences is the proof that a rational prime is unramified in a field if it does not divide the discriminant of the field.

The definition of an ideal is motivated by a tentative description of an ideal in a number field K as the set of all integral multiples in K of some algebraic integer which does not necessarily lie in K. In Chapter 10 it is demonstrated that every ideal can in fact be so described, the proof depending on the finiteness of the class number of a field. The application of algebraic number theory to the Fermat conjecture ($x^p+y^p=z^p$) is then discussed, on the basis of previous illustrative analysis of the cyclotomic field generated by the pth roots of unity. The last chapter presents the Minkowski proof of the Dirichlet theorem on the units of a number field. The material is well arranged, with frequent appropriate examples, and is self-contained except for the fundamental theorem of algebra, the theorem on symmetric functions, and the basic facts about simultaneous homogeneous linear equations.

The reviewer regrets that the author did not take a little additional space to point out the fascinating relations between the specific concepts arising in algebraic number theory and the more general concepts of algebra. Thus, the fact that an ideal is precisely the kernel of a homomorphism is not mentioned. A field means exclusively a field of numbers; hence the revealing circumstance that the congruence classes of integers modulo a prime ideal constitute a finite field cannot be brought out. In keeping with an ancient tradition in number theory, the word "group" is never mentioned, in the midst of numerous arguments of a group-theoretic character. The fact that the ideal classes form a group is thus not clearly stated, and the familiar argument that a subgroup of a free abelian group is free appears in disguise at least twice (for units, and for bases of an ideal).

This book provides a clear and elementary introduction to its subject, in keeping with the purpose of the Carus Monographs. If supplemented as indicated, it would be of use in elementary graduate courses.

SAUNDERS MACLANE

BRIEF MENTION

Collected mathematical papers. By G. D. Birkhoff. Vol. 1, 43+754 pp.; vol. 2, 8+983 pp.; vol. 3, 8+897 pp. New York, American Mathematical Society, 1950. $18.00.

In addition to the collected papers, volume 1 contains obituaries by R. E. Langer, O. Veblen, and M. Morse, and volume 3 contains a list of Birkhoff's publications. The obituary by Morse, reproduced from Bull. Amer. Math. Soc. vol. 52 (1946) pp. 357–391, contains a detailed review of Birkhoff's work.

Contributions to Fourier analysis. By A. Zygmund, W. Transue, M.

Morse, A. P. Calderón, and S. Bochner. (Annals of Mathematics Studies, no. 25.) Princeton University Press, 1950. $3.00.

The titles of the contributions are as follows. I, Localization of best approximation (Bochner); II, Dirichlet problem for domains bounded by spheres (Bochner); III, The Fréchet variation and Pringsheim convergence of double Fourier series (Morse and Transue); IV, Norms of distribution functions associated with bilinear functionals (Morse and Transue); V, Note on the boundary values of functions of several complex variables (Calderón and Zygmund); VI, On the theorem of Hausdorff-Young and its extensions (Calderón and Zygmund).

NEW PUBLICATIONS

ALEKSANDROV, A. D. *Vypuklye mnogogranniki.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1950. 428 pp.

FORTET, R. *Calcul des probabilités. Centre d'études mathématiques en vue des applications.* Vol. 1. *Application des théories mathématiques.* Paris, Centre National de la Recherche Scientifique, 1950. 330 pp. 1200 fr.

H. FREUDENTHAL, D. VAN DANTZIG, J. DE GROOT, B. L. VAN DER WAERDEN, A. VAN HEEMERT, G. HIRSCH, and J. C. H. GERRETSEN. *Zeven voordrachten over topologie.* Ed. by H. Freudenthal and W. Peremans. Gorinchem, Noorduijn, 1950.

HILL, R. *The mathematical theory of plasticity.* Oxford, Clarendon, 1950. 9+356 pp.

KAGAN, V. F. *Osnovaniya geometrii. Učenie ob obosnovanii geometrii v hode ego istoričeskogo razvitiya.* Part 1. *Geometriya Lobačevskogo i ee predistoriya.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1949. 492 pp.

LEVITAN, B. M. *Razloženie po sobstvennym funkciyam differencial'nyh uravnenii vtorogo poryadka.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1950. 159 pp.

McLACHLAN, N. W. *Ordinary non-linear differential equations in engineering and physical sciences.* Oxford, Clarendon, 1950. 7+201 pp. $4.25.

MICHAEL, W. *Ortskurvengeometrie in der komplexen Zahlenebene.* Basel, Birkhäuser, 1950. 95 pp. 11.50 Swiss fr.

SCHRÖDINGER, E. *Space-time structure.* Cambridge University Press, 1950. 8+119 pp. $2.75.

VAN DER WAERDEN, B. L. *Moderne Algebra.* Vol. 1. 3d ed. Berlin, Springer, 1950. 8+292 pp. 27 DM.

NOTES

Mr. D. L. Herr of Hughes Aircraft Company, Culver City, California, was awarded an A. Cressy Morrison prize for 1950, by the New York Academy of Sciences.

Dr. R. C. Briant of Mellon Institute of Industrial Research has accepted a position as division director with Union Carbide and Carbon Corporation, Oak Ridge, Tennessee.

Dr. Eugenio Calabi of Princeton University has been appointed to an assistant professorship at Louisiana State University.

Dr. A. P. Calderón has been appointed to a visiting associate professorship at Ohio State University.

Professor F. R. Bamforth of Ohio State University has been appointed to a professorship at Otterbein College.

Professor W. E. Cederberg of Augustana College and Theological Seminary has retired with the title emeritus.

Professor H. B. Curtis of Lake Forest College has retired with the title emeritus.

Mr. Arie Gaalswyk of the University of California has accepted a position as mathematician in the Aeronautical Research Laboratories of General Mills, Inc., Minneapolis, Minnesota.

Mr. J. E. Hanson has been appointed mathematician at the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Professor L. A. Hopkins of the University of Michigan has retired with the title emeritus.

Dr. E. R. Keown has been appointed to an assistant professorship at Florida State University.

Associate Professor C. K. Payne of New York University has retired.

Professor W. L. Pickard of the State University of New York has been appointed to an associate professorship at Northern Illinois State Teachers College.

Associate Professor L. V. Robinson of the University of South Carolina has accepted a position as mathematician at the Aberdeen Proving Ground, Maryland.

Mr. C. W. Vickery has accepted a position as mathematician with V. L. Beavers and Associates, Consulting Engineers, San Antonio, Texas.

Mr. J. L. Nagle has been appointed director of the Real Property Acquisition and Utilization Division, U. S. Public Buildings Service.

The following promotions have been announced:

Warren Ambrose, Massachusetts Institute of Technology, to an associate professorship.

R. E. Anderson, Northern Illinois State Teachers College, to an associate professorship.

B. E. Gatewood, U.S.A.F. Institute of Technology, to a professorship of mechanics.

Wilfred Kaplan, University of Michigan, to an associate professorship.

J. A. Larrivee, University of Vermont, to an associate professorship.

C. V. Newsom, to Associate Commissioner for Higher and Professional Education, New York State Education Department.

Dr. Z. I. Mosesson, Prudential Insurance Company of America, to chief actuarial assistant.

The following appointments to instructorships have been announced: Cornell University: Dr. A. S. Shapiro; University of Michigan: Dr. J. R. Büchi; New York University: Mr. J. M. Patterson; Rutgers University: Mr. R. F. Gabriel.

Professor H. A. Bohr of the University of Copenhagen died January 21, 1951 at the age of sixty-three years. He had been a member of the Society for nineteen years.

Professor J. H. Binney of the Agricultural and Mechanical College of Texas died on May 29, 1950 at the age of fifty-five years.

Professor Emeritus Lennie P. Copeland of Wellesley College died on January 11, 1951 at the age of sixty-nine years. She had been a member of the Society for thirty-five years.

Dr. Alfred Hume, chancellor emeritus of the University of Mississippi, died on December 25, 1950 at the age of eighty-four years. He had been a member of the Society for twenty-three years.

Assistant Professor J. G. Millar of the University of Alberta died December 9, 1950 at the age of twenty-eight years.

Professor J. F. Ritt of Columbia University died on January 5, 1951 at the age of fifty-seven years. He had been a member of the Society for thirty-five years.

Professor Abraham Wald of Columbia University died December 13, 1950 at the age of forty-eight years. He had been a member of the Society for eleven years.

Professor Emeritus F. S. Woods of the Massachusetts Institute of Technology died December 1, 1950 at the age of eighty-six years. He had been a member of the Society for thirty-seven years.

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The initiation fee and annual dues of members of the Society (see this BULLETIN, November, 1950, pp. 567–568) are payable to the Treasurer of the Society, A. E. Meder, Jr., 531 West 116th St., New York 27, N.Y.

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THE FEBRUARY MEETING IN CINCINNATI

The four hundred sixty-fifth meeting of the American Mathematical Society was held at the University of Cincinnati on Friday and Saturday February 23–24, 1951. There were about 85 registrations including the following 57 members of the Society:

J. E. Adney, E. S. Allen, Emilio Baiada, I. A. Barnett, W. R. Baum, L. G. Belai, W. D. Berg, Dewey Blair, Louis Brand, R. W. Bryant, E. D. Cashwell, Lamberto Cesari, V. F. Cowling, E. H. Crisler, R. E. Fullerton, Franklin Haimo, O. G. Harrold, Hugo Heermann, Carl Holtom, W. E. Jenner, H. K. Justice, Frank Levin, D. J. Lewis, H. D. Lipsich, Lee Lorch, C. I. Lubin, R. W. MacDowell, H. B. Mann, William Marcaccio, Gaylord Merriman, E. J. Mickle, H. L. Miller, C. N. Moore, W. B. Morgan, Tadasi Nakayama, Zeev Nehari, H. S. Pollard, O. W. Rechard, P. V. Reichelderfer, Haim Reingold, M. E. Rice, R. A. Roberts, C. H. Rust, H. J. Ryser, W. C. Sangren, K. C. Schraut, M. E. Shanks, Edward Silverman, T. F. Smith, W. S. Snyder, V. C. Stechschulte, E. G. Swafford, Otto Szász, E. F. Trombley, J. A. Ward, E. F. White, J. W. T. Youngs.

The Committee to Select Hour Speakers for Western Sectional Meetings had issued two invitations, one to Professor Zeev Nehari of Washington University and the second to Professor Tadasi Nakayama of Nagoya University and the University of Illinois. Professor Nehari spoke on *Bounded analytic functions* at 2:00 P.M. Friday while Professor Nakayama addressed the Society on *Galois theory of rings* at 10:30 A.M. Saturday.

Presiding officers for the various sessions were Professors Otto Szász, W. S. Snyder, E. S. Allen, and I. A. Barnett.

The University of Cincinnati entertained the Society at a tea on Friday afternoon and there was a dinner on Friday evening at the Sinton Hotel. Dean Greene welcomed the guests on behalf of the University of Cincinnati and Professor Shanks responded. The dinner was characterized by an air of friendly informality, possible perhaps only at one of the regional meetings.

The list of papers presented follows. Those abstracts indicated by the letter "*t*" were presented by title. Professor Koenig was introduced by Professor J. W. T. Youngs and Professor Fortet by Professor G. E. Forsythe.

ALGEBRA AND THEORY OF NUMBERS

186*t*. A. A. Albert: *On simple alternative rings.*

The purpose of this note is that of proving the following result. Let C be a simple alternative ring which contains an idempotent not its unity quantity. Then C is either associative or is a Cayley algebra over its center. The result is proved by a procedure similar to that used in recent work on simple power-associative commutative rings and no restriction on the characteristic of C is required. (Received January 15, 1951.)

187t. Bjarni Jónsson: *An infinite Boolean algebra without proper automorphisms.*

A simply ordered set C is constructed which has the following properties: (i) Each member of C has a character of the form $(\omega_\alpha, \omega_0^*)$ for some $\alpha > 0$. (ii) No two members of C have the same character. (iii) Each gap of C has a character of the form $(\omega_\alpha, \omega_0^*)$ for some $\alpha \geqq 0$. (iv) C is cofinal with ω_0 and coinitial with ω_0^*. Replacing each member of C by an ordered pair of elements and completing the resulting set by cuts, one obtains a simply ordered set S whose interval topology is compact and zero-dimensional and has no proper homeomorphisms onto itself. Hence the Boolean algebra of open and closed subsets of S has no proper automorphisms. This solves Problem 74 in G. Birkhoff, *Lattice theory* (rev. ed.). (Received January 10, 1951.)

ANALYSIS

188. Lamberto Cesari: *Eilenberg's inequality for Lebesgue area.*

In 1943 S. Eilenberg and O. Harrold proved a general form of inequality involving the α-dimensional measures of a set H in a general space. Eilenberg, L. C. Young, and the author have shown that in a particular case, involving the elementary area of a polyhedral surface in E_3 and the lengths of elementary curves, a simple formulation and proof are possible. The author has also applied such a case to the calculus of variations with the aim of extending to parametric continuous surfaces an existence theory based only upon geometrical considerations and the concept of lower semicontinuity. The author has proved the following independent formulation of the statement of Eilenberg and Harrold, involving the concepts of Lebesgue area and of ordinary length: Given a parametric continuous surface S: $x = x(w)$, $w \in Q\,[w = (u, v)$, $x = (x^1, x^2, x^3)$, Q unit square$]$ and a real one-valued function $f(x)$, $x \in E_3$, such that $|f(x) - f(x')| \leqq |x - x'|$, let $t_0 = \max f[x(w)]$, $w \in Q^*$, $t_1 = \max f[x(w)]$, $w \in Q$, and suppose $t_0 < t_1$. For any $t_0 < t < t_1$ let $A(t)$ be the maximal connected open set $A(t) \subset Q$ containing Q^* and such that $f[x(w)] < t$. The part of the boundary $A^*(t)$, besides Q^*, has an image $m(t)$ on S. By extension of the ordinary length, calling $l(t)$ the "length" of $m(t)$, the following inequality holds: $L(S) \geqq \int_{t_0}^{t_1} l(t)\,dt$, where $L(S)$ is the Lebesgue area of S. (Received November 29, 1950.)

189. J. F. Koenig: *Regions of the zeros of polynomials with real or complex coefficients.*

Using the Bilharz-Frank theorem, it can be determined whether or not all zeros of a polynomial with complex coefficients lie within the half-plane to the left of the imaginary axis. Several generalizations of the Bilharz-Frank theorem are given. With these generalizations it can be determined whether or not the zeros of polynomials with real or complex coefficients lie within regions such as: Sectors and sub-sectors of the complex plane; any rectangular infinite strip symmetric about the real axis; region where the real part of Z is less than $-\alpha$ (α real). When the region of the zeros of a polynomial with real coefficients is to be determined, the real polynomial is transformed into a polynomial with complex coefficients. The Bilharz-Frank theorem is then applied to the transformed polynomial. Various linear transformations can be used. When the region of the zeros of a polynomial with complex coefficients is to be determined, the complex polynomial is transformed into another complex polynomial. The Bilharz-Frank theorem is then applied to the transformed polynomial. Various linear transformations can be used. These generalizations also give the conditions on the polynomial coefficients such that all zeros will lie within a prescribed region. (Received December 26, 1950.)

190. H. B. Mann: *Some theorems on difference sets.*

A set of different residues $a_1, \cdots, a_k$ mod v is called a difference set (v, k, λ) $(v>k>\lambda)$ mod v if the congruence $a_i - a_j \equiv r(v)$ has exactly λ solutions for every $r \neq 0$. A number t is called a multiplier of (v, k, λ) if $\{ta_i\} = \{a_j + s\}$ for some s. The following theorems are proved. (1) If a multiplier t is of even order with respect to a prime divisor g of v, then $n = k - \lambda$ is a square if $(t/g) = -1$. If $(t/g) = +1$, then n is either a square or of the form a^2q^3: If moreover $g = v$, then $k = v - 1$. We further put $\theta(x) = \sum_1^k x^{a_i}$, $T(x) = \sum_0^{v-1} x^r$. Then (2) a prime p is multiplier of (v, k, λ) if and only if $\theta(x)^p \equiv x^s\theta(x)(p, x^v - 1)$ for some s. If moreover $n \not\equiv 0(p)$, then $\theta(x)^{p-1} \equiv x^s(p, x^v - 1)$ if $k \not\equiv 0(p)$ and $\theta(x)^{p-1} \equiv x^s - T(x)(p, x^v - 1)$ if $k \equiv 0(p)$. Both theorems have several corollaries. Theorem 1 implies for instance that n is a square if for some divisor p of n we have $p > \lambda$ and $((-1)^{(v-1/2)}\lambda/p) = -1$. Moreover if $p > \lambda$, $((-1)^{(v-1/2)}\lambda/p) = -1$, and v is a prime, then (v, k, λ) is impossible. Theorem 2 implies: If 2 is a multiplier of (v, k, λ), then 2 divides n. If 3 is a multiplier mod $(v, k, 1)$ then 3 divides $n = k - 1$. (Received December 22, 1950.)

191. M. E. Shanks: *On the existence of measures.*

Let ϕ be a finite measure on a set X. Let F be the set of all real functions on X and let $\mathcal{J}$ be the set of equivalence classes of F under ϕ. Then (see A. Rosenthal and H. Hahn, *Set functions*, University of New Mexico Press, 1948, p. 137) the asymptotic ϕ-metric makes $\mathcal{J}$ a topological group. In this paper the following converse to the above theorem is proved. Suppose: (1) $\rho(f_1, f_2)$ is a quasi-metric on F, and $\mathcal{J}$ the set of equivalence classes under ρ. (2) With metric ρ, $\mathcal{J}$ is a topological group. (3) $\rho(f, 0) = 0$ implies $\rho(|f|, 0) = 0$. (4) $0 \leqq g_n \leqq f_n$ and $\rho(f_n, 0) \to 0$ imply $\rho(g_n, 0) \to 0$. (5) If f_n is a Cauchy sequence in $\mathcal{J}$ and $f_n \to f$ pointwise, then $\rho(f_n, f) \to 0$. Then there exists an invariant metric ρ^* on $\mathcal{J}$ such that if we define $\phi(A) = \rho^*(f_A, 0)$, where f_A is the characteristic function on A, ϕ is a finite measure on X and the asymptotic metric gives the same equivalence classes in F as does ρ and the same topology to $\mathcal{J}$. (Received January 11, 1951.)

192. Otto Szász: *On the Gibbs' phenomenon for a class of linear transforms.*

Given a function $\phi(t)$ such that $\phi(+0) = 1$ and $\int_0^\infty |d\phi(t)|$ is finite. The series $\phi(h) = \sum_1^\infty a_n\phi(nh)$ is then a regular transform of the series $\sum a_n$. For $a_n = (\sin nt)/n$ write $\phi(h) = T(h, t)$; the constant $G = \limsup (2/\pi)T(h, t)$, when h and t tend to zero, is called the Gibbs' ratio of the transform. The transform presents a Gibbs' phenomenon if $G > 1$. The main result of this paper is that $G = \max_{\zeta>0}(2/\pi)\int_0^\infty \phi(t)((\sin \zeta t)/t)dt$. A related result is due to Kuttner. Some particular summability methods and the corresponding Gibbs' phenomenon are then discussed. (Received January 5, 1951.)

APPLIED MATHEMATICS

193. Louis Brand. *The algebra of motors.*

The author has shown that a motor M can be regarded as a dual multiple of a unit line vector (*vector and tensor analysis*): $\mathbf{M}=\mathbf{m}+\epsilon\mathbf{m}_0=(\lambda+\epsilon\lambda')(\mathbf{a}+\epsilon\mathbf{a}_0)$, where λ, λ' are real scalars, $\mathbf{a}$, $\mathbf{a}_0$ vectors such that $\mathbf{a}\cdot\mathbf{a}=1$, $\mathbf{a}\cdot\mathbf{a}_0=0$ and ϵ a unit such that $\epsilon^2=0$. In the algebra of motors, regarded as dual vectors, the scalar and motor product of two motors are defined as $\mathbf{M}\cdot\mathbf{N}=\mathbf{m}\cdot\mathbf{n}+\epsilon(\mathbf{m}\cdot\mathbf{n}_0+\mathbf{m}_0\cdot\mathbf{n})$ and $\mathbf{M}\times\mathbf{N}=\mathbf{m}\times\mathbf{n}+\epsilon(\mathbf{m}\times\mathbf{n}_0+\mathbf{m}_0\times\mathbf{n})$; the former definition differs from the scalar product of von Mises (Zeitschrift für Angewandte Mathematik und Mechanic vol. 4, p. 163), namely $\mathbf{M}\cdot\mathbf{N}=\mathbf{m}\cdot\mathbf{n}_0+\mathbf{m}_0\cdot\mathbf{n}$. In the algebra of dual vectors thus obtained, *all* the rules of ordinary vector algebra are still valid. Any three motors $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3$ for which $\mathbf{m}_1\cdot\mathbf{m}_2\times\mathbf{m}_3\neq 0$ have a unique reciprocal set $\mathbf{M}^1$, $\mathbf{M}^2$, $\mathbf{M}^3$, where $\mathbf{M}_i\cdot\mathbf{M}^j=\delta_i^j$. The axes of two reciprocal sets afford a constructive proof of the Theorem of Peterson and Morley. Motor dyadics can be introduced; and the unit motor dyadic is the idemfactor of Gibbs, which can be expressed in the form $\mathbf{M}_1\mathbf{M}^1+\mathbf{M}_2\mathbf{M}^2+\mathbf{M}_3\mathbf{M}^3$. Motor dyadics of this type can be reduced to the form $\Psi+\epsilon\Psi_0$, where Ψ and Ψ_0 are ordinary dyadics and $\Psi_P=\Psi_0+PO^{\rightarrow}\times\Psi-\Psi\times PO^{\rightarrow}$. If Ψ is complete, the motor dyadic $\Psi+\epsilon\Psi_0$ has the unique reciprocal $\Psi^{-1}-\epsilon\Psi^{-1}\cdot\Psi_0\cdot\Psi^{-1}$. (Received January 10, 1951.)

GEOMETRY

194*t*. D. O. Ellis: *Notes on abstract distance geometry.* II. *Implications of basality in generalized semimetric spaces.*

With the exception of the term *basality* the present note is couched in the terminology of the foundational paper (David Ellis, *Geometry in abstract distance spaces*, Publicationes Mathematicae (Debrecen) vol. 2 (1951)). A distance space is called *basal* if every point of the space forms a complete metric base for the space. In this note it is shown that *any basal generalized semimetric space has the property of free mobility and is* (1) *metrically irreducible*, (2) *distancially irreducible*, (3) *irreducible over the ground set*. An example of a basal generalized semimetric space is the autometrized Boolean algebra (cf. David Ellis, *Autometrized Boolean algebras*, I, to appear in the Canadian Journal of Mathematics). (Received November 30, 1950.)

195*t*. Ira Rosenbaum: *The theory of the congruence of point pairs.*

By Veronese and R. L. Moore the theory of geometric congruence is based on that of the congruence of open intervals or point pairs. A formal, logical development of the theory of the congruence of point pairs is elaborated employing the logical machinery of the author's *Introduction to mathematical logic and its applications*, University of Miami Press, 1950. Axioms essentially equivalent to those for the congruence of segments in Hilbert's *Foundations of geometry* are employed. A further axiom, "If $zz\cong xy$, then $x=y$," is needed if theorems are to be freed from tacit restrictions and given added generality and precision. The definitions of the relation "less than" among open intervals and of the "sum" of two intervals employed, for example, by R. L. Moore (Trans. Amer. Math. Soc. (1908)) require modification, for the reason Hilbert's axioms require supplementation. The emended definitions are: $xy<zw:\rightleftarrows:[x=y.z\neq w.v.](Ep)(pe(zw).zp\cong xy)$ on the one hand, and on the other: (a) $x\neq y:.\rightarrow:.(z$ is in the extension of xy beyond y[or $z=y$] and $yz\cong uv:\rightleftarrows:xy+uv=xz$, and (b) $[x=y.\rightarrow.xy+uv=uv+xy.]$. Novel are the bracketed portions of the above definitions. (Received January 10, 1951.)

Statistics and Probability

196t. R. E. Fortet: *Additive functionals of a Markoff process.*

Let $X(t)$ be a Markoff process of a continuous (for instance) parameter t; X takes its values in a space $\mathfrak{X}$, and $F(t, x; \tau, e)$ (e: subset of $\mathfrak{X}$) is its transition probability. Let $L(t, \tau)$ be a random variable corresponding to each interval $(t, \tau)(\tau<t)$ with the following properties: (a) $L(t, \tau)=L(t, t')+L(t', \tau)$ $(t<t'<\tau)$; (b) Conditionally if $X(t)=x$, $X(\tau)=\xi$, $L(t, \tau)$ and $X(t')$ are independent if $t'<t<\tau$ or $t<\tau<t'$; (c) Conditionally when $X(t)=x$, $X(t')=x'$, $X(\tau)=\xi$ $(t<t'<\tau)$, $L(t, t')$ and $L(t', \tau)$ are independent. $L(t, \tau)$ will be called an additive functional of $X(t)$; if $\psi(t, x; \tau, \xi; v)$ is the characteristic function of $L(t, \tau)$, conditionally when $X(t)=x$, $X(\tau)=\xi$ $(t<\tau)$, and if one puts: $\mu(t, x; \tau, e; v)=\int_e\psi(t, x; \tau, \xi; v)d\xi F(t, x; \tau, e)$, one obtains the fundamental relation: $\mu(t, x; \tau, e; v)=\int_{\mathfrak{X}}\mu(t', x', \tau, e; v)d_{x'}\mu(t, x; t', e')$ $(t<t'<\tau)$. Assume that: $\lim_{\Delta t\to+0}$ $(\mu(t, x; t+\Delta t, \mathfrak{X}; v)-1)/\Delta t=\lim_{\Delta t\to+0}$ $(\mu(t-\Delta t, x; t, \mathfrak{X}; v)-1)/\Delta t=A(t, x; v)$ exists; under some conditions, one can prove that (1) $\mu(t, x; \tau, e; v)=F(t, x; \tau, e)$ $+\int_t^{\tau}du\int_{\mathfrak{X}}\mu(u, y; \tau, e; v)A(u, y; v)d_yF(t, x; u, e'')$. If $X(t)$ is a real number, under some conditions it follows from (1) that $\partial\mu/\partial t+(\alpha/2)(\partial^2\mu/\partial x^2)+\beta\partial\mu/\partial x+A\mu=0$, where α and β are the coefficients of the Kolmogoroff equation of $X(t)$. (Received January 15, 1951.)

197t. R. E. Fortet: *Estimation of eigenvalues by Monte Carlo methods.*

Given an integral equation of the form (1) $f(t)=\lambda\int_a^b\Gamma(t, \tau)f(\tau)d\tau$, where $\Gamma(t, \tau)$ is a definite positive kernel, one obtains an estimation of the Fredholm determinant $D(\lambda)$ of (1) in the following way: let $X^*(t)$ be the Gaussian stochastic process with covariance equal to $\Gamma(t, \tau)$; the characteristic function $\phi(v)$ of the random variable $Y=\int_a^bX^{*2}(t)dt$ is equal to $[D(2iv)]^{-1/2}$; the difficulty is to "realize" $X^*(t)$; a procedure is suggested which replaces $X^*(t)$ by an approximate process of the form $X(t)$ $=(1/m^{1/2})\int_{-\infty}^{+\infty}R(t, \tau)dN^*(\tau)$ where $N^*(\tau)$ is a centered Poisson process of parameter m; m has to be sufficiently large; R is any solution of $\Gamma(t, \tau)=\int_{-\infty}^{+\infty}R(t, u)R(\tau, u)du$ $(a\leqq t, \tau\leqq b)$. If $\Gamma(t, \tau)$ depends on $(\tau-t)$ only, it is easy to find the solutions R which depend on $(\tau-t)$ only; in the general case one can prove the existence of at least one solution under weak conditions. (Received January 15, 1951.)

198t. R. E. Fortet: *Random determinants.*

The following investigations are an attempt to solve problems arising in so-called "linear programming." Let D_n be a determinant, of any order n, of the form: D_n $=\|x_{ij}+X_{ij}\|$, where the x_{ij}'s are given numbers, and the X_{ij}'s random variables; one can compute $E(D_n)$ and $E(D_n^2)$, under some assumptions, for instance if the X_{ij}'s are independent and if $E(X_{ij})=0$, $E(X_{ij}^2)=\sigma_i^2<+\infty$. In a Euclidean n-dimensional space E_n, let $P_1, \cdots, P_n$ and ϕ be $(n+1)$ given points, and $M_1, \cdots, M_n$ be n random points; let P and Π be the planes $(P_1, \cdots, P_n)$ and $(M_1, \cdots, M_n)$; let h be the algebraic distance from P to ϕ; if $n \rightarrow +\infty$, and if h remains >0 and constant, the probability that ϕ is above Π tends toward 1, under some assumptions [for instance, if the $(P_iM_i)^{\rightarrow}$ are independent Laplacian vectors with $E((P_iM_i)^{\rightarrow 2})$ equal to a constant smaller than 1/2, and if $P_iP_j = 1$ for any i, j, n]; the preceding results on D_n can be applied to a more precise study of the distance from Π to ϕ. (Received January 15, 1951.)

199t. R. E. Fortet: *On some functionals of Laplacian processes.*

Let $N^*(t)$ be a centered Poisson process, with parameter m; and $R(t, \tau)$ a (not random) function such that $\int_{-\infty}^{+\infty} R(t, \tau)^2 d\tau < +\infty$ for any given t. The process $X(t)$ defined by: $X(t) = \lim \text{m.q.}_{\alpha \rightarrow -\infty,\ \beta \rightarrow +\infty}$ i.a.c. $(1/m^{1/2}) \int_\alpha^\beta R(t, \tau) dN^*(\tau)$ is called a random function derived from a Poisson process (f.d.P.). Assume that $R(t, \tau) = R(\tau - t)$ depends on $(\tau - t)$ only; and let $V(x)$ be a function such that (1) $|V(x) - V(x')| \leqq M|x - x'|^\alpha$ $(0 < \alpha \leqq 1$; M a constant depending on $V)$. We put $Y(t) = V[X(t)]$, $L(t, \tau) = \int_t^\tau Y(u) du$. If $\int_{-\infty}^{+\infty} [\int_{|u|>a} R^2(u) du]^{\alpha/2} da < +\infty$ and, except in an exceptional case, for any fixed t, the distribution function of $L(t, t+T)$, when properly normed, tends toward the Laplace law if $T \rightarrow +\infty$. Other results of the same kind can be obtained. These results are extended to the case where $X(t)$ is a Laplacian stationary process instead of an f.d.P., and to cases where $X(t)$ (f.d.P. or Laplacian process) is not stationary. Assumption (1) can be weakened. (Received January 12, 1951.)

TOPOLOGY

200. W. R. Baum: *The group of reduced null-spherods.*

Let $\Psi^n(K)$ be the group of n-dimensional null-spherods of a complex K, $\Psi^n(K) \cong \Gamma^n(K^n)/P_0^n(K, K^n)$. (Cf. W. R. Baum, Bull. Amer. Math. Soc. Abstract 57-2-162.) If $\Pi_0^n(K^n)$ is the subgroup of the Hurewicz homotopy group $\pi_n(K^n)$ generated by all the differences $\alpha - \alpha'$, where the elements $\alpha, \alpha' \in \pi_n(K^n)$ are free-homotopic on the n-dimensional skeleton K^n of K (cf. H. Hopf, Comment. Math. Helv. vol. 17 (1945) pp. 307–326), consider the subgroup $\Psi_0^n(K)$ of $\Psi^n(K)$ which corresponds to $\Pi_0^n(K^n)/P_0^n(K, K^n)$. For $n = 1$, $\Psi_0^n(K) = \Psi^n(K)$. The factor-group $\Omega^n \cong \Psi^n/\Psi_0^n$ depends only on the skeleton K^n, but is a topological invariant of K. The group Ω^n can be considered as the group of the "reduced" n-null-spherods (in a certain geometric sense). If K is aspherical in the dimensions r $(1 < r < n)$ it follows from a theorem due to H. Hopf (loc. cit., Theorem I) that the structure of Ω^n $(n > 1)$ is determined by the structure of the fundamental group of K. (Received January 10, 1951.)

201. P. V. Reichelderfer: *On the barycentric homomorphism in a singular complex.*

Radó has introduced a singular complex R in which no a priori identifications are made (T. Radó, *On identifications in singular homology theory*, to appear in Rivista di Matematica della Universitá di Parma), and has devised the following technique for studying the effect of certain a priori identifications on the theory (T. Radó, *An approach to singular homology theory*, unpublished). A system of subgroups G_p of the chain groups C_p is termed an identifier if the relation $\partial G_p \subset G_{p-1}$ is satisfied, so that the G_p form a subcomplex G of R. The identifier is said to be unessential in R if the natural homomorphism π: $R \rightarrow R/G$ induces an isomorphism onto between the corresponding homology groups. The main purpose of this note is to show that the nuclei of the barycentric homomorphisms constitute an unessential identifier in R, and to improve certain results obtained by Radó. In order to establish these results, explicit formulas for the barycentric homomorphisms and for a modified barycentric homotopy operator are developed. (Received December 18, 1950.)

J. W. T. YOUNGS,
Associate Secretary

THE FEBRUARY MEETING IN NEW YORK

The 466th meeting of the American Mathematical Society was held at Columbia University, New York City, on Saturday, February 24, 1951. The meeting was attended by over 200 persons, including the following 189 members of the Society:

Milton Abramowitz, M. I. Aissen, E. J. Akutowicz, C. B. Allendoerfer, R. D. Anderson, R. L. Anderson, T. W. Anderson, R. G. Archibald, M. C. Ayer, K. Y. Bal, Iacopo Barsotti, M. F. Becker, E. G. Begle, H. B. Belck, A. H. Berger, Stefan Bergman, Lipman Bers, Nicholas Bilotta, D. W. Blackett, W. E. Bleick, Raoul Bott, Samuel Bourne, A. D. Bradley, F. E. Browder, C. T. Bumer, J. H. Bushey, Jewell H. Bushey, F. P. Callahan, W. R. Callahan, H. E. Campbell, P. G. Carlson, Paul Chessin, Joshua Chover, W. L. Chow, K. L. Chung, F. E. Clark, E. A. Coddington, L. W. Cohen, R. M. Cohn, Philip Cooperman, Natalie Coplan, L. M. Court, W. H. H. Cowles, H. F. Cullen, P. M. Curran, J. H. Curtiss, Vittorio Dalla Volta, G. B. Dantzig, M. D. Darkow, R. J. De Vogelaere, J. A. Dieudonné, Avron Douglis, Nelson Dunford, Aryeh Dvoretzky, Joanne Elliott, J. M. Feld, W. E. Ferguson, F. A. Ficken, D. T. Finkbeiner, R. S. Finn, J. B. Freier, Gerald Freilich, Bernard Friedman, T. C. Fry, Bent Fuglede, David Gale, Abe Gelbart, Irving Gerst, B. P. Gill, G. H. Gleissner, Sidney Glusman, H. E. Goheen, Oscar Goldman, Laura Guggenbuhl, Harish-Chandra, G. A. Hedlund, Alex Heller, Erik Hemmingsen, A. D. Hestenes, T. W. Hildebrandt, W. M. Hirsch, A. J. Hoffman, S. P. Hoffman, Banesh Hoffmann, Heinz Hopf, S. T. Hu, E. M. Hull, T. R. Humphreys, L. C. Hutchinson, W. S. Jardetzky, M. L. Juncosa, R. V. Kadison, Shizuo Kakutani, S. N. Karp, M. E. Kellar, D. E. Kibbey, J. F. Kiefer, H. S. Kieval, Morris Kline, E. G. Kogbetliantz, E. R. Kolchin, Saul Kravetz, Wouter van der Kulk, Benjamin Lepson, Howard Levi, Norman Levinson, Charles Loewner, E. R. Lorch, S. S. McNeary, L. A. MacColl, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, Murray Mannos, A. J. Maria, M. H. Maria, H. G. Mazurkiewicz, A. N. Milgram, Joseph Milkman, J. M. Miller, K. S. Miller, W. H. Mills, C. R. Morris, T. S. Motzkin, W. R. Murray, D. S. Nathan, J. D. Newburgh, Katsumi Nomizu, P. B. Norman, A. M. Peiser, R. S. Phillips, E. L. Post, Walter Prenowitz, F. M. Pulliam, J. F. Randolph, H. E. Rauch, Moses Richardson, R. E. Roberson, J. H. Roberts, Robin Robinson, S. L. Robinson, I. H. Rose, Saul Rosen, R. A. Rosenbaum, Maxwell Rosenlicht, J. E. Rosenthal, H. D. Ruderman, J. P. Russell, Bernard Sachs, James Sanders, Arthur Sard, L. R. Sario, A. T. Schafer, Henry Scheffé, Leonard Schieber, Pincus Schub, Abraham Schwartz, B. L. Schwartz, Seymour Sherman, James Singer, P. A. Smith, J. J. Sopka, G. L. Spencer, F. M. Stewart, Walter Strodt, R. L. Swain, P. M. Treuenfels, C. A. Truesdell, J. W. Tukey, Annita Tuller, A. H. Van Tuyl, D. H. Wagner, R. M. Walter, W. R. Wasow, M. A. Weber, J. V. Wehausen, H. F. Weinberger, B. A. Welch, David Wellinger, M. E. Wells, J. G. Wendel, M. E. White, Albert Wilansky, Jacob Wolfowitz, Albert Wolinsky, E. S. Wolk, M. A. Woodbury, N. J. Zabb, H. J. Zimmerberg.

Professor Heinz Hopf of the Swiss Federal School of Technology and Princeton University delivered an invited address *On complex and almost complex manifolds* at the general session presided over by Professor G. A. Hedlund. It was announced that the Duke Mathematical Journal had lifted its "moratorium" on the acceptance of manuscripts.

The sessions for contributed papers in the morning were presided over by Dr. A. D. Hestenes and Professor Bernard Friedman. Dr. T. S. Motzkin presided at the afternoon session.

Abstracts of the papers presented are listed below, those with a "*t*" after their numbers having been read by title. Papers numbered 188, 213, 215 were presented by Professor Gale, Dr. Gerst, and Dr. Motzkin, respectively. Mr. Gaier was introduced by Professor J. F. Randolph.

ALGEBRA AND THEORY OF NUMBERS

202*t*. Eckford Cohen: *A theorem on polynomial sums.*

Let r be a positive integer and let A, B, F be polynomials over $GF(p^n)$ of degree less than r with A, B primary, $(A, B)=1$. The author considers the problem of determining the number of solutions N of $F=AU+BV$ in polynomials U, V with deg $AU<r$, deg $BV<r$. Placing $a=\deg A$, $b=\deg B$, $r=a+b+s$, the following result is proved: Let X denote the unique polynomial of degree less than a satisfying $BX\equiv 1 \pmod{A}$. Then if $s\geqq 0$ it follows that $N>0$ if and only if deg $(FX \pmod{A})<a-s$, in which case $N=1$. If $s<0$, then $N=p^{-ns}$. The result for $s<0$ was proved earlier (Duke Math. J vol. 14 (1947) pp. 253–254); the present proof is based on the fundamental theorem of arithmetic functions (L. Carlitz, Duke Math. J. vol. 14 (1949) p. 1124). (Received January 10, 1951.)

203*t*. Harvey Cohn: *Stable lattices.*

The author unifies some of his earlier work (Bull. Amer. Math. Soc. Abstracts 57-1-15, 57-1-16, and 57-2-82), with recent researches of Mahler on star bodies and earlier researches of Korkine and Zolatareff. Let $x_i=\sum a_{ij}m_j$ be a lattice, m_j integral, with $\det=|a_{ij}|>0$, $1\leqq i, j\leqq n$. Let $\phi(x_i)$ be a homogeneous function of degree h. Define $F^{(k)}=F(m_j^{(k)}; a_{ij})=\phi(x_i^{(k)})/|a_{ij}|^{h/n}$ for a set of n-tuples $(m_j^{(k)})$. Consider the differentials $d|F^{(k)}|$ in the variables a_{ij} about a fixed value of a_{ij}. They form a set whose dimension is determined by the set of $(A^{(k)})$ for which $\sum A^{(k)}d|F^{(k)}|=0$. If the variable k indexes *all* n-tuples, the dimension is independent of the lattice and is called Q, the *free dimension*. If a set of $Q+1$ n-tuples are such that $|F^{(k)}|$ equals its minimum $\mu(a_{ij})$ for the lattice and such that the (one-dimensional) family of $A^{(k)}$ are proportional to positive constants, then the lattice is called *stable*. The $A^{(k)}$ are also determined by $\sum A^{(k)}R_{ij}^{(k)}=0$, $1\leqq i, j\leqq n$, where $R_{ij}=x_i\partial|\phi|/\partial x_j-(h/n)|\phi|\delta_{ij}$. A measure of the distance between two lattices can be defined so that for stable lattices (a_{ij}^*), for constants ϵ and ρ, $\mu(a_{ij}^*)\leqq\mu(a_{ij})-\rho\|a_{ij}-a_{ij}^*\|$ if $\|a_{ij}-a_{ij}^*\|<\epsilon$. The last result has a partial converse. Minkowski's extremal lattices are in general not stable. (Received January 8, 1951.)

204. David Gale and F. M. Stewart: *Infinite games with complete information.*

Von Neumann and Morgenstern have defined the notion of a finite, two-person, zero-sum game with complete information and have proved the fundamental theorem to the effect that all such games are strictly determined. It is shown here that the theorem no longer holds if the game is allowed to be infinite. The following simple example illustrates the situation: The unit interval $[0, 1]$ is partitioned into two sets S_{I} and S_{II}. Players I and II alternately pick a digit from 0 to 9 thus giving rise to an infinite sequence of digits which corresponds in an obvious way to an infinite decimal, hence a number on $[0, 1]$. The winner is player I or player II as this number lies in S_{I} or S_{II}. It is shown that S_{I} and S_{II} may be so chosen that this game is not strictly determined! In the other direction some general theorems on infinite games show that if S_{I} and S_{II} are sufficiently well-behaved (for example, open or closed sets) then the game is determined. (Received December 15, 1950.)

205*t*. Oscar Goldman: *Hilbert rings and the Hilbert Nullstellensatz.*

For commutative rings with unit element, two notions of radical have been defined: the ideal of nilpotent elements and the intersection of the maximal ideals. A ring R is called a *Hilbert ring* if for every ideal I in R, the two notions of radical coincide in R/I. A homomorphic image of a Hilbert ring is one also. Furthermore, R is a Hilbert ring simultaneously with $R[x]$, x transcendental over R. From this it follows that $k[x_1, \cdots, x_n]$ is a Hilbert ring, when k is a field. An important characterization of this class of rings is the following: R is a Hilbert ring if, and only if,every maximal ideal in $R[x]$ contracts to a maximal ideal in R. This implies immediately the Hilbert Nullstellensatz. (Received November 21, 1950.)

206*t*. Stanley Katz: *On the representation of powerfree integers by systems of polynomials.*

Take c integers $r_i \geqq 2$, and corresponding polynomials $f_i(x)$ with integral coefficients, $1 \leqq i \leqq c$. Let each $f_i(x)$ have as divisors exactly γ_i distinct monic rationally irreducible polynomials $g_{ij}(x)$ of positive degree with rational coefficients, where $g_{ij}(x)$ is of degree t_{ij} and appears in the factorization of $f_i(x)$ to the power d_{ij}, $1 \leqq j \leqq \gamma_i$. Set $\sigma = \max_{1 \leqq i \leqq c, 1 \leqq j \leqq \gamma_i} \{t_{ij}/-[-r_i/d_{ij}]\}$, where $-[-\theta]$ is the smallest integer greater than or equal to θ. Denote by $\mu_r(N)$ that function of the integer N which is 0 if N is divisible by the rth power of a prime and 1 otherwise. Then, if $\phi(n)$ is any complex-valued function of the integer n with positive integral period, it is shown that $\sum_{1 \leqq n \leqq z} \phi(n) \prod_{1 \leqq i \leqq c} \mu_{r_i}\{f_i(n)\} = \lambda z + o(z)$ as $z \rightarrow \infty$, provided $\sigma \leqq 1$. The constant λ is given explicitly by an absolutely convergent series, and estimates are given for the error term $o(z)$. Also, if the periodic function $\phi(n)$ is real and non-negative, it is shown that $\lambda > 0$ if and only if $\phi(\nu) \prod_{1 \leqq i \leqq c} \mu_{r_i}\{f_i(\nu)\} > 0$ for some integer ν, provided again that $\sigma \leqq 1$. (Received January 5, 1951.)

ANALYSIS

207. M. I. Aissen: *Cyclically-ordered sets.*

The classical separation theorems concerning the zeros of pairs of consecutive orthogonal polynomials are generalized. The methods involve a Sturm-type analysis of certain linear difference equations. The concept of cyclically-ordered sets is introduced to simplify the proofs. A family of k finite sets of real numbers is said to be *cyclically-ordered* if and only if they are disjoint in pairs, and if any k consecutive elements of their union contain one element from each of the sets. Some of the theorems proved are the following. In a set of symmetric orthogonal polynomials (each polynomial is either an odd or an even function) the sets consisting of the *positive* zeros of any *three* consecutive polynomials are cyclically-ordered. If we define the polynomials $p_n^{(k)}(t)$ by the generating function $\exp(x + tx^k) = \sum_{n=0}^{\infty} p_n^{(k)}(t) x^n$, then the sets consisting of the zeros of any $k+1$ consecutive polynomials (k is fixed) are cyclically-ordered. This is used to show that, for fixed k, if $p_n^{(k)}(t) \geqq 0$ for all n, then $t \geqq 0$. (Received January 15, 1951.)

208*t*. Stefan Bergman: *Representation of singular solutions of a linear differential equation.*

The author considers a solution of an equation of the form $L(\psi) \equiv \Delta\psi + 4N(\lambda)\psi = 0$, where N belongs to a certain class. Let $\lambda_0 < 0$, and $\lambda' = \lambda - \lambda_0$. If, for a solution $\psi(\lambda', \theta)$, $\psi(0, \theta) = \kappa_1(\theta) = \sum \beta_n \theta^n$ and $\psi_{\lambda'}(0, \theta) = \kappa_2(\theta) = \sum \gamma_n \theta^n$ are given, the author determines the associate $g = \sum (\mu_{n,1} + i\mu_{n,2}) Z'^n$, $Z' = \lambda' + i\theta$, of the second kind. One then obtains $\mu_{2n,1} = (-1)^n \beta_{2n}$, $\mu_{2n+1,2} = (-1)^{n+1}\beta_{2n+1}$, $\mu_{2n+2-k,k} = \sum_{\nu=0}^{n} D_\nu S_\nu$, $k = 1, 2$, $D_\nu = (-1)^n \gamma_{2n/(2n+1)} - \sum_{j=1}^{n} S_j^* \beta_{2n-2j}$, where S_n and S_ν^* are certain constants which depend upon N. ψ is regular in every domain $B \in [(\lambda - \lambda_0)^2 + \theta^2 < 4\lambda^2, \lambda < 0]$ in which g is regular. If g has poles or branchpoints, in B, ψ has pole-like singularities or branch points of the second kind. If the $\mu_{n,k}$, $k = 1, 2$, satisfy certain conditions of the Hadamard type, ψ can be represented in B in the form $\psi = \int_{t=-1}^{1} E(Z', \bar{Z}', t)[f_1(Z'(1-t^2)/2) \cdot (f_2(Z'(1-t^2)/2))^{-1}]dt$, where f_1 and f_2 are analytic functions which are regular in B, and E (the generating function of the operator) depends only on N. (Received February 23, 1951.)

209*t*. Stefan Bergman: *The initial value problem in the large for differential equations with singular coefficients.*

Using the integral operator of the second kind, the initial value problem in the large for the solutions of the equations $L(\psi) \equiv \Delta\psi + 4N(\lambda)\psi = 0$ is considered. Here, $N(\lambda) = (-\lambda)^{-1}[-(1/12) + \sum_{\nu=1}^{\infty} B_\nu (-\lambda)^{2/3\nu}]$, $B_1 \geqq 0$, is a (real) analytic function for $-\infty < \lambda < 0$, such that $[-\int_{-\infty}^{\lambda} N(t)dt]$ exists for all $\lambda < 0$, and satisfies some other additional conditions. The author determines from the given initial data $\psi(0, \theta) = \chi_1(\theta)$, $\lim_{\lambda \to 0}[(-\lambda)^{1/3} \partial(\lambda, \theta)/\partial\lambda] = \chi_2(\theta)$, the domain B situated in $[|\lambda| < 3^{1/2}|\theta|, \lambda < 0]$, where the solution $\psi(\lambda, \theta)$ is regular. This domain depends only on χ_1 and χ_2, but is independent of the B_ν's. The author also gives sufficient conditions that ψ has singularities on some points of the boundary of B. The proof consists in reducing the above problems to the question of determining the regularity domain of an analytic function $f(z) = \sum a_n z^n$ from the coefficients a_n. These results can also be interpreted as theorems for the behavior of the solution of the initial value problem in the large for certain equations of mixed type, with the initial data given on the transition line. (Received February 23, 1951.)

210*t*. S. D. Bernardi: *A new upper bound for the fourth coefficient of a schlicht function.*

Let (S) denote the class of functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$, which are regular and schlicht for $|z| < 1$. It is proven that if $f(z) \subset (S)$ then $|a_4| < 4.0891$. This is an improvement upon Friedman's estimate (see B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. vol. 13 (1946)) of $|a_4| < 4.16$. The method used is as follows: Let $f(z) \subset (S)$. Form $[f(z)/z]^{-\alpha/2} = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then by Prawitz (see H. Prawitz, *Über Mittlewerte analytischer Funktionen*, Arkiv for Matematik, Astronomi och Fysik vol. 20 (1927–1928)) it follows that $\sum_{n=1}^{\infty} (2n-\alpha)|b_n|^2/\alpha \leqq 1$, $\alpha > 0$. The author considers $\sum_{n=1}^{4} (2n-2)|b_n|^2/2 = H_1(a_2, a_3, \bar{a}_3, a_4, a_4) \leqq 1$. By a result by Marty (see F. Marty, *Sur le module des coefficients de MacLaurin d'une fonction univalente*, C. R. Acad. Sci. Paris vol. 198 (1934) pp. 1569–1571) if $f(z) \subset (S)$ and has maximum $|a_n|$, then $(n+1)a_{n+1}-2a_2a_n=(n-1)\bar{a}_{n-1}$; letting $n=4$ the author considers $H_1(a_2, a_3, \bar{a}_3, a_4, 2/5a_2a_4-3/5\bar{a}_3)\leqq 1$. Solving for $|a_4|$, the author proves that $|a_4|\leqq H_2(a_2, a_3, \bar{a}_3)$ takes its maximum value when both a_2 and a_3 are real, and hence proceeds to maximize $|a_4|=H_3(a_2, a_3)$ over the Peschl region (see E. Peschl, *Zur Theorie der schlichten Funktionen*, J. Reine Angew. Math. vol. 176 (1937) pp. 61–94) of variability of (a_2, a_3). (Received December 26, 1950.)

211*t*. S. D. Bernardi: *Schlicht functions with integral coefficients in a quadratic field.*

Let $K((-N)^{1/2})$ be the imaginary quadratic extension generated over the rational field by a root of $x^2+N=0$, where $N\geqq 1$ is a rational integer having no square factor. Let $K(1)$ denote the field of rationals. Let (S) denote the class of functions $f(z)=\sum_{n=1}^{\infty}a_nz^n$, $a_1=1$, which are regular and schlicht for $|z|<1$. Let $S[K((-N)^{1/2})]$ denote the class of functions $f(z)\subset(S)$ whose coefficients are algebraic integers in $K((-N)^{1/2})$. It is proven that the class $S[K((-N)^{1/2})]$ is finite for $N\geqq 1$ and consists of exactly forty-five functions of the form $f(z)=z/1-a_2z+(a_2^2-a_3)z^2$, where $|a_2^2-a_3|=0$ or 1, and the roots of $H(z)=1-a_2z+(a_2^2-a_3)z^2=0$ lie among the nonprimitive 24th roots of unity. These functions are explicitly given. The existence of the functions follows from the generalized "area principle" of Prawitz (see H. Prawitz, *Über Mittlewerte analytischer Funktionen*, Arkiv for Matemik, Astronomi och Fysik vol. 20 (1927–1928)), while the proofs for uniqueness follow the methods of Friedman (see B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. vol. 13 (1946)) who found the class $S[K(1)]$. All results apply equally well to the larger class of mean 1-valent functions. For real quadratic fields the class $S[K(-N^{1/2})]$ is infinite for each $N\geqq 2$. (Received December 18, 1950.)

212. R. S. Finn: *A new proof and generalization of a result of Bers.*

A simple criterion on the coefficients is used to characterize a class Ψ of quasilinear elliptic partial differential equations of the form $(\rho\phi_x)_x+(\rho\phi_y)_y=0$ for which a solution, single-valued in a domain D, admits only removable isolated singularities in D. The case $\rho=(1+\phi_x^2+\phi_y^2)^{-1/2}$ has been treated by Bers (Bull. Amer. Math. Soc. Abstract 54-7-239). It is shown that this criterion distinguishes a broad class Ω of nonlinear equations, for which the behavior of a single-valued solution at an isolated singular point is severely restricted. In particular, such a function is bounded at every isolated singular point. This is true also for solutions of equations of a class $\Omega', \Psi\subset\Omega'\subset\Omega$, which are finitely multivalued in a neighborhood of an isolated singular point. (Received November 13, 1950.)

213. Deiter Gaier: *Behavior of power series on the border of summability.*

If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is convergent for $z=1$ to the sum s, a well known Abelian theorem says that $f(z)\rightarrow s$ for $z\rightarrow 1-0$. The author proved some theorems, which include extensions of this Abelian theorem in three directions: (1) Instead of the circle of convergence he takes the border of summability for some transformation methods. (2) Some kinds of complex approach are admitted in addition to real-axis approach. (3) Instead of concluding a convergent behavior of $f(z)$ from convergent behavior of the series he assumes more generally a certain asymptotic behavior of the transformation of the series and infers an asymptotic behavior of $f(z)$. For the transformation of the series the methods of Cesàro, Euler, Borel, and Meyer-König are used.

Similar results are true for Dirichlet's series and Laplace integrals, for which the methods of Riesz and Cesàro are used, respectively. (Received December 27, 1950.)

214t. J. J. Gergen and F. G. Dressel: *Mapping by p-regular functions.*

Let Γ denote a circle with interior S. If $W(z)\in C'$ on S, $z=x+iy$, and if there exists a constant M such that $|W_x+iW_y|\leq M|W|$ on S, then $W\in\mathfrak{M}$ on S. The following are examples of results obtained for functions of class $\mathfrak{M}$. Let $W\in\mathfrak{M}$ on S and C^0 on $\bar{S}=S+\Gamma$; if $W\not\equiv 0$ on S there corresponds to each zero z_0 of W in S a positive integer $k=k(z_0)$ and a number $L=L(z_0)\neq 0$ such that $W/(z-z_0)^k\to L$ $z\to z_0$; if $W\equiv 0$ on an arc of Γ, then $W\equiv 0$ on $\bar{S}$. By means of properties of functions of class $\mathfrak{M}$ the authors extend their previous mapping results for p-regular functions (see Bull. Amer. Math. Soc. Abstract 54-11-464) to the case $p\in C^0$ on $\bar{S}$, $p>0$ on $\bar{S}$, p_x, p_y exist and satisfy uniform Hölder conditions on S. (Received January 8, 1951.)

215. H. E. Goheen: *On the two-point boundary value problem for the equation $u''=f(u)$.*

The author discusses the two-point boundary value problem for the nonlinear ordinary differential equation $u''=f(u)$. Results include a bound for the magnitude of the solution in the interval and a rather remarkable inequality on the gamma function. (Received January 3, 1951.)

216. R. V. Kadison: *Order properties of bounded self-adjoint operators.*

It is well known that the self-adjoint operators in a commutative C^*-algebra (that is, a uniformly closed self-adjoint algebra of operators on a Hilbert space) form a lattice in the usual operator order. S. Sherman [to appear in the Amer. J. Math.] has shown the converse to be true. Some examples were known of pairs of self-adjoint operators which did not have a greatest lower bound in all bounded operators. It is shown that the only time two self-adjoint operators have a greatest lower bound is when they are comparable (that is, one is greater than or equal to the other). A partially ordered system with the property that two elements have a greatest lower bound in the system only if they are comparable is called "an anti-lattice." It is proved more generally that the class of factors (weakly closed C^*-algebras whose centers consist of multiples of the identity operator) is identical with the class of weakly closed C^*-algebras whose self-adjoint elements form anti-lattices. Thus, in a certain sense, the strength of the lattice structure of an operator algebra varies with the degree of commutativity of the algebra. (Received January 8, 1951.)

217. P. D. Lax: *Second order eliptic equations.* Preliminary report.

The alternative for the first boundary value problem is demonstrated: if $L[u]=0$ in D, $u=0$ in D, $u=0$ on B has n linearly independent solutions, then the inhomogenerous problem $L[u]=f$ in D, $u=\phi$ on B has a solution for at least one pair in any $(n+1)$-dimensional subspace of pairs (f,ϕ). L is any second order elliptic operator on an m-dimensional differentiable manifold with Hölder continuous coefficients; the boundary B of D is suitably restricted. The case where the coefficient of u in L is negative is handled first, by the method of superfunctions. For general L one writes $L=M+ku$ where M is of previous type; $u=T[w]$ is determined as the solution of

$M[u]=-kw+f$, $u=\phi$ on B. The solvability of $w=T[w]$ is studied in view of the complete continuity of T in the metric $\|u\|=\max|u|$, a byproduct of the method of superfunctions. These methods can also be used to construct singular solutions and solutions to nonlinear problems. (Received November 10, 1950.)

218. Benjamin Lepson: *Note on hyperdirichlet series with complex exponents.* Preliminary report.

Consider the series (1) $\sum_{n=0}^{\infty}P_n(z)e^{-\lambda_n z}$ and (2) $\sum_{n=0}^{\infty}A_n e^{-\lambda_n z}$, where $P_n(z)$ is a polynomial in the complex variable z of degree μ_n, A_n is the maximum of the moduli of the coefficients of $P_n(z)$, λ_n is complex, and $\lim_{n\to\infty}\mu_n/\lambda_n=0$. The set of points in the plane where (2) is absolutely convergent was shown by Ritt and Hille to be the interior plus a portion of the boundary of a convex domain D. It is shown here that the interior of the set of points where (1) is absolutely convergent is the domain D, and that (1) converges uniformly in any closed bounded set contained in D. Let E be the set of limit points of the roots of the $P_n(z)$, including those roots which appear infinitely often. Then the set of points exterior to D at which (1) converges absolutely is a subset of E of first category in the plane. This generalizes results of Valiron and of the author. (Received January 11, 1951.)

219t. L. B. Robinson: *A complete system of tensors.*

Having computed a complete system of semitensors attached to system (I) (Bull. Soc. Math. France (1940) p. 129), the author will compute a complete system of tensors where $r=2$. Set up a system of Riquier (Ω) whose solutions give the tensors. The tensors are obtained after (Ω) is solved directly. It is possible to obtain the same result by symbolic multiplication like that of Clebsch-Gordan. Let $r=1$. From a system of Riquier three solutions involving I are found, that is, the tensors $f_1\equiv\phi_1I_2-\phi_2I_1$, $f_2\equiv\psi_1I_2-\psi_2I_1$, $f_3\equiv\chi_1I_3$. The Greek letters are functions not containing I. For $r=2$, $f_1\times f_2\equiv\phi_1\psi_1I_{22}-(\phi_2\psi_1+\phi_1\psi_2)I_{12}+\phi_2\psi_2I_{11}$, $f_1\times f_1\equiv\phi_1^2I_{22}-2\phi_1\phi_2I_{12}+\phi_2^2I_{11}$, $f_2\times f_2\equiv\psi_1^2I_{22}-2\psi_1\psi_2I_{12}+\psi_2^2I_{11}$, $f_1\times f_3\equiv(\phi_1I_{23}-\phi_2I_{13})\chi_1$, $f_2\times f_3\equiv(\psi_1I_{23}-\psi_2I_{13})\chi_1$, $f_3\times f_3\equiv\chi_1^2I_{33}$. U is the aggregate of covariants $f_i\times f_j=\Phi_{ij}(U)$, Φ are arbitrary functions. Solve for I_{ij}. This gives a complete system of tensors. (Received January 29, 1951.)

220t. L. B. Robinson: *On a functional integral equation with a bearing on the group theory.*

Consider the equation $u(x, y)=f(x, y)+\lambda\int_1^{x^2}\int_1^{y^2}(1/(1+x_1^2))(1/(1+y_1^2))N(x, y; x_1, y_1)\cdot u(x_1, y_1)dy_1dx_1$. x, y are any points in the complex plane. $f(x, y)$, $N(x, y; x_1, y_1)$ are finite, even at infinity. The author, following the method of Fredholm and also making use of a cyclic and a permutation group, solves the above equation. (Received January 11, 1951.)

221t. Walter Rudin: *Green's second identity for generalized Laplacians.*

Let P denote a point in the plane domain D. Put $\Delta F(P)=\lim_{r\to 0}4(m(F; P, r)-F(P))/r^2$, where $m(F; P, r)$ is the mean of the function F on the circle of radius r about P. Δ^*F, Δ_*F are defined likewise, with lim sup, lim inf in place of lim. Theorem: Suppose (1) U is continuous in D; (2) $\Delta^*U(P)>-\infty$, $\Delta_*U(P)<+\infty$ except possibly on a closed set of capacity zero; (3) there exists a function $y\in L$ on every compact subset of D, such that $y(P)\leq\Delta^*U(P)$ in D; (4) V also satisfies (1), (2), (3), and

$V(P)=0$ outside a compact subset K of D. Then $\int\int_D U(P)\Delta V(P)dP=\int\int_D V(P) \cdot \Delta U(P)dP$. The proof is based on an earlier result concerning generalized Laplacians (Trans. Amer. Math. Soc. vol. 68 (1950) p. 279). (Received January 9, 1951.)

222*t*. Walter Rudin: *Positive infinities of potentials.*

Let R denote Euclidean n-space ($n\geqq 2$). Put $g(r)=r^{2-n}$ ($n>2$), $g(r)=-\log r$ ($n=2$). Theorem: Let E be a closed set of capacity zero in R, let G be an open set containing E. Then there exists a non-negative function f, summable on R, such that the superharmonic function (that is, the potential) $F(M)=\int_R g(MP)f(P)dP$ is infinite at every point of E, is continuous in $R-E$, and is harmonic in $R-\overline{G}$. If this result is compared with a theorem of Evans (Monatshefte für Mathematik und Physik vol. 43 (1936) p. 421), the main difference is that the potential obtained here is that of an absolutely continuous mass distribution. (Received January 9, 1951.)

223. James Sanders: *A class of partial differential equations of the fourth order.*

The equation (1) $L(u)=\sigma_1^2(x)u_{xx}+(\sigma_1(x)\sigma_2(x)\tau_1(y)/\tau_2(y))u_{yy}+\sigma_1\sigma_1' u_x+\sigma_1\sigma_2\tau_1 \cdot (1/\tau_2)'u_y=0$ has been considered by Bers and Gelbart (Trans. Amer. Math. Soc. vol. 56 (1944)). In an analogous manner this paper considers the equation (2) $L'[L(u)]=0$, where L' is derived from L by replacing σ_1 by $1/\sigma_2$ and σ_2 by $1/\sigma_1$. This equation generalizes the biharmonic equation and occurs in mathematical physics. It is shown that every solution of (2) is of the form (3) $u(x, y)=(\int_{x_0}^{x}dx/\sigma_1(x))P_1(x, y)+P_2(x, y)$, where $P_1(x, y)$ and $P_2(x, y)$ are solutions of (1). Conversely, every function of the form (3) is a solution of (2). Next, it is shown that every solution $u\in C^{(4)}$ of (2) in a domain D can be expanded in the neighborhood of every $z_0\in D$ in the form $u(x,y)=\sum_{n=0}^{\infty}\sum_{i=0}^{3}a_{ni}u_{ni}(x, y)$ where the a_{ni} are uniquely determined by u and explicit expressions for the u_{ni} are given. Finally, the solution of the boundary value problem $L'L(u)=0$, $u=f(s)$, $\partial u/\partial n=g(s)$, on the boundary C of D, is shown to be unique. (Received November 13, 1950.)

224. F. M. Stewart: *Uniqueness theorems for the equation $y'=f(x, y)$.*

By considering Lipschitz conditions and their generalizations on certain cones, new criteria for the uniqueness of solutions of $y'=f(x, y)$ are established. One somewhat special, but useful result is: If (i) $\mathfrak{M}$ is the set of all (x, y) at which the Lipschitz constant of f is infinite; (ii) $L(\rho)$ is the infimum of the Lipschitz constants of f at points whose distance from $\mathfrak{M}$ is at least ρ; (iii) $\int_{0+}^{a}L(\rho)d\rho$ converges; (iv) for each (x_0, y_0) neither of the half-lines $x=x_0\pm\theta$, $y=y_0\pm\theta f(x_0, y_0)$, $\theta\geqq 0$, is a half-tangent to $\mathfrak{M}$; then solutions of $y'=f(x, y)$ are unique. This result applies, as previously published theorems do not, to equations like $y'=g(x)+|y|^{\alpha}$ with $g(x)>0$, $0<\alpha<1$. (Received December 26, 1950.)

225*t*. Ferran Sunyer i Balaguer: *Values of entire functions represented by gap Dirichlet series.*

In two former notes (C. R. Acad. Sci. Paris vol. 224 (1947) pp. 1609–1610 and vol. 225 (1947) pp. 21–23) the author pointed out that if the entire function $F(z)$ is represented by a Taylor series which verifies certain gap conditions, the zeros of $F(z)-f(z)$ cannot be exceptional with respect to the proximate order of $F(z)$ by any meromorphic function $f(z)\not\equiv\infty$ of lower order. In the present paper the proof is given that an entire function, represented by a Dirichlet series verifying certain gap conditions, has no exceptional value, with respect to the proximate order (Ritt) of the function, in any strip of width greater than a quantity which depends only on the order (R) of the function. (Received January 9, 1951.)

226. H. F. Weinberger: *The connection between one-dimensional eigenvalue problems with interior boundary conditions.*

By an application of Green's theorem, a method is found for computing the eigenvalues and eigenfunctions of a regular self-adjoint differential problem of the form $p_n(x)u^{(n)}+\cdots+p_1(x)u'+p_0(x)u+\lambda u=0$ with boundary conditions at a finite number of points in terms of the Green's function and eigenfunctions of another problem with the same differential equation and any other self-adjoint boundary conditions at these points. The eigenvalues are determined by the behavior of a determinant of the Green's function and its derivatives at the "boundary points." The corresponding eigenfunctions are linear combinations of the eigenfunctions and of the Green's function and its derivatives. These results are an extension of a paper (Weinberger and Weinstein, *On the connection between the eigenvalues and eigenfunctions of some Sturm-Liouville problems*, Bull. Amer. Math. Soc. Abstract 57-2-138. (Received January 12, 1951.)

227. Albert Wilansky: *On norms of matrix type for* (*c*) *and* (*m*).

We redefine $\|x\| = \sup_n |\sum_k a_{nk}x_k|$ for $x \in (c)$ or $x \in (m)$, and inquire conditions of the matrix A that (c) and (m) be complete with the new norms. A known result (Toeplitz) implies that $\|A\| = \sup_n \sum_n |a_{nk}| < \infty$ is necessary. It is sufficient but not necessary that $\|A^{-1}\| < \infty$. If in addition $\lim_n a_{nk}^{-1}$ (element of matrix A^{-1}) exists for each k, the condition is also necessary; A will then be conservative. If (m) is complete, so is (c). The techniques used are the Banach-Steinhaus theorem and summability theory of the author, see Trans. Amer. Math. Soc. vol. 67, Bull. Amer. Math. Soc. vol. 55. (Received January 12, 1951.)

APPLIED MATHEMATICS

228. René De Vogelaere: *A new family of periodic orbits in the cosmic rays problem: horseshoe orbits.*

The study of the motion of primary cosmic rays leads to a reversible Lagrangian system with two degrees of freedom. The three families first discovered are the equatorial family, the principal one, and the family of ovals; they were investigated mainly by Störmer, Lemaître, Vallarta, Lifschitz, and De Vogelaere. Here is a fourth family studied, two orbits of which were calculated by Störmer. A first approximation of the starting points is obtained by interpolation of the results of the orbits of Störmer and the supposed terminations of the family, the differential equations being solved by numerical integration. Most of the orbits have the form of a horseshoe and, on both sides, the family terminates on a line which corresponds to the equator. The stability and instability of the orbits are determined as well as the form of other periodic orbits near those of the family that have a zero characteristic exponent. (Received January 10, 1951.)

229. A. D. Fialkow and Irving Gerst: *The transfer function of a three terminal network.*

The transfer function of a three terminal (grounded) network is the ratio of the voltage between the output terminal and ground to the voltage between the input terminal and ground. The necessary and sufficient conditions that a real rational function $A(p)$ given by $A(p)=KN/D=K(p^n+a_1p^{n-1}+\cdots+a_n)/(p^m+b_1p^{m-1}+\cdots+b_m)$ be the transfer function of a passive three terminal network containing resistance and capacitance only are: (1) The roots of D are distinct negative numbers. (2) The roots of N may not be positive real but are otherwise arbitrary. (3) $m\geqq n$. (4) The number K satisfies the inequalities $0\leqq K<K_0$ where K_0 is the least of the three quantities K_d, b_m/a_n, 1 if $m=n$ and of the first two quantities if $m>n$. If $K_0\neq K_d$, then K may equal K_0. Here K_d is the least positive value of K (if it exists) for which the equation (*) $D-KN=0$ has a positive double root. (It follows that K_d is a certain root of the equation in K obtained by equating the discriminant of (*) to zero.) An algorithm for the synthesis of the network corresponding to $A(p)$ is given. It is possible to take account of any resistive source and load. (Received November 20, 1950.)

230t. A. D. Fialkow and Irving Gerst: *The transfer function of a four terminal network.*

The transfer function of a four terminal network is the ratio of the voltage at the output terminals to the voltage at the input terminals. The necessary and sufficient conditions that a real rational function $A(p)$ be the transfer function of a passive four terminal network containing resistance and capacitance only are identical with those stated in the preceding abstract except that condition (2) is lacking (that is, the roots of N are arbitrary) and condition (4) must be modified to read: (4′) The number K satisfies the inequalities $-K_0<K<K_0$ where K_0 is the least of the three quantities $|K_d|$, $|b_m/a_n|$, 1 if $m=n$, and of the first two quantities if $m>n$. If $K_0\neq|K_d|$, then K may actually equal $\pm K_0$. Here K_d is that real value of K of smallest absolute value (if it exists) for which the equation $D-KN=0$ has a positive double root. A synthesis procedure is given. Analogous results obtain for networks which contain resistance and inductance only or inductance and capacitance only. (Received November 20, 1950.)

231. G. E. Forsythe and T. S. Motzkin: *Asymptotic properties of the optimum gradient method.*

To minimize a well-behaved real-valued function F of an n-dimensional vector x, start with x_0 sufficiently near to a minimum x^*, and successively determine the $x_{k+1}=x_k-\alpha\,\mathrm{grad}\,F(x_k)$ which minimizes $F[x_{k+1}(\alpha)]$ (Cauchy, 1847; see H. B. Curry, *The method of steepest descent for non-linear minimization problems*, Quarterly of Applied Mathematics vol. 2 (1944) pp. 258–261). Commonly x_k will ultimately behave as though $F(x)=|Ax-b|^2$, where A and b are an appropriate matrix and vector. For $n=3$ the authors prove that x_k-x^* is asymptotically a linear combination of the eigenvectors belonging to the largest and smallest eigenvalue of A^TA (A^T=transpose of A). Furthermore $v_{2k}=\mathrm{grad}\,F(x_{2k})/|\mathrm{grad}\,F(x_{2k})|$ has a limit v^*, and $|v_{2k}-v^*|\to 0$ like some q^k. (Received January 8, 1951.)

232. Jenny E. Rosenthal: *Inapplicability of standard techniques of conformal mapping to certain types of physical problems.*

The method of conformal mapping to determine the potential distribution near the edge of a parallel plate condenser, first used by Helmholtz in 1868, consists of the transformation of two infinite lines into two semi-infinite ones. It has since been given in numerous texts on applied mathematics and mathematical physics. It is shown in the present paper that the Helmholtz "solution" is not unique. Infinitely many functions will transform two infinite lines into two semi-infinite ones. Various classes of such functions are constructed. It is shown that the basic difficulty is due to the fact that while all such "solutions" satisfy the boundary conditions on the condenser plates, none of them gives a physically reasonable potential distribution at infinity. By means of a standard successive transformation it is also shown that the Helmholtz transformation gives spirals as equipotential lines in the case of two intersecting portions of straight lines raised to different potentials, a manifest impossibility physically. This standard transformation is a simple exponential known to transform two parallel infinite lines into two semi-infinite lines diverging from a vertex. (Received January 11, 1951.)

GEOMETRY

233. Iacopo Barsotti: *Local properties of algebraic correspondences.*

Let D be an irreducible algebraic correspondence between the irreducible varieties F, V; if G is an irreducible subvariety of F, let D' be the algebraic correspondence between G and V obtained by "reducing" D. The multiplicity with which a component D^* (having the right dimension) of D' has to be "counted" is shown to be equal to the ratio between the multiplicities of the local rings of D^* and G with respect to any set of parameters of the latter. Other expressions are obtained for this multiplicity, in terms of the ramification of certain valuations. These expressions remain valid when G is fundamental for D, provided that F is analytically irreducible at G. Let W be an irreducible subvariety of V; it is proved (under certain restrictions) that D can be "reduced" to a correspondence between G and W by reducing D' from V to W, and that the result is the same if the two reductions are performed in the opposite order. The method used is algebro-analytical, that is, completion of local rings and valuations; a short proof of the associativity formula for geometric domains is supplied. The results have immediate application (not given in the paper) to the theory of intersection of cycles. The paper closes with a jacobian criterion for multiplicity one (for multiplicity p^m in the language of Weil). (Received January 5, 1951.)

234t. Stefan Bergman. *On models of four-dimensional domains.*

It is useful (as indicated in Amer. Math. Monthly vol. 53 (1946) p. 20) to employ in teaching and research models of domains of four-dimensional space which arise in the theory of functions of two complex variables. The procedures for constructing different geometrical configurations and performing geometrical operations can be represented analytically by certain formulas, for example, various geometrical operations (rotation, stretching, and so on) are equivalent to linear transformations, $X_k = \sum_{\nu=1}^{4} a_{k\nu}x_\nu$, $k = 1, 2, 3, 4$. Evaluation of the above formulas for a great number of points, needed in these procedures, can be performed conveniently by using punch-card machines or other computing devices. The author indicates procedures for the performance of these operations and the preparation of the stills by the use of automatic computing devices such as punch-cards or a differential analyzer. (Received February 7, 1951.)

STATISTICS AND PROBABILITY

235. W. R. Wasow: *On the duration of random walks.*

In a bounded domain B of n-dimensional Euclidean space E consider a discrete random walk that begins at a point x of B. The nature of the random walk is characterized by a transition probability function $F(y, x)$. The duration N_x of the walk, that is, the number of steps before the moving particle leaves B for the first time, is shown to possess, under mild hypotheses, a moment generating function $\phi(s, x)$ that is the unique solution of the integral equation $\int_B \phi(s, y) d_a F(y, x) - e^{-s}\phi(s, x) = \int_{E-B} d_y F(y, x)$. If $F(y, x)$ depends on a parameter μ in such a way that the transition probability is increasingly concentrated around x as $\mu \rightarrow 0$, then $\phi(s, y)$ can, under certain assumptions, be asymptotically approximated by the solution of a boundary value problem for an elliptic differential equation. This equation involves only the first and second moments of $F(y, x)$. It is further shown that the distribution function of μN_x tends, as $\mu \rightarrow 0$, to a solution of a related parabolic differential equation problem. As an application, asymptotic expressions for the variance of N_x in special cases and estimates for the likelihood of very long walks are obtained. (Received December 26, 1950.)

TOPOLOGY

236. R. D. Anderson: *Hereditarily indecomposable plane continua.*

The author demonstrates the existence of a hereditarily indecomposable plane continuum, not separating the plane, which is not homeomorphic to a pseudo-arc (that is, a chained hereditarily indecomposable plane continuum). There are, in fact, uncountably many distinct such continua including one which contains no pseudo-arc and another all of whose proper subcontinua are pseudo-arcs. As a corollary of the above, it follows that there exist uncountably many mutually exclusive continua in the plane such that no one separates the plane and no one is chainable. (Received January 10, 1951.)

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Associate Secretary

BOOK REVIEWS

Sur la fonction noyau d'un domaine et ses applications dans la théorie du transformations pseudoconformes. By Stefan Bergman. (Mémorial des Sciences Mathématiques, no. 108.) Paris, Gauthier-Villars, 1948. 250 fr.

This memoir is the continuation of the book *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques* (Interscience, 1941, and Mémorial des Sciences Mathématiques, no. 106, 1947), by the same author (see Bull. Amer. Math. Soc. vol. 48 (1942)).

The starting point for the considerations outlined in the following is the kernel function $K_G(z, \bar{t})$, $z=(z_1, z_2)$, $\bar{t}=(\bar{t}_1, \bar{t}_2)$, uniquely associated with any given domain G, which has been introduced by the author and used with ever growing success. $K_G(z, \bar{t}) = \sum_{\nu=1}^{\infty} \phi^{(\nu)}(z)\overline{\phi^{(\nu)}(t)}$, where $\{\phi^{(\nu)}\}$ represents an arbitrary system of functions which are analytic in G and complete and orthonormal over G in the L^2 metric. K is a relative invariant under analytic transformations of the four-dimensional (z_1, z_2)-space.

Chapter I contains an investigation of the behavior of $K_G(z, \bar{t})$ on the boundary of G. To this end: (1) K_G is identified as the solution of the following minimum problem: The normalized function K_G minimizes $\int_G |f|^2 d\omega_z$ ($d\omega_z$ is the four-dimensional volume element) under the condition $f(t_1, t_2)=1$, where the value $\lambda_G(t)$ of the minimum is

$$\lambda_G(t) = 1/K_G(t, \bar{t}); \tag{1}$$

(2) K_G is compared with the kernel functions of such domains, both inscribed in and circumscribed about G, that have the property that their kernel functions can be constructed explicitly.

The author lists a number of such domains and their respective kernel functions. Thus, one obtains bounds for K_G; these bounds are then particularly valuable as we approach the boundary of G. Boundary points R are classified according to the smallest number n, such that $[\rho(z)]^n K(z, \bar{z})$ has a limit as the point (z) approaches R. (Here, ρ denotes the euclidian distance from (z) to R.) If R is a point totally pseudo-convex in the sense of E. E. Levi, then $n=3$. Cases are also considered where $n=0$, 1, or 2.

The core of the entire presentation is in Chapter II. By means of minimum problems, two covariant mapping functions are assigned to every region in such a way that to every couple consisting of a domain G and any fixed interior point, there is associated a unique representative domain. All proper (m, p) domains, in the sense of H. Cartan, that is, domains which admit automorphisms of the form

$$z_1' = z_1 e^{im\theta}, \qquad z_2' = z_2 e^{ip\theta}, \qquad (m, p) = 1,$$

are such representative domains. Next, using the properties of the invariant Hermitian form

$$ds^2 = \sum T_{mn} dz_m d\bar{z}_n, \qquad T_{mn} = \frac{\partial^2 \log K(z, \bar{z})}{\partial z_m \partial \bar{z}_n},$$

the author demonstrates the connection between these results and the work of Elie Cartan on homogeneous domains (domains which always admit an automorphism interchanging any two given points).

In Chapter III, a given domain is mapped analytically onto domains interior to a fixed domain A. The kernel function of A then serves to prove bounds for various geometric magnitudes associated with the mapping function.

These distortion theorems are obtained by the use of the following procedure, which is denoted by the author as "the method of the minimum integral." It is shown that various quantities connected with the invariant metric, such as the expression $\sum T_{mn} u_m \bar{u}_n$, the curvature of the metric, and so on, can be expressed by the value of the minimum $\lambda_G(t)$ of the integral $\int_G |f|^2 d\omega$ under various normalization conditions. This fact permits us to obtain bounds for them, using conveniently chosen interior and exterior domains of comparison.

As a special case, choosing $f(t_1, t_2) = 1$ as the normalization condition, one obtains the relation (1), and thus, if $B \supset G$,

$$K_B(t, \bar{t}) \leqq K_G(t, \bar{t}).$$

(Since, in the case of one variable, $ds_B^2(t) = K_B(t, \bar{t}) |dt|^2 \leqq K_G(t, \bar{t}) |dt|^2 = ds_G^2(t)$, the above inequality is the lemma of Schwarz-Pick.)

The usefulness and power of the method of the kernel function is particularly evident if one specializes this approach to the case of one complex variable.

In Chapter IV, it is shown that the domain functions, such as the Green and Neumann functions, $G(z, t)$ and $N(z, t)$, the harmonic measures, as well as functions mapping simply- or multiply-connected domains into canonical domains, can be expressed in an unexpectedly simple manner by the kernel function. In particular, it is shown that $N_B(z, t) - G_B(z, t) = 2\pi k_B(z, t)$, where $k_B(z, t)$ is the kernel function for the Laplace equation.

Of especial interest to the geometer will be the frequent recourse to fundamental domains defined by the invariant metric.

Throughout this memoir, numerous examples are calculated for special domains. Extensive bibliography and detailed indications of original papers are given.

H. BEHNKE

La théorie de la relativité restrainte. By O. Costa de Beauregard. Paris, Masson, 1949. 6+174 pp. 800 fr.

In this little volume, the author presents a textbook of the special theory of relativity, with special emphasis on those aspects of the theory to which he himself has contributed. As a result, he has shown that it is possible to this day to write a textbook on the subject that is not repetitive.

The preface contributed by Professor de Broglie is most illuminating. It appears that the author's principal contribution has been the thorough investigation of three-dimensional integrals in Minkowski space, particularly as regards their transformation properties. While the results are probably well known to differential geometers, their consistent application to physically interesting questions affords the physicist an introduction to relativistically invariant three-dimensional (space-like) integrals.

Space-like integrals are frequent in physical theories. The integral over the electric charge density must be extended over a three-dimensional space-like domain to yield the total charge; the integral over the entropy density in three dimensions gives the total entropy, and so on. The corresponding four-dimensional integrals lack physical significance. In the standard physical literature Schwinger was among the first to discover that one cannot examine the properties of such integrals conveniently if one restricts oneself to plane surfaces. The reason is that in going over from one surface to a neighboring one (connected with each other by means of an infinitesimal transformation) one finds that the variation of the integral consists of terms having the form of a (three-dimensional) volume integral and additional terms that appear in a (two-dimensional) surface integral. Now if the only domains to be considered are space-like coordinate hypersurfaces in Minkowski space, the surface integrals are to be taken at infinity, and the convergence considerations that must be carried out, though feasible, are artificial. It is much more convenient to consider at first neighboring domains that coincide everywhere except in a bounded domain. And that point of view requires the consideration of curved hypersurfaces. All this has, of course, been well known to workers in general relativity, but has been relatively unfamiliar to the community of physicists until the appearance of the Schwinger papers.

In this book the author devotes a great deal of attention to those aspects of special relativity in which the consideration of three-dimensional integrals is suggested by the subject matter, that is, the theory of the electromagnetic field, including continuous charge and current distribution, relativistic thermodynamics, and relativistic fluid dynamics, including the dynamics of viscous and "spinning" fluids. Most of this material appears to be interesting and well presented.

There are altogether five chapters. The first is devoted to historical and mathematical antecedents, the second to relativistic kinematics and (geometrical) optics, the third to relativistic electromagnetic theory, and the last two chapters to relativistic dynamics (including thermodynamics).

In its early portions, the presentation is enlivened by a discussion of all the major experiments bearing on the kinematics and the optics of moving systems. In the latter portions of the work, this reviewer was struck by the phenomenological approach to fluid dynamics and particularly to thermodynamics. In extending thermodynamics to relativity, there is a fundamental conceptual paradox: Relativistic thermodynamics may be carried out in the sense that one considers only the before-after of the two systems interacting (both are isolated before and after the "collision"), but all the results obtainable in that case have already been reported by Planck in 1908; the other approach, chosen by Costa de Beauregard among others, is to treat the thermodynamics of continuous matter, introducing temperature, entropy density, and so on, as *fields*. With the latter approach, there is no local equilibrium, hence no canonical ensemble, and the usual definition of temperature in statistical mechanics breaks down. That is why, in the reviewer's opinion, relativistic thermodynamics, if it is to have physical significance, should be built up from a new approach to statistical mechanics.

Altogether, this work appears a must for workers in any of the fields enumerated above. References to the literature are very complete. This reviewer regretted the absence of an index, but the table of contents is very detailed. Apart from the language difficulties, the differences in notation from those customary in this country and the generous use of rather difficult differential-geometrical concepts militate against its use as a principal text in a course taught in an American university.

P. G. BERGMANN

Funktionentheorie. By C. Carathéodory. 2 vols. Basel, Birkhäuser, 1950.

The appearance of a book by a master of the stature of Carathéodory which is concerned for the most part with the classical aspects of a classical subject is an occasion of great interest. This is particularly so in the present case because it is reported that Carathéodory himself regarded his *Funktionentheorie* as his finest achievement. In apparaising this work it is well to recall that on the one hand, Carathéodory was a mathematician of very broad interests and on the other hand the theory of functions of a complex variable was an ever returning theme in his research.

It is therefore natural to anticipate that the present book would be written from a catholic point of view, that the treatment of general questions of convergence, continuity, and so on, would have a strong "real variable" flavor. This is indeed the case. Throughout the book there is constant reference to his *Reelle Funktionen.* One meets everywhere striking formulations of concepts which conventionally are phrased in other ways. For this reason the *Funktionentheorie* will be of considerable interest to the specialist who likes to compare notes.

Carathéodory's contributions to the theory of analytic functions are many and of lasting importance. We need call to mind only his early work on the Picard theorem which in turn led to the study of the coefficient problem for analytic functions with positive real part, the boundary behavior of conformal maps, the conformal mapping of variable regions, to mention but a few of his contributions. These interests are reflected in the second volume where the theory of bounded analytic functions, conformal mapping, the triangle functions and the Picard theorem are treated.

The author places great emphasis on the geometric point of view. The Weierstrassian aspects find less prominence than is often the case; instead the Riemannian aspect dominates. In fact, to lend point to this geometric tendency, some sixty pages of the beginning of the book are devoted to the geometry of the circle and non-euclidean geometry. In the introduction, Carathéodory says that he regards this chapter of geometry as the best entry to the theory of functions of a complex variable and he cites the role that these methods played in the achievements of Schwarz. In the treatment of this material synthetic and analytic methods are interwoven.

The first volume is divided into five parts. The first of these is concerned with the geometry of the circle, as we have just mentioned. The second part gives a brief resumé of material on convergence, continuity, connectedness, the topology of the plane, line integrals.

Emphasis is placed on the chordal metric and the unity it lends to the study of meromorphic functions.

The third part is concerned with the classical Cauchy theory and its immediate consequences. The maximum principle enters early and is used to establish the interiority of the mapping defined by an analytic function in a manner which is modeled on G. Kowalewski's treatment of continuously differentiable point transformations with nonvanishing Jacobian. The Poisson integral is then studied and the relation between harmonic and analytic functions is developed. In the definition of harmonicity, differentiability demands are reduced to a minimum. This part concludes with an introductory account of meromorphic functions, partial fractions, Liouville's theorem, and the fundamental theorem of algebra.

The author's concept of continuous convergence plays a central role in the fourth part: the generation of analytic functions by limiting processes. Normal families of meromorphic functions are considered and here the theorems of Vitali and Osgood find their place. A short chapter is devoted to power series. The Mittag-Leffler theorem and the calculus of residues terminate the section.

The elementary transcendental functions and the gamma function are the concern of the fifth part. The elementary functions are studied with reference to their classical expansions and the mappings they define. Mention should be made of the elegant treatment of the partial fraction expansions of the cotangent which was communicated orally to the author by Herglotz.

The second volume contains parts six and seven. Part six, *Foundations of geometric function theory*, treats the theory of bounded analytic functions, conformal mapping, and behavior of the mapping function on the boundary. Here one finds the Schwarz-Pick lemma with applications, the Julia lemma, Fatou's theorem (Carathéodory's 1912 proof), the Riemann mapping theorem, the Koebe distortion theorem. The general theory of prime ends is not considered but the mapping of Jordan regions is studied in detail.

The final part, which is introduced by a chapter on functions of several complex variables and differential equations, gives a detailed account of the triangle functions. The Picard theorems are then developed via Landau and Schottky. An account of Carathéodory's 1947 paper (Comment. Math. Helv. vol. 19, pp. 263–278) on the cluster values of meromorphic functions completes the book.

Carathéodory did not aim at writing a comprehensive treatise of modern geometric function theory. Many topics such as the general theories of Riemann surfaces and uniformization are frankly omitted.

On the other hand, it would appear that Carathéodory wished to revive certain classical aspects of the theory of analytic functions which generally do not receive much attention nowadays.

Here is a book which will be of permanent interest not only to the specialist but to all who are inclined to graze in function-theoretic pastures.

MAURICE HEINS

Lezioni de geometria moderna. Vol. 1. *Fondamenti di geometria sopra un corpo qualsiasi.* By B. Segre. Bologna, Zanichelli, 1948. 4+195 pp. 1200L.

This admirable little book comprises a course given by the author at the University of Bologna. It will be followed by two volumes devoted, respectively, to non-linear projective geometry and invariants of birational transformations.

Since an objective is to have the basis of (projective) geometry reflect the great generality achieved in recent years by abstract algebra, almost half of the 180 pages of text (twelve of the seventeen chapters) are exclusively algebraic. In a rapid but clear manner the reader is presented with the essentials of residue classes of integers, groups, rings, corpora and fields, homomorphisms, sub-rings and ideals, zeros and decomposability of polynomials, algebraic and transcendental extensions of fields, finite corpora, and Galois fields.

The word "corpus" (plural, corpora) requires an explanation. The author avoids the contradiction in terms current in English (and other languages) that refers to an algebraic structure which has all the properties of a field except that commutativity of multiplication is not assumed (and may even be denied) as a *noncommutative field.* He calls such a structure a corpus (corpo) and reserves "campo" for a field. The reviewer feels that this terminology might well be generally adopted.

The algebraic preliminaries disposed of, the remaining five chapters proceed at a still brisk but somewhat slower pace. In Chapter 13 a *(right) linear space over a corpus* γ is defined as a set S of "points" in which certain subsets (subspaces) are distinguished, and the following two properties subsist.

I. There is a one-to-one correspondence between the points ξ of S and ordered (right) homogeneous n-tuples $(x_1, x_2, \cdots, x_n)$ of elements of γ (not all zero); that is

$$\xi \sim (x_1, x_2, \cdots, x_n) = (x_1c, x_2c, \cdots, x_nc) \neq (0, 0, \cdots, 0), \qquad c \in \gamma.$$

II. A subspace S' of S consists of all points ξ of S representable by

$$\xi = \xi^{(1)}c_1 + \xi^{(2)}c_2 + \cdots + \xi^{(h)}c_h \qquad (h \geqq 1)$$

where $c_1, c_2, \cdots, c_n$ vary over γ and $\xi^{(r)} \in S$ $(r=1, 2, \cdots, h)$.

When γ is a finite corpus (and $n \geqq 2$) a linear space S over γ is a finite linear space.

Right linear dependence and independence are defined in the usual manner and a subspace has dimension h if it may be defined by $h+1$ linearly independent points. It is seen that if $\lambda, \mu, \nu, \rho \in S_1$, (line) $\nu = \lambda + \mu$, $\rho = \lambda a + \mu b$, and $r = ab^{-1}$, there is attached to the ordered quadruple λ, μ, ν, ρ as cross ratio $(\lambda\mu\nu\rho)$ the class $\{r\}$ of elements of γ consisting of the conjugates $c^{-1}rc$ of r by the nonzero elements c of γ. The correspondence between ρ and the cross ratio $(\lambda\mu\nu\rho)$ is thus one-to-one if and only if γ is a field.

The reviewer found Chapter 14 to be exceedingly well done. It deals with so-called graphic spaces (spazi grafici) and their relations to linear spaces. A set S forms a graphic space of n dimensions $(n \geqq 2)$ provided certain proper subsets (subspaces) are defined, to each of which is attached as dimension a number $<n$ such that the following properties subsist.

I. For each $h = -1, 0, 1, \cdots, n-1$ subspaces S_h of dimension h exist. (S_{-1} is the null subspace and S_0 the subspace of one point.)

II. From $S_h \subset S_k$ follows $h \leqq k$, with the equality holding if and only if $S_h = S_k$.

III. The points common to two arbitrary subspaces S_h, S_k form a subspace S_p. It follows that there is one and only one S_s of minimum dimension containing S_h and S_k.

IV. With the above notation, the dimension formula $h+k=p+s$ holds.

Every linear space is graphic, but not conversely. It is shown that Desargues' Theorem is valid in every graphic space of dimension greater than 2 which is irreducible (that is, each line (S_1) has at least three points). An irreducible graphic space is proved linear if and only if the Desargues Theorem holds in the space (and hence every irreducible graphic space of more than two dimensions is linear). A simple example of a two-dimensional, irreducible graphic space without the Desargues property is given.

The validity of the Pascal-Pappus Theorem in a linear space is proved equivalent to commutativity of multiplication in the base corpus γ, and the Desargues Theorem is shown to be a consequence, in any graphic plane, of the first named theorem.

Collineations, homographies, involutions, correlations are discussed in graphic spaces with the Pascal property, and the book ends with a brief account (8 pages) of finite linear spaces. There are no exercises.

The book treats with clarity and precision an astonishing amount of material, and is a very welcome addition to the literature of the subject.

LEONARD M. BLUMENTHAL

The location of critical points of analytic and harmonic functions. By J. L. Walsh. (American Mathematical Society Colloquium Publications, vol. 34.) New York, American Mathematical Society, 1950. 8+384 pp. $6.00.

As the title indicates, the book is concerned with the critical points of analytic functions $f(z)$ of the single complex variable $z=x+iy$ and of harmonic functions $u(x, y)$ of the two real variables x and y. As is well known, a critical point of $f(z)$ means a zero of its derivative $f'(z)$, and a critical point of $u(x, y)$ means a point where both partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ vanish. The former are the points where the map by $w=f(z)$ fails to be conformal and are the multiple points of the curves $|f(z)|=\text{const.}$ and $\arg f(z)=\text{const.}$ The latter are the equilibrium points in the force field having $u(x, y)$ as force potential and are the stagnation points in the flow field having $u(x, y)$ as velocity potential. Thus the subject matter of the book is one of considerable importance in both pure and applied mathematics.

In this book the analytic functions considered are largely polynomials, rational functions, and certain periodic, entire, and meromorphic functions. The harmonic functions considered are largely Green's functions, harmonic measures, and various linear combinations of them. The interest in these functions centers about the approximate location of their critical points. The approximation is in the sense of determining minimal regions in which lie all the critical points or maximal regions in which lies no critical point.

This book not only has a unity of subject matter, but it also has very nearly a unity of method. The method is based upon the observation that, with $z-a=re^{i\theta}$ and thus $(\bar{z}-\bar{a})^{-1}=r^{-1}e^{i\theta}$, the vector $(\bar{z}-\bar{a})^{-1}$ has the direction of the line segment from the point a to the point z and a magnitude equal to the reciprocal of the length of this line segment. Accordingly, the vector $m(\bar{z}-\bar{a})^{-1}$ may be regarded as force with which a particle of mass m repels a unit particle at z; the sum $F(z)=\sum m_k(\bar{z}-\bar{a}_k)^{-1}$ may be regarded as the resultant force upon a unit particle at z due to the system of discrete masses m_k at a_k, and the integral $J(z)=\int[\bar{z}-\bar{a}(t)]^{-1}dm(t)$ may be regarded as the resultant force at z due a continuous spread of matter. Now, it turns out that the critical points of the various functions studied in this book are the zeros of functional forms $F(z)$ and $J(z)$. Hence, the critical points in question may be sought as positions of equilibrium in the above fields of force.

This method of studying critical points was introduced by Gauss during the first third of the nineteenth century. (The exact date is in doubt, some books including Walsh's taking it as 1816, but others including the reviewer's taking it as nearer 1836.) Gauss identified the critical points of polynomials with the equilibrium points in a field of positive discrete particles. In generalization, Bôcher identified the critical points of rational functions with the equilibrium points in a field of both positive and negative discrete particles. But no one has carried this method quite as far and as skillfully as Walsh. On reading this book, those who have not seen Walsh's earlier work will be amazed at the breadth, depth, and beauty of the results which he is able to derive by this method of force fields.

Walsh's book is divided into nine chapters. The first five cover the polynomials and rational functions. The sixth has to do with certain entire, meromorphic, and periodic functions. The final three chapters deal with harmonic functions.

Chapter I, entitled *Fundamental results*, opens with the definition of terms such as Jordan arc, curve, and configuration, and with a statement of certain theorems needed subsequently, such as the Principle of Argument and Harnack's theorm. The Principle of Argument is then used to derive Rouché's Theorem which in turn is used to derive Hurwitz' Theorem and the theorem on the continuity of the zeros of a polynomial as functions of the coefficients of the polynomial.

After these preliminaries, Chapter I proceeds to the subject matter proper by establishing the four basic theorems due to Gauss, Lucas, Jensen, and Walsh. The *theorem of Gauss* states that every critical point of a polynomial $P(z) = \prod_{j=1}^{p} (z-z_j)^{m_j}$ lies at a position of equilibrium in the field of force due to particles of mass m_j at z_j repelling a unit particle at z according to the inverse distance law. From this theorem there follows at once *Lucas' Theorem* that every critical point of $P(z)$ lies in the smallest convex polygon enclosing the zeros z_j of $P(z)$. When $P(z)$ is real, the consideration of the resultant of the forces due to the equal particles at the two conjugate zeros z_j and $\bar{z}_j$ leads to *Jensen's Theorem* that every non-real critical point of real polynomial lies in at least one of the so-called Jensen circles of $P(z)$. The Jensen circles of $P(z)$ are those whose diameters are the line-segments joining the pairs of conjugate imaginary zeros

z_j and $\bar{z}_j$ of $P(z)$. Finally, there is *Walsh's Theorem* that, if all the z_j, $1 \leq j \leq q$ $(q<p)$, lie in or on a circle C_1 with center at α_1 and radius r_1 and if all the z_j, $q+1 \leq j \leq p$, lie in or on a circle C_2 with center α_2 and radius r_2, then all the critical points of $P(z)$ lie in or on C_1, C_2, and a third circle C_3, with center $(n_1\alpha_2+n_2\alpha_1)/(n_1+n_2)$ and radius $(n_1r_2+n_2r_1)/(n_1+n_2)$, where $n_1=m_1+\cdots+m_q$ and $n_2=m_{q+1}+\cdots+m_p$. This theorem is proved by showing that at any point z outside C_1 and C_2 the resultant force due to the masses m_j at the z_j is the same as that due to a particle of mass n_1 at a suitable point in C_1 and a particle of mass n_2 at a suitable point in C_2.

The last topics discussed in Chapter I are the elementary properties of the lemniscates $|P(z)|=\text{const.}$ and their orthogonal trajectories $\arg P(z)=\text{const.}$

Chapter II, entitled *Real polynomials*, includes also certain nonreal polynomials which possess a real polynomial factor. The chapter opens with some theorems that sharpen Rolle's Theorem as applied to polynomials all of whose zeros are real. Then the author returns to Jensen's Theorem. After restating it in an equivalent form involving equilateral hyperbolas instead of circles, he adds the following result of his own: Let K be a configuration comprised of the segment (a, b) of the real axis $(P(a)P(b)\neq 0)$ and the closed interiors of the Jensen circles that intersect this segment; then, if k z_j lie in K, then $k-1$, k, or $k+1$ critical points lie in K. This result is deduced by a study of the force field and use of the Principle of Argument. Further theorems of the Jensen type are also developed for polynomials of the forms $P(z)=(z^2+1)^k p_1(z)$ where all the zeros of $p_1(z)$ are real, $P(z)=p_1(z)p_2(z)$ where $p_1(z)$ is real and $p_2(z)$ has all its zeros in the upper half-plane, and $P(z)=p_1(z)p_2(z)$ where the zeros of $p_1(z)$ are symmetric in the origin and lie in a double sector R of opening less than $\pi/2$ and the zeros of $p_2(z)$ lie in one of the sectors complementary to R.

In Chapter II, also, Walsh introduces the novel idea of a W-curve for a given polynomial $P(z)$ whose zeros z_j are subject to some prescribed symmetry S. This curve is the locus of the critical points of $P(z)$ when the "multiplicities" of the z_j are allowed continuously to take on all possible values consistent with S. Thus, for $P(z)=(z^2+1)^k(z-a)^m$ with $a>0$, the W-curve is the line-segment $0\leq x\leq a$ of the real axis plus that arc of the circle: $a(x^2+y^2-1)+2x=0$ which lies inside the unit circle. More generally, if the zeros z_j of $P(z)$ fall into two groups G_1 and G_2 each satisfying S, the corresponding W-curve consists of G_1 and G_2 and the set of all points Q at which the force due to G_1 has a direction opposite to the force due to G_2; for, by suitable assignment of multiplicites to the z_j in G_1 and G_2, point Q can be made a point of equilibrium and hence a critical point. This fact leads to the following broad generalization of Lucas' Theorem: If T denotes the set of the W-curves corresponding to all possible groups G_1 and G_2 for the given z_j and the given symmetry S, then no point Q is a critical point of $P(z)$ if it can be joined to the point at infinity by a Jordan arc which does not intersect T.

Chapter III entitled *Polynomials, continued*, falls into three parts. First, there are developed three analogs and extensions of Walsh's Theorem: (1) the analog when the exterior instead of the interior of circle C_2 is used; (2) the analog when two half-planes are used instead of the circles C_1 and C_2; (3) the generalization when, instead of just two circles, there are n circles C_j having a common external (finite or infinite) center of similitude. Secondly, some results analogous to those in Chapter II are developed for polynomials $P(z)$ symmetric in the origin or, more generally, having a k-fold symmetry in the origin. These new results are derived by applying the previous results to the polynomial $p(w)=[P(z)]^k$ where $w=z^k$. Thirdly, a brief study is made regarding the zeros of higher derivatives of a polynomial $P(z)$. By iteration of the results for the first derivative, Lucas' Theorem is found to be valid also for the kth derivative, but in Jensen's Theorem the Jensen circles must be replaced by the ellipses whose minor axes are the line segments joining the pairs of conjugate imaginary zeros z_j and $\bar{z}_j$ of $P(z)$ and whose eccentricity is $(1-k^{-1})^{1/2}$.

Chapter IV, entitled *Rational functions*, opens with Bôcher's extension of Gauss' Theorem to fields containing both positive and negative particles and thus to rational functions $R(z)$. This force field is invariant under linear transformation provided that a particle of suitable mass is placed at the image of the point at infinity if the latter is a zero or pole of $R(z)$. From the fact that the force due to a pair of particles of masses m and $-m$ is directed along the circle through the particles towards the negative mass, one deduces the following (*Bôcher's Theorem*): If C_1 and C_2 are two disjoint circular regions which contain respectively all the zeros and all the poles of a rational function $R(z)$, then C_1 and C_2 also contain all the critical points of $R(z)$.

This theorem, as well as Walsh's Theorem, are given the following beautiful generalization (*Walsh's Cross-Ratio Theorem*): Let C_1, C_2, and C_3 be closed circular regions with C_3 disjoint from C_1+C_2. Let k zeros of a rational function $R(z)$ of degree $n>k$ have C_1 as their common locus, the remaining zeros have C_2 as their common locus and all the poles have C_3 as their common locus. Then the locus of the critical points $R(z)$ consists of C_1 if $k>1$, C_2 if $n-k>1$, the interior points of C_3, and a fourth circular region C_4. This fourth circular region is locus of the point z_4 defined by the cross-ratio $(z_1, z_2, z_3, z_4) = n/k$, as z_1, z_2, and z_3 vary independently over C_1, C_2, and C_3. Walsh's proof of his theorem is quite geometrical.

Chapter IV closes with a statement of *Marden's Theorem.* This is a generalization of Walsh's Cross-Ratio Theorem to rational functions of the form $R(z) = [f_1(z)f_2(z) \cdots f_q(z)]/[f_{q+1}(z)f_{q+2}(z) \cdots f_p(z)]$, where $f_k(z)$ is a polynomial whose zeros all lie in a given circular region C_k. Any critical point of $R(z)$ which lies outside all the given regions C_k lies in a point-set bounded by some or all of the ovals of a certain curve whose equation has the form $A(x^2+y^2)^{p-1}+\phi(x, y)=0$ where in general $A\neq 0$ and where $\phi(x, y)$ is a polynomial in x and y of combined degree less than $2(p-1)$.

Chapter V, entitled *Rational functions with symmetry*, is the longest chapter in the book. It includes some of Walsh's more recent results.

The first symmetry studied is that in the real axis. Somewhat parallel to the earlier sections of Chapter III, the earlier sections of Chapter V deal with rational functions, all of whose zeros are real and with the extension of Jensen's theorem to arbitrary real rational functions and especially to real rational functions whose only poles are $\pm i$.

The second symmetry, that in the unit circle C, is even more interesting as it involves hyperbolic, non-euclidean (abbreviated: NE) geometry. One example is that of a rational function $R(z)$ whose zeros and poles are interlaced on C. For such an $R(z)$ a Lucas-type theorem is shown to hold: namely, that all the critical points within C lie in the NE convex polygon whose sides are the NE lines for C joining consecutive zeros and those joining consecutive poles.

Another example of symmetry in the unit circle is the rational function of the form

$$R(z) = \lambda \prod_{k=1}^{m} \left(\frac{z-\alpha_k}{1-\bar{\alpha}_k z}\right), \qquad |\alpha_k| < 1, \ |\lambda| = 1.$$

For this $R(z)$ a Lucas type of theorem is proved to the effect that all the critical points of $R(z)$ in C lie in the smallest closed NE polygon Π enclosing the α_k. Also, for this same $R(z)$ if real, a Jensen-type theorem is established with the ordinary Jensen circles replaced by circles having as NE diameters the NE line segments joining the pairs of conjugate imaginary α_j. Likewise, for a quotient of two such $R(z)$, a Bôcher type theorem is established; namely, that under certain conditions the critical points in C of the quotient lie within two disjoint NE polygons.

As to the method of proving the results just mentioned, that used for the Lucas-type theorem is typical. A point z_0 is chosen in C outside Π and, through z_0, a NE line L is drawn not containing any point in or on Π. Now region C is mapped upon itself so that z_0 goes into the origin O, L into the real axis, and the α_j into the points α_j' of the upper half plane. At O the force due to unit particles at α_j' and $1/\alpha_j'$ will for all j have a component in the negative direction of the axis of imaginaries and hence O cannot be an equilibrium point. Hence, no point z_0 in C outside Π can be a critical point of $R(z)$.

In addition to rational functions symmetric in the real axis and the unit circle, Chapter V includes a treatment of those symmetric in the origin, those skew-symmetric in the origin, and those symmetric in z and $1/z$. The results are too numerous to mention.

Chapter VI, entitled *Analytic functions*, falls into three parts.

The first part is concerned with the extension of Lucas', Jensen's, and Walsh's Theorems to entire functions $f(z)=\prod_1^\infty (1-z/\alpha_k)$ of genre zero and of Bôcher's Theorem to quotients of two entire functions of genre zero. Also, in the first part, infinite Blaschke products are considered, and a Bôcher type theorem as applied to a double sector is developed for

$$\Psi(z) = \prod_{m=-\infty}^{\infty} \prod_{k=1}^{n} \frac{(z - \rho^m\alpha_k)(1 - \rho^m\beta_k)}{(1 - \rho^m\alpha_k)(z - \rho^m\beta_k)},$$

a function with a multiplicative period of ρ.

In the second part of Chapter VI, conformal mapping is employed to extend the previous results on rational functions over to more general types of functions. For example, let $f(w)$, an analytic function of $w=u+iv$ with a period of $2\pi i$, have a period strip that has no singularity except a pole at its right-hand end point. Then, by mapping the period strip upon a half-plane and by using Lucas' Theorem, one may prove that if all the zeros of $f(w)$ satisfy the inequality $u \leqq c$, or $a \leqq v \leqq b$ with $b-a<2\pi$, so do also the critical points of $f(w)$. In the same manner, a Bôcher-type theorem is derived for meromorphic functions that are simply or doubly periodic.

In the third part of Chapter VI, the possibility of the use of Cauchy's Integral Formula in the investigation of critical points is explored. If $f(z)$ is analytic in a region R bounded by two disjoint Jordan configurations C_1 and C_2; and if $|f(z)|=M_1$ on C_1 and $|f(z)| = M_2$ on C_2 where M_1 and M_2 are constants with $0<M_1<M_2$, then

the Cauchy Integral Formula leads to the representation

$$f'(z)/f(z) = (2\pi i)^{-1}\left\{\int_{C_1}(z-t)^{-1}d\mu_1 + \int_{C_2}(z-t)^{-1}d\mu_2\right\},$$

where $d\mu_k(t) = -d\,[\arg f(z)]$, a quantity positive on C_1 and negative on C_2 with $\int_{C_1} d\mu_1 + \int_{C_2} d\mu_2 = 0$. The critical points are therefore the equilibrium points in the field of force due to a spread of positive matter on C_1 and of negative matter on C_2, with the total mass zero. Consequently, the Bôcher-type theorem is valid; namely, if C_1 and C_2 lie respectively in the disjoint circular regions K_1 and K_2, all the critical points of $f(z)$ also lie in K_1 and K_2.

Chapter VII, entitled *Green's functions*, is devoted largely to the Green's function $G(x, y)$, with pole at infinity corresponding to an infinite simply-connected region R bounded by a finite Jordan configuration B. For such functions is developed the representation

$$G(x_0, y_0) = \int_B (\log r)d\sigma + g,$$

where (x_0, y_0) is a point in R, r is the distance of a variable point on B from (x_0, y_0), g is a constant, and $d\sigma > 0$ with $0 \leqq \sigma \leqq 1$. On introducing the harmonic conjugate $H(x, y)$ to $G(x, y)$ and defining $F(z) = G(x, y) + iH(x, y)$, one may write

$$F'(z) = \int_B (z-t)^{-1}d\sigma(t).$$

Thus, the critical points are the points of equilibrium in the field of force due to a spread of positive matter on B. Therefore, theorems of the Lucas, Jensen and Walsh types are valid. In particular, the Walsh type theorem reads: If $B = B_1 + B_2$ where B_1 and B_2, two disjoint Jordan configurations, lie respectively in the circles C_1: $|z-\alpha_1| = r_1$ and C_2: $|z-\alpha_2| = r_2$, then the critical points lie in C_1, C_2, and a third circle C_3: $|z-(m_2\alpha_1+m_1\alpha_2)| = m_2r_1+m_1r_2$ where $m_1 = \int_{B_1}d\sigma$, $m_2 = \int_{B_2}d\sigma$, and $m_1+m_2 = 1$. Furthermore, if B_1 and B_2 consist respectively of k_1 and k_2 components and if the closed regions C_1, C_2, and C_3 are disjoint, then C_1, C_2, and C_3 contain respectively k_1-1, k_2-1, and one critical point. While the first portion of the theorem may be expected, the second portion shows a surprising contrast with *Walsh's Theorem* for polynomials. For, here the number of components in B_1 and B_2, rather than the total mass on B_1 and B_2, determines the number of critical points in C_1, C_2, and C_3. The latter result, however, follows from a study of the variation in direction of the force on each component of

B_1 and B_2 and the use of the Principle of Argument. Some of the above theorems have a counterpart in the case of Green's functions, with pole at a finite point for a doubly-connected region R containing the point, the counterpart being obtained by a mapping which sends this point to infinity.

Chapter VIII, entitled *Harmonic functions*, falls into two parts.

The first part considers a function $u(x, y)$ which is harmonic in a region R bounded by two sets of mutually disjoint Jordan configurations C and D, is continuous on $R+C+D$, is zero on C, and is unity on D. By definition, $u(x, y)=\omega(z, D, R)$, the harmonic measure of D with respect to R. At any point (x_0, y_0) in R, $u(x, y)$ may be written in the form $u(x_0, y_0)=\int_C \log r\, d\sigma-\int_D \log r d\sigma$, where $d\sigma>0$ and $\int_C d\sigma=\int_D d\sigma$. Now, introducing $v(x, y)$, the harmonic conjugate of $u(x, y)$, defining $f(z)=u(x, y)+iv(x, y)$, and differentiating, one obtains the representation

$$f'(z) = \int_C (z - t)^{-1} d\sigma - \int_D (z - t)^{-1} d\sigma.$$

Hence, the critical points of $u(x, y)$ are the points of equilibrium in the force field due to a spread of positive matter on C and of negative matter on D, with the total mass zero. Therefore, a Bôcher-type theorem is valid: if the disjoint circular regions M and N contain respectively C and D, then they also contain all the critical points of $u(x, y)$ in R with $m-1$ critical points in M and $n-1$ in N where m and n are the number of components of C and D respectively. Equally valid is also a result similar to Walsh's Cross-Ratio Theorem. Furthermore, if C is assumed to consist of a single Jordan curve J_0 and if J denotes the region bounded by J_0 and containing R, the region R may be provided with an NE geometry by mapping J upon the interior of the unit circle. Hence, the smallest convex NE polygon in J containing D must also contain all the critical points of $\omega(z, D, R)$. Finally, these theorems involving the harmonic measures of the closed components of D may be generalized to cover the harmonic measures of a finite number of arcs on D.

The second part of Chapter VII is concerned with linear combinations of Green's functions. In a region R, bounded by a Jordan curve C, there are given the distinct points $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n$. The linear combination

$$w(z) = \sum_1^m \lambda_k G(z, \alpha_k) - \sum_1^n \mu_k G(z, \beta_k)$$

is formed with $\lambda_j \geqq 0$ and $\mu_k \geqq 0$ for all j and k with $G(z, \gamma)$ denoting the Green's function for R with pole at γ. By applying earlier theorems with D as the locus $w(z) = M$, M sufficiently large, and with $u(z) = w(z)/M$, one may deduce, for the critical points of $w(z)$, a Lucas-type theorem in the special case $\mu_k = 0$ for all k and a Bôcher-type theorem in the general case. Furthermore, these theorems may be extended to regions of higher connectivity.

Chapter IX, entitled *Further harmonic functions*, is concerned with more general harmonic measures, linear combinations of harmonic measures and Green's functions, superharmonic functions, and brief applications of some other methods for investigating critical points.

Chapter IX begins by generalizing the previous integral representations. If $u(x, y)$ denotes a function, harmonic in a region R bounded by a finite Jordan configuration B and continuous on $R+B$, and if $f(z) = u(x, y) + iv(x, y)$ where $v(x, y)$ is the harmonic conjugate of $u(x, y)$, then

$$f'(z) = -(2\pi)^{-1}\int_B (z-t)^{-1}dv - i(2\pi)^{-1}\left[u(z-t)^{-1}\right]_B$$
$$+ i(2\pi)^{-1}\int_B (z-t)^{-1}du.$$

Thus, the critical points of $u(x, y)$ in this general case are the points of equilibrium in a force field due not only to a spread $-dv/2\pi$ of ordinary matter on B, but also to a spread $du/2\pi$ of skew matter on B and to skew particles corresponding to the middle term in the above formula. Without first mapping R upon the unit circle as was necessary in Chapter VIII, one may now obtain directly the following theorem. If $B = J + C$ where J is a Jordan curve and C a Jordan configuration disjoint from J, if A is an arc of J and if K is a circle which separates the interior points of A from points on C but not on K and from the interior points of $J - A$, then no critical point of $\omega(z, A, R)$ lies in R on K. This theorem may be extended to the harmonic measure of several arcs of Jordan configurations and, in greater generality, to linear combinations including either or both the harmonic measures of the separate Jordan arcs or configurations and the Green's functions for R with poles at several given points in R.

In the second part of Chapter IX, methods other than that of force fields are studied briefly. For example, the method of symmetry alone leads to regions free of the critical points in the case of functions which assume at a given point set greater values than at the corresponding reflected points. As another example topological methods may be used to prove that, if $u(x, y)$ is harmonic a region R bounded by a Jordan curve J, and continuous on $R+J$, and if $u(x, y) \geqq 0$ on an arc C of J and $u(x, y) \leqq 0$ and $J-C$, then no critical point of $u(x, y)$ lies in R on the locus $u(x, y)=0$.

The final part of Chapter IX deals with the potential of single and double layer distributions. The most striking result concerns a function $u(x, y)$ which is superharmonic in a region R interior to a Jordan curve C and harmonic in a subregion R' of R bounded by C and by a closed set B in R. If, in addition, $u(x, y)$ is continuous near C in $R+C$ and zero on C, then all the critical points of $u(x, y)$ in R' lie in the smallest NE convex set of R containing B. This theorem is a consequence of F. Riesz' integral representation for a function superharmonic in the interior of a unit circle. For, this representation permits $u(x, y)$ to be regarded as the potential of a certain spread of matter on B.

Thus, over the 376 pages of his text, the author adheres, with a remarkable, nearly perfect, consistency, not only to the single objective of studying the location of the critical points of various functions, but also to the single method of regarding these critical points as equilibrium points in fields of force due to suitable distributions of matter. He not only treats each individual topic with a local thoroughness but also indicates its relation to the preceding and succeeding topics. It is obvious that the manuscript and proofs were prepared and examined with unusual care, for the book seems to be not only free of mathematical errors, but also almost entirely of typographical errors. In short, this is a well-organized, thorough treatment of the subject. In the opinion of the reviewer, it is destined to serve, for a long time to come, as the principal reference book on the location of critical points.

Only in a minor way does Walsh's book overlap the reviewer's recent book in the Mathematical Surveys Series, entitled *The geometry of the zeros of a polynomial in a complex variable.* Specifically, the overlapping occurs between parts of Walsh's Chapters I and IV and of the reviewer's Chapters I, II, III and V. Such overlapping is of course unavoidable in any book aiming to be self-contained and in Walsh's book is not objectionable since Walsh usually treats the duplicated material by methods different from those of the reviewer.

With an exposition that is clear, complete, and well-illustrated with drawings and examples, the book may be recommended even to the general reader in analysis, geometry, or applied mathematics. Indeed, this reader will find in Walsh's book a refreshing change from the extreme abstractness of some present-day mathematics and perhaps he too will find it encouraging that so much new and important mathematics can still be discovered by relatively elementary methods.

MORRIS MARDEN

Anwendung der elliptischen Funktionen in Physik und Technik. By F. Oberhettinger and W. Magnus. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 55.) Berlin, Springer, 1949. 8+126 pp.

The authors' collection, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik,* which appeared some years ago, is now supplemented by a treatment of those applications of elliptic functions and integrals which arise in the study of a wide variety of physical and engineering problems. With the exception of some conformal maps, none of the results of the theory of elliptic functions are proved. All formulas used in the applications are, however, collected in the first chapter and, wherever desirable, have been supplemented by useful comments.

In the first chapter the authors study elliptic integrals of the first and second kind in Legendre's normal form, as well as the corresponding complete integrals; the elliptic integral of the third kind is not considered. A variety of expansions and transformation formulas are listed, together with a large number of integrals reducible to them. This is followed by the four theta-functions and their properties and by a similar treatment of the Jacobian elliptic functions. A somewhat briefer treatment is accorded the Weierstrass theory. The second chapter deals with the conformal mapping of ellipses and certain types of polygons. The third chapter is devoted to a large number of examples of electrostatic distributions in two dimensions which may be treated by means of elliptic functions. The fourth chapter deals with similar applications to problems in fluid dynamics. In particular, there are some problems on wind tunnels, such as the airfoil in a wind tunnel of elliptic cross-section. The fifth chapter is a collection of various unrelated physical problems, such as the pendulum and the potential due to a charged ellipsoid. Finally, the authors consider a problem of Chebyshev approximation which leads to elliptic functions.

A short, but useful, bibliography follows each chapter and the book concludes with a short set of tables of the Legendre integrals of the first and second kind.

The book should prove valuable for physicists and engineers with a limited mathematical background who wish to acquire quickly a working knowledge of elliptic functions.

W. SEIDEL

Space-time structure. By Erwin Schrödinger. Cambridge University Press, 1950. 8+119 pp. $2.75.

In this book Professor Schrödinger reviews some of the main ideas underlying Einstein's theory of gravitation and the mathematical apparatus for expressing these ideas. The entire book is written in a clear and interesting manner. The main ideas of differential geometry which are needed for an understanding of the theory of relativity are expounded in a simple and lively fashion. The author develops his subject in "three stages, namely, (1) when only general invariance is imposed; (2) when in addition an affine connection is imposed; (3) when this is specialized to carry a metric." The discussion is organized in three parts corresponding to these stages.

Part I is concerned with tensor algebra and invariant integrals. Part II deals with covariant differentiation, parallel displacement, the curvature tensor, geodesics, and a chapter on the hypotheses about gravitation. Part III deals with affine connections derivable from metrics, the meaning of the metric according to the special theory of relativity, conservation laws and variational principles, and generalizations of Einstein's theory.

The last chapter deals with two recent attempts to formulate a unified field theory, one by Einstein and one by the author himself. These theories are given in a brief outline form and fundamental questions concerning them are not discussed. Thus the physical interpretations of the various quantities entering in these theories is not stated nor is there any discussion of the consistency of the equations resulting when the field equations are supplemented by conditions which would determine quantities left arbitrary in the existing theory. This latter question is of paramount importance in the latest formulation of the "Einstein-Strauss-theory." (See Appendix II in *The meaning of relativity*, 3d ed., by Albert Einstein, reviewed below.)

In spite of the short treatment of the newer aspects of the subject which still abounds with difficult and unresolved questions, this is an interesting and stimulating book, especially for the "general" reader.

A. H. TAUB

BRIEF MENTION

Moderne Algebra. By B. L. van der Waerden. Vol. 1, 3d. ed. Springer, Berlin, 1950. 8+292 pp.

Modern algebra. By B. L. van der Waerden. Vol. 1. Trans. by Fred Blum. New York, Ungar, 1949. 12+264 pp. $5.50.

Modern algebra. By B. L. van der Waerden. Vol. 2, Trans. by T. J. Benac. New York, Ungar, 1950. 9+222 pp. $5.00.

Volume 1 of the third German edition of this basic text remains in essence the same as in the previous editions. Sections 42 and 60 have been dropped from the second edition; some simplifications and fuller explanations have been introduced in the definitions of polynomials and norms and traces; some exercises are modified or replaced by new ones. But there are two major changes: (1) the reintroduction of the axiom of choice, well-ordering, and transfinite induction (as in the first edition) with the resulting elimination of denumerability hypotheses for the existence of algebraic closures and real closed fields and finiteness hypotheses in the Steinitz theory of extension fields; (2) an expansion of the theory of valuations with more detail on valuations of algebraic extensions (in particular of the rational field) and the addition of three sections on valuations of function fields.

The two volumes in English are a translation of the second German edition (1937) with the exception of some additions and revisions by the author in vol. 1 on the subjects of polynomials (as in the German edition treated above) and valuations of algebraic number fields. The translation of vol. 2 is very readable. Unfortunately the translation of vol. 1 suffers from clumsiness of English style, some literal translations of German terms where standard English terms would be preferable or necessary, occasional inconsistencies, and some simple errors. The reader would be warned especially about chap. 10 which is full of too literal translations, particularly the erroneous use of "perfect" for "complete"—and this error is not made consistently. In any case, van der Waerden is now available to the student who reads only English, though he will miss some of the lucidity in vol. 1.

DANIEL ZELINSKY

Généralités sur les probabilités. Éléments aléatoires. By Maurice Fréchet. 2d ed., rev. and enlarged, with a note by P. Lévy. Paris, Gauthier-Villars, 1950. 16+355 pp.

The second edition of this book is essentially the same as the first [reviewed in Bull. Amer. Math. Soc. vol. 43 (1937) pp. 602–603] aside from more complete treatments of a few points and a new chapter outlining the theory of abstract valued random variables. Although it lacks a systematic and unifying approach, the book remains a useful source book, containing valuable material on many parts of probability theory.

J. L. Doob

An introduction to the calculus of variations. By C. Fox. Oxford University Press, 1950. 8+271 pp. $4.50.

This book is designed as a text for undergraduate students. It includes a large number of examples, and devotes Chapters 5, 6, and 7 to applications to mechanics, relativity, and elasticity. Unfortunately it is not written so as to give the student clear ideas about the calculus of variations. Much of it is phrased in the language of pre-Weierstrassian days. A basic difficulty is the author's failure to define what is meant by a weak relative extreme. This leads him to state proofs of sufficient conditions for such extrema which are quite unsatisfactory. An instance of the author's methods of reasoning may be quoted from page 32, where he deduces the equation $f_{y'}=\int_{x_1}^{x} f_y dx+c$ by integrating $d/dx(f_{y'})=f_y$, and so concludes that $f_{y'}$ must be continuous along an extremizing curve. In Chapter 2, Jacobi's transformation of the second variation is carried out without assuming that the required solution of the Jacobi equation does not vanish on the interval (x_1, x_2). On p. 64 and at other places, the author shows ignorance of the conditions for a quadratic form to be definite. His discussion of multiple integrals is exceedingly vague, and can be of no possible use to the students. In particular the Jacobi condition is not properly stated. Reference is made to conjugate curves, with a suggestion of analogy with conjugate points, but no clear statement. In the discussion of isoperimetric problems, there is no proof of the multiplier rule, and there are lacunae in the discussion of the multiplier rule for the problem of Lagrange. Most of the references are to Forsyth's *Calculus of variations.* (Cf. the review of Forsyth's book by G. A. Bliss, Bull. Amer. Math. Soc. vol. 34 (1928) p. 512.)

L. M. Graves

The meaning of relativity. By Albert Einstein. 3d ed., rev. Princeton University Press, 1950. 4+165 pp. $2.50.

The first edition (1923) was reviewed in this Bulletin, vol. 30, p. 71. The second edition (1945) contained an additional appendix discussing certain advances since 1921, and the third edition added a further appendix on the author's generalized theory of gravitation; this appendix has been revised in the present printing.

Oeuvres mathématiques d'Évariste Galois publiées en 1897, suivies d'une notice sur Évariste Galois et la théorie des équations algébriques par G. Verriest. 2d. ed., rev. and corrected. Paris, Gauthier-Villars, 1951. 10+63+56 pp.

This volume contains photographic reproductions of the 1897 edition of Galois's works and of an article by Verriest reprinted from Revue des Questions Scientifiques (4) vol 25 (1934), pp. 341–366; vol. 26 (1934) pp. 12–39.

The French Bibliographical Digest. No. 6. New York, The Cultural Division of the French Embassy, 1950. 3+109 pp.

This issue of the *French bibliographical digest* is devoted to mathematics and astronomy. It lists most of the mathematical books and papers published in France between 1940 and 1948 and contains abstracts of many of these. Copies can be obtained free of charge from the Cultural Division of the French Embassy, 934 Fifth Avenue, New York 21, New York.

NOTES

The Oak Ridge Summer Symposium for 1951 will be held from August 27 to September 6 and will be devoted to *Nuclear engineering*.

Assistant Professor W. E. Barnes of the College of William and Mary is on leave of absence and has accepted a position as mathematician at the U. S. Naval Proving Ground, Dahlgren, Virginia.

Professor F. W. Beal of the University of Pennsylvania has retired with the title emeritus.

Associate Professor Lipman Bers of Syracuse University is on leave of absence and has been appointed to a visiting professorship at the Institute of Mathematics and Mechanics, New York University, for the spring semester of 1951.

Mr. L. F. Boron of the University of Kentucky has accepted a position as mathematician in the Bureau of Ordnance, U. S. Navy Department, Washington, D. C.

Dr. J. M. Danskin of the Office of Naval Operations has accepted a position as mathematician with the Rand Corporation, Santa Monica, California.

Assistant Professor H. F. DeFrancesco of the College of Charleston has accepted a position as analyst with the U. S. Department of Defense, Washington, D. C.

Mr. C. E. Diesen has accepted a position as dynamics engineer with the Bell Aircraft Company, Buffalo, New York.

Professor J. A. Dieudonné of the University of Nancy has been appointed to a visiting professorship at Johns Hopkins University for the spring semester.

Dr. Aryeh Dvoretzky of Hebrew University has been appointed to a visiting professorship at Columbia University.

Assistant Professor W. L. Fields of the University of Louisville has been appointed to an associate professorship at Hampton Institute.

Mr. D. R. Fulkerson has accepted a position with the Rand Corporation, Santa Monica, California.

Mr. G. H. Gleissner of Columbia University has accepted a position as mathematician at the U. S. Naval Proving Ground, Dahlgren, Virginia.

Mr. G. E. Gourrich has accepted a position as research engineer with Northrop Aircraft, Inc., Hawthorne, California.

Mr. J. B. Kelly has been appointed a member of the Institute for Advanced Study.

Miss Louise M. Knifley of Marshall College has been appointed to an assistant professorship at Grand Canyon College.

Dr. Kunihiko Kodaira of the Institute for Advanced Study has been appointed to a visiting associate professorship at Johns Hopkins University.

Assistant Professor F. W. Light of Johns Hopkins University has accepted a position in the Biophysics Section, Medical Division, at the Army Chemical Center, Maryland.

Mr. E. E. Osborne has accepted a position as mathematician with the Institute for Numerical Analysis.

Dr. Clarence Ross of the Naval Proving Ground, Dahlgren, Virginia has accepted a position as mathematician at the Office of Air Research, Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. Herbert Ruderfer has accepted a position as mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Mr. E. L. Schlain has accepted a position as mathematician with the Department of the Army, Washington, D. C.

Mr. L. R. Schlauch has accepted a position as analyst with the Department of Defense, Armed Forces Security Agency, Washington, D. C.

Mr. M. H. Slud has accepted a position as mathematician with the Cornell Aeronautical Laboratory, Buffalo, New York.

Professor B. L. van der Waerden of the University of Amsterdam has been appointed to a professorship at the University of Zurich.

Dr. W. R. Wasow of the National Bureau of Standards, University of California, is on leave of absence and has been appointed a visiting lecturer at the Massachusetts Institute of Technology.

Professor J. W. T. Youngs of Indiana University is on leave of absence and has accepted a position as research consultant for the Atomic Energy Commission at Sandia Corporation, Albuquerque, New Mexico.

Mr. Arthur Zeichner of Oklahoma Agricultural and Mechanical College has accepted a position as associate mathematician at the Reeves Instrument Corporation, New York, New York.

The following promotions have been announced:

W. B. Caton, DePaul University, to an associate professorship.

W. H. Fagerstrom, City College, New York, New York, to an associate professorship.

S. T. Hu, Tulane University of Louisiana, to an associate professorship.

W. J. Wells, State Teachers College, Mankato, Minnesota, to an assistant professorship.

The following appointments to instructorships are announced: DePaul University: Mr. E. P. Merkes; Milwaukee School of Engineering: Mr. C. W. Larson; Tufts College: Mr. Harold Weintraub.

Dr. Daniel Buchanan of the University of British Columbia died on December 1, 1950 at the age of seventy years. He had been a member of the Society for forty years.

Associate Professor Emeritus Otto Dunkel of Washington University died on January 15, 1951 at the age of eighty-two years. He had been a member of the Society for thirty-eight years.

Sister Edward Joseph Hackett of Saint Mary's College, Notre Dame, Indiana, died in November 1950 at the age of fifty years.

Professor G. A. Miller of the University of Illinois died on February 10, 1951 at the age of eighty-seven years. He had been a member of the Society for fifty-one years.

Professor Emeritus W. H. Roever of Washington University died on January 31, 1951 at the age of seventy-six years. He had been a member of the Society for thirty-eight years.

Correction: The announcement of the First National Congress of Applied Mechanics which appeared in the Bulletin, vol. 57 (1951) p. 99, is in error in stating that the American Mathematical Society is a sponsor. The second sentence of the announcement should have read "The U. S. National Committee on Theoretical and Applied Mechanics, on which the American Mathematical Society is represented, is a sponsor."

The following one hundred and fifty-four doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1950 in universities in the United States and Canada. The university, month in which degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

L. U. Albers, Michigan, June, *An application of the Leray-Schauder method to quasi-linear partial differential equations of parabalor type.*

J. J. Andrews, St. Louis, June, *Admissibility theory applied to time series with variable probability.*

M. G. Arsove, Brown, June, *Functions representable as differences of subharmonic functions.*

D. F. Atkins, Kentucky, June, *On pseudo-doubly periodic functions.*

R. G. Ayoub, Illinois, June, minor in physics, *Extensions of the Waring-Siegel theorem in algebraic fields.*

R. R. Bahadur, North Carolina, June, *On a class of decision problems in the theory of R populations.*

Gerald Berman, Toronto, June, *Finite projective geometries.*

C. R. Blyth, California, June, I. *Contribution to the statistical theory of the Geiger-Müller counter.* II. *On minimax statistical decision procedures and their admissibility.*

J. E. Brock, Minnesota, June, *Generalized Blaschke products.*

Eleazer Bromberg, New York, June, *Non-linear bending of a circular plate under normal pressure.*

Marjorie L. Browne, Michigan, February, *Studies of one-parameter subgroups of certain topological and matrix groups.*

Bernard Budiansky, Brown, October, *Fundamental theorems and consequences of the slip theory of plasticity.*

K. A. Bush, North Carolina, August, minor in economics, *Orthogonal arrays.*

A. P. Calderón. Chicago, December, I. *On the ergodic theorems.* II. *On the behavior of harmonic functions on the boundary.* III. *On the theorem of Marcenkiewicz and Zygmund.*

R. L. Calvert, Illinois, October, minor in astronomy, *An analysis of galactic structure in the direction of Aquila.*

R. E. Chamberlin, Harvard, June, *On the mapping of a 4-complex into certain simply connected spaces.*

K. T. Chen, Columbia, June, *Integration in free groups.*

P. P. Chen, Brown, October, *The panel matrix method for analyzing plane structures.*

J. T. Chu, Iowa State, December, minor in physics, *Generalized Hermitean operators in Hilbert space.*

J. H. Chung, Toronto, June, *Modular representations of the symmetric group.*

H. G. Cohen, Brown, October, *Subharmonic synchronization for the forced van der Pol equation.*

H. D. Colson, Minnesota, March, minor in physics, *An existence theorem for a generalized Riemann-Stieltjes integral.*

George Copp, Texas, June, *Some convergence regions for a continued fraction.*

J. B. Crabtree, Harvard, June, *Dual rings and derivations.*

W. S. H. Crawford, Minnesota, July, minor in physics, *The evaluation of a certain class of Wiener integrals.*

Helen F. Cullen, Michigan, February, *A set of parabolic regular curve families filling the plane and certain related Riemann surfaces.*

Felice H. Davidson, Michigan, February, *Algebras with radical; an investigation of the class* QF1–3.

H. C. Davis, Harvard, June, *Lattices and model operators.*

M. D. Davis, Princeton, June, *On the theory of recursive unsolvability.*

J. C. E. Dekker, Syracuse, September, *The constructivity of maximal ideals in Boolean algebras.*

J. E. Denby-Wilkes, California Institute of Technology, June, minor in aeronautics, *Structure of commutative normed rings.*

Allen Devinatz, Harvard, June, *Transformations in reproducing kernel spaces.*

N. J. Divinsky, Chicago, June, *Power-associativity and crossed extension algebras.*

W. L. Doyle, California Institute of Technology, June, minor in physics, *An arithmetical theorem for partially ordered sets.*

H. A. Dye, Chicago, September, *Radon-Nikodým theorems for operator algebras.*

Franz Edelman, Brown, June, *On the coincidence of plasticity solutions obtained with incremental and deformation theories.*

Joanne Elliott, Cornell, June, *On some singular equations of the Cauchy type.*

D. O. Ellis, Missouri, June, *On distance geometries of algebraic structures.*

E. S. Elyash, Cornell, June, minor in physical chemistry, *Several limiting laws of the Kolmogorov-Smirnov type.*

W. B. Evans, Illinois, February, *Uniqueness properties of general monogenic functions.*

Chester Feldman, Chicago, September, *Real Banach algebras.*

J. C. Flack, Kentucky, August, *On reducible plane algebraic curves.*

L. E. Fuller, Wisconsin, June, *The Hermite canonical form for a matrix with elements in the ring of integers modulo m.*

L. M. Fulton, Duke, June, minor in physics, *Decompositions induced under finite-to-one closed mappings.*

J. R. Garrett, Duke, June, minor in philosophy, *Normal equations and resolvents in fields of characteristic p.*

Samuel Goldberg, Cornell, September, minor in genetics, *On a singular diffusion equation.*

J. K. Goldhaber, Wisconsin, June, *Conditions ordering the characteristic roots of matrices.*

N. A. Goldsmith, Illinois, June, minor in education, *Differential invariants of ruled surfaces.*

M. M. Gordon, Washington University, June, *Nucleon-deuteron scattering.*

Daniel Gorenstein, Harvard, June, *An arithmetic theory of adjoint plane curves.*

R. N. Goss, Iowa State, March, minor in mathematical physics, *Center of flexure of beams of triangular cross section.*

L. C. Graue, Indiana, June, minor in physics, I. *A necessary and sufficient condition that a curve lie on a quadric surface.* II. *A necessary condition that a curve lie on a hyperquadric.*

Emil Grosswald, Pennsylvania, February, *On the structure of some subgroups of the modular group.*

N. B. Haaser, Brown, October, *The viscous flow past a flat plate.*

Violet G. Hachmeister, Wisconsin, June, *A study of the parastrophic matrices of a group.*

Max Halperin, North Carolina, August, *Estimation in truncated sampling processes.*

V. C. Harris, Northwestern, May, minor in physics, *A system of linear difference equations and an associated boundary value problem.*

Stuart Haywood, Maryland, June, minor in electrical engineering, *On the structure of locally connected plane continua on which it is possible to define a pointwise periodic homeomorphism which is not almost periodic.*

P. W. Healy, Kentucky, August, *Derivatives and orders of convergence of Fourier sine series.*

Alex Heller, Columbia, June, *Equivariant maps of space with operators.*

W. R. Heller, Washington University, June, *Kinetic-statistical theory of dielectric breakdown in non-polar crystals.*

Abraham Hillman, Princeton, June, *Algebraic and manifold theoretic properties of a single algebraic differential equation.*

A. K. Hinds, North Carolina, June, *On the second order differential system.*

A. J. Hoffman, Columbia, June, *On the foundations of inversion geometry.*

F. S. Holt, Massachusetts Institute of Technology, June, minor in electrical engineering, *Extension of integral equation methods to heat flow problems.*

T. C. Holyoke, Ohio State, June, *An embedding problem for transitive permutation groups.*

R. T. Hood, Wisconsin, June, *On the asymptotic representation of the solutions of linear ordinary differential equations of the third order relative to a large parameter.*

H. M. Hughes, California, September, *Estimation of the variance of the bivariate normal distribution.*

P. E. Irick, Purdue, February, minor in psychology, *A geometric study of the exact sampling distribution of standard deviations when the sampled population is arbitrary.*

S. L. Isaacson, Columbia, June, *On the theory of unbiased tests of simple statistical hypotheses specifying the values of two or more parameters.*

Lloyd Jackson, California, Los Angeles, August, *Sub-functions and elliptic partial differential equations.*

S. L. Jamison, California, September, *Perturbation of normal operators.*

Meyer Jerison, Michigan, February, *The space of bounded maps into a Banach space.*

S. M. Johnson, Illinois, June, minor in physics, *On the representations of an integer as the sum of k n-tuple products of positive integers.*

R. V. Kadison, Chicago, June, *A unified representation theory for topological algebras.*

Kazar Kazarian, California, Los Angeles, June, *Properties of linearly absolutely continuous functions.*

E. R. Keown, Massachusetts Institute of Technology, June, minor in physics, *Hilbert algebras.*

I. I. Kolodner, New York, June, *On the application of the Boltzmann equations to the theory of gas mixtures.*

H. W. Kuhn, Princeton, June, *Subgroup theorems for groups presented by generators and relations.*

K. C. Kuo, Illinois, June, *The imbedding problem for systems with an incomplete, commutative addition.*

R. M. Lakness, California, June, *Green's theorem and subharmonic functions for multiple spaces.*

R. L. Lane, Texas, June, minor in business administration, *Sequences of points in the complex plane.*

J. R. Lee, Yale, June, *Addition theorems in abstract spaces.*

Patrick Leehey, Brown, June, *On the existence of not necessarily unique solutions of the classical hyperbolic boundary value problems for non-linear second order partial differential equations in two independent variables.*

R. B. Leipnik, California, June, *Heaviside operational calculus.*

Benjamin Lepson, Columbia, June, *Series of analytic functions and polynomial Dirichlet series.*

D. J. Lewis, Michigan, June, *Cubic homogeneous polynomials over a p-adic number field.*

J. A. Lewis, Brown, June, *Free convection in commercial insulating materials.*

R. E. Lowney, Wisconsin, June, *A boundary value problem involving an exponential turning point.*

R. D. Luce, Massachusetts Institute of Technology, June, minor in aeronautical engineering, *On semigroups.*

Harold Luxenberg, California, Los Angeles, August, *The torsion of anisotropic elastic cylinders by forces applied on the lateral surfaces.*

Sister Mary Ferrer McFarland, Notre Dame, June, *On quasi-homomorphisms of the Gauss type and their application to the theory of binary quadratic forms.*

J. E. McLaughlin, California Institute of Technology, June, minor in physics, *Projectivities in relatively complimented lattices.*

A. M. MacBeath, Princeton, June, *The geometry of non-homogeneous lattices.*

R. E. MacKenzie, Princeton, June, *Root numbers and class group relations in cyclotomic fields.*

Nathaniel Macon, North Carolina, June, *Some theorems on the approximation of irrational numbers by the convergents of their continued fractions.*

K. S. Miller, Columbia, June, *On iterative methods in linear differential equations.*

E. F. Moore, Brown, June, I. *Convexly generated k-dimensional measures.* II. *Density ratios and* $(\phi-1)$ *rectifiability in n-space.*

D. R. Morrison, Wisconsin, February, *On the extension to rings of the regular matric representations.*

Leo Moser, North Carolina, August, *On sets of integers which contain no three in arithmetical progression, and on sets of distances determined by finite pointsets.*

W. L. Murdock, Cornell, September, *On the distribution of eigenvalues of kernels and matrices.*

J. F. Nash, Princeton, June, *Non-cooperative games.*

S. W. Nash, California, June, I. *Contribution to the theory of experiments with many treatments.* II. *On the law of the iterated logarithm for dependent random variables.*

Alexander Orden, Massachusetts Institute of Technology, June, minor in physics, *Correction of errors in orbital flight.*

Simon Ostrach, Brown, October, *A boundary layer problem in the theory of free convection.*

Margaret Owchar, Minnesota, June, *Wiener integrals of multiple variations.*

A. J. Owens, Florida, minor in physics, *Effect of a rigid elliptic disk on the stress distribution in an orthotropic plate.*

Athanasios Papoulis, Pennsylvania, February, *On the strong differentiation of the indefinite integral.*

L. E. Payne, Iowa State, June, minors in physics and mechanical engineering, *Torsion and flexure of composite sections.*

J. E. L. Peck, Yale, June, *The embedding of a topological semigroup in a topological group and its generalizations, and an ergodic theorem for a non-commutative semigroup of linear operators.*

A. J. Penico, Pennsylvania, February, *The Wedderburn principal theorem for Jordan algebras.*

A. J. Perlis, Massachusetts Institute of Technology, June, minor in physics, *On integral equations.*

A. D. Perry, Massachusetts Institute of Technology, September, minor in electrical engineering, *Conditionally stable feedback amplifiers.*

R. P. Peterson, California, Los Angeles, June, *Certain optimum statistical decision methods.*

D. H. Porter, Indiana, June, minor in physics, *Two-dimensional bounded variation and absolute continuity.*

C. M. Price, Chicago, June, *Jordan division algebras and their arithmetics.*

J. K. Reckzeh, Kentucky, June, *Modification of the first necessary condition in the calculus of variations for the parametric non-homogeneous case.*

Russell Remage, Pennsylvania, June, *Invariance and periodicity properties of non-alternating in the large transforms.*

R. P. Rich, Johns Hopkins, June, *Contributions to the theory of partially ordered groups.*

Florence V. Rohde, Kentucky, August, *On large deflections of beams.*

Saul Rosen, Pennsylvania, February, *Modular transformation of certain series.*

M. A. Rosenlicht, Harvard, June, *Equivalence concepts on an algebraic curve.*

C. H. Rust, St. Louis, June, *On natural boundaries of a generalized Lambert series.*

Judson Sanderson, Illinois, June, minor in electrical engineering, *Study of the generalized potential integral.*

W. C. Sangren, Michigan, February, *Generalized Fourier expansions.*

A. E. Scheidegger, Toronto, June, minor in physics, *Gravitational radiation and equations of motion.*

E. V. Schenkman, Yale, June, *A theory of subinvariant Lie algebras.*

R. W. Schmied, Brown, June, *Limiting forms of Green's functions* *and Harnack's inequality for multiply connected plane regions.*

George Seifert, Cornell, June, minor in physics, *Some third order boundary value problems.*

A. S. Shapiro, Chicago, September, *Cohomology relations in fiber bundles.*

Bernard Sherman, Princeton, June, *A random variable related to the spacing of sample values.*

S. S. Shrikhande, North Carolina, August, *Construction of partially balanced designs and related problems.*

I. M. Singer, Chicago, June, *Lie algebras of unbounded operators.*

J. W. Smith, California, June, *Two classes of self-adjoint boundary conditions for the Laplacian operator.*

J. J. Sopka, Harvard, June, *On the characterization of Reynolds operators on the normed ring of all continuous real-valued functions defined on a compact Hausdorff space.*

J. A. Sullivan, Indiana, October, minor in physics, *Contractions in hyperbolic spaces.*

Wanda Szmielew, California, June, *Arithmetical properties of Abelian groups.*

J. T. Tate, Princeton, June, *Fourier analysis in number fields and Hecke's zeta-functions.*

H. E. Taylor, Rice, June, *Determination of the type and properties of the mapping function for certain classes of Riemann surfaces.*

H. E. Teicher, Columbia, June, *On the factorization of distributions.*

W. B. Thompson, Toronto, June, *Thermal convection and the earth's magnetic field.*

G. D. Torres, Princeton, June, *Some properties of the Alexander polynomial.*

W. A. Vezeau, St. Louis, June, *On the product distribution of normally distributed variables.*

S. A. Vora, North Carolina, June, *Bounds on the distribution of chi-square.*

M. C. Waddell, Johns Hopkins, June, *On properties of regular rings.*

S. S. Walters, California, Los Angeles, January, *Locally bounded linear topological spaces with applications to classes of analytic functions.*

Jack Warga, New York, June, *On the representation of large integers as sums of primes.*

Gerhard Washnitzer, Princeton, June, *A Dirichlet principle for analytic functions of several complex variables.*

H. F. Weinberger, Carnegie Institute of Technology, June, minor in physics, *Fourier transforms of Moebius series.*

G. N. White, Brown, June, I. *Application of the theory of perfectly* *plastic solids to the stress-analysis in strain-hardening solids.* II. *A quantitative comparison of flow and deformation theories of plasticity.*

Alice Winzer, Brown, June, *Discontinuous solutions in problems in the theory of plasticity.*

Henry Wolf, Brown, June, *The propagation of torsional plastic waves in circular cylindrical tubes and shafts.*

D. M. Young, Harvard, June, *Iterative methods for solving partial difference equations of elliptic types.*

J. L. Zemmer, Wisconsin, August, *On the subalgebras of finite division algebras.*

A. D. Ziebur, Wisconsin, June, *The asymptotic solutions of a certain type of ordinary differential equation of the second order, with an application to Whittaker's function* $M_{k,m}(z)$.

NEW PUBLICATIONS

Allcock, H. J., Jones, J. R., and Michel, J. G. L. *The nomogram. The theory and practical construction of computation charts.* 4th ed. New York, Pitman, 1950. 10+238 pp. $3.75.

Aumann, G. See Haupt, O.

Bilimovič, A. See *Euclid's elements.*

Bremmer, H. See van der Pol, B.

Bulgakov, B. V. *Kolebaniya.* Vol. 1. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi-Literatury, 1949. 464 pp.

Četaev, N. G. See *Tablicy značeniĭ funckiĭ Besselya.*

Charrueau, A. *Sur les congruences de droites ou de courbes et sur une transformation de contact lié à ces congruences.* (Mémorial des Sciences Mathématiques, no. 115.) Paris, Gauthier-Villars, 1950. 72 pp. 500 fr.

Churchill, K. V. See *Elasticity.*

Contributions to the theory of games. (Annals of Mathematics Studies, no. 24.) Ed. by H. W. Kuhn and A. W. Tucker. Princeton University Press, 1950. 210 pp. $3.00.

Davis, M. See Landau, E.

Dubyago, A. D. *Opredelenie orbit.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1949. 444 pp.

Elasticity. (Proceedings of Symposia in Applied Mathematics, vol. 3.) Ed. by R. V. Churchill. New York, McGraw-Hill, 1950. 6+233 pp. $6.00.

Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. 1. *Algebra und Zahlentheorie.* Part 1. A. *Grundlagen.* B. *Algebra.* No. 1. Part II. *Schmidt, Arnold. Mathematische Grundlagenforschung.* Leipzig, Teubner, 1950. 48 pp. $1.34.

Esnault-Pelterie, R. *Dimensional analysis and metrology.* Lausanne, Rouge, 1950. 14+112 pp.

Euclid's elements. Book II. Trans. and annotated by A. Bilimovič. Belgrade, Matematički Institut, 1950. 29 pp. (Serbo-Croatian)

Fan, K. *Les fonctions définies-positives et les fonctions complètement monotones. Leurs applications au calcul des probabilités et à la théorie des espaces distanciés.* (Mémorial des Sciences Mathématiques, no. 114.) Paris, Gauthier-Villars, 1950. 48 pp. 400 fr.

Federhofer, K. *Dynamik des Bogenträgers und Kreisringes.* Vienna, Springer, 1950. 12+179 pp. $5.50.

Finikov, S. P. *Teoriya kongruènciĭ.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoreticeskoi Literatury, 1950. 528 pp.

Fox, C. *An introduction to the calculus of variations.* New York, Oxford University Press, 1950. 8+271 pp. $4.50.

Fréchet, M. *Généralités sur les probabilités. Éléments aléatoires.* 2d ed. Paris, Gauthier-Villars, 1950. 16+355 pp.

Ghaffari, A. G. *The hodograph method in gas dynamics.* (University of Teheran. Faculty of Science Publication, no. 85.) Teheran, 1950. 2+4+129 pp.

Gonçalves, J. V. *Curso de álgebra superior.* Vol. 2. Lisbon, 1950. 6+512 pp.

Guinier, G. *Éléments de physique moderne théorique.* Paris, Bordas. Vol. 1, 1949, 160 pp. Vol. 2, 1950, 149 pp.

Hasse, H. *Vorlesungen über Zahlentheorie.* Berlin, Springer, 1950. 12+474 pp. 45.00 DM.

HAUPT, O., AUMANN, G., and PAUC, C. *Differential- und Integralrechung unter besonderer Berücksichtigung neuerer Ergebnisse*. Vol. 2, 2d ed. Berlin, de Gruyter, 1950. 210 pp. 16.50 DM.

HAUSNER, M. See LANDAU, C.

HEMELRIJK, J. *Symmetrietoetsen en andere toepassingen van de theorie van Neyman en Pearson*. Gravenzande, Excelsiors, 1950. 91+4 pp.

HORN, J. *Partielle Differentialgleichungen*. 4th ed. Berlin, de Gruyter, 1949. 8+228 pp. 14.00 DM.

HUMBERT, P. See MCLACHLAN, N. W.

JONES, J. R. See ALLCOCK, H. J.

KEMPERMAN, J. H. B. *The general one-dimensional random walk with absorbing barriers with applications to sequential analysis*. Gravenzande, Excelsiors, 1950. 7+111 +4 pp.

KUHN, H. W. See *Contributions to the theory of games*.

LANDAU, E. *Differential and integral calculus*. Trans. by M. Hausner and M. Davis. New York, Chelsea, 1951. 7+366 pp.

LENSE, J. *Kugelfunktionen*. Leipzig, Akademische Verlagsgesellschaft, 1950. 13+294 pp. 26.00 DM.

LEVI-CIVITA, T. *Le problème des n corps en relativité générale*. (Mémorial des Sciences Mathématiques, no. 116.) Paris, Gauthier-Villars, 1950. 111 pp. 800 fr.

MCLACHLAN, N. W., and HUMBERT, P. *Formulaire pour le calcul symbolique*. 2d. ed. (Mémorial des Sciences Mathématiques, no. 100.) Paris, Gauthier-Villars, 1950. 67 pp. 350 fr.

MCLACHLAN, N. W., HUMBERT, P., and POLI, L. *Supplément au formulaire pour le calcul symbolique*. (Mémorial des Sciences Mathématiques, no. 113.) Paris, Gauthier-Villars, 1950. 62 pp. 450 fr.

MAAK, W. *Fastperiodische Funktionen*. Berlin, Springer, 1950. 8+240 pp. 24.60 DM.

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CLASSES OF TRANSFORMATIONS AND BORDERING TRANSFORMATIONS

D. G. BOURGIN

Introduction. I do not propose to give a complete or closed theory of any of the phases of the subjects I touch on in this address. In fact one of my primary purposes is to indicate what the natural unsolved problems are and what it would be reasonable to expect, and I shall raise as many questions as I answer. Many of my comments bear on results not yet published or on work in progress in connection with a book on fixed point theory. In a general way the first part of my remarks will interpret bordering transformations as those metrically near a canonical type, usually an isometry, while my later remarks will interpret bordering in a homotopy sense.

Our initial considerations will be devoted to the implications of what we may term uniform departures from standard transformations. The first to formulate some problems of this type was Ulam. In order to bring out their interconnections we present them as special cases or modifications of a general form and free the formulation from metric dependence although the metric cases alone have been treated. Suppose then that X and Y are uniform spaces with the vicinities of the diagonals denoted by $\mathfrak{N}$ and $\mathfrak{M}$. We assume the existence of a multiplication in these spaces denoted by $\bigcirc$ and $\oplus$. That is to say, $\bigcirc$ maps $X \times X$ into X, and $\oplus$ maps $Y \times Y$ into Y. Let T, S be transformations on X to Y satisfying

$$(S(x_1 \bigcirc x_2),\ T_{x_1} \oplus T_{x_2}) \subset \mathfrak{M}. \tag{1}$$

We assume also a class of standard transformations from X to Y which we denote by U. We may then formulate a significant problem in the following way: Does there exist a standard transformation U and a vicinity $\mathfrak{N}'$ dependent on $\mathfrak{M}$ alone such that

$$(Tx,\ Ux) \subset \mathfrak{N}', \tag{2}$$

and what is the "best" $\mathfrak{N}'$?

1. **Approximate additivity and isometry.** As illustration we can take for X a Banach space in its norm topology and for Y the real line. Let $S = T$ and let $\bigcirc$ and $\oplus$ be the respective vector addition

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An address delivered before the Evanston meeting of the Society on November 24, 1950 by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors January 27, 1951.

operations. Here U is additive and $\mathfrak{M}$ is the ϵ sphere. In more familiar guise then we rewrite (1) as

$$\|T(x+y) - Tx - Ty\| < \epsilon, \tag{3}$$

and ask if an additive transformation U exists with

$$\|Tx - Ux\| \leqq k\epsilon, \tag{4}$$

where k is a fixed constant for the space. Later references to k invariably imply k satisfies (4). This problem was suggested by Ulam and solved in the affirmative by Hyers [15].[1] He showed k could be taken as 1 and designated T as an "ϵ linear transformation." A direct generalization of this result can be obtained by replacing ϵ by $\psi(x, y)$, a positive monotone nondecreasing symmetric function of $\|x\|$ and $\|y\|$ such that with $(\phi(x) = \psi(x, x), \sum_{i=1}^{\infty} (\phi(2^i x)/2^i) = \mu(x) < \infty$. One can then show by analysis similar to that of Hyers that (4) is valid with $k\epsilon$ replaced by $\mu(x)$. On the other hand (3) and (4) cannot be replaced by

$$\overline{L}_{\|x\|,\|y\| \to \infty} \|T(x+y) - Tx - Ty\| / (\|x\| + \|y\|) = 0, \tag{3'}$$

$$\overline{L}_{\|x\| \to \infty} \|Tx - Ux\| / \|x\| = 0. \tag{4'}$$

Indeed with E the real line and $[|x|]$ the integer part of $|x|$ the choice $Tx = x(1 + \sum_{j=1}^{j=[\log_2|x|]} 1/j)$ or x, according as $|x| \geqq 4$ or $|x| < 4$, is consistent with (3′). Yet (4′) cannot be satisfied. Obvious trifling modification yields a continuous $T(x)$ of this type.

Suppose now that X and Y refer to the same Banach space in its norm topology, that S is the identity operator, that $T\theta = \theta$, that $\mathfrak{M}$ is again an ϵ vicinity, and $\ominus$ and $\oplus$ are the vector difference operations. Then (1) is

$$|\|Tx - Ty\| - \|x - y\|| < \epsilon. \tag{5}$$

The standard transformation U is now an isometry. For convenience we refer to T as an ϵ isometry. The first result here was for X a separable Hilbert space. Again U exists satisfying (4) with $k = 10$ provided T is an onto transformation. If this last proviso is not met, it is easy to give counter examples to show no k is adequate in (4). Another result in this direction was for the case of $X = C(S)$, the Banach space of continuous functions over a compactum, S, where T was required to be single-valued and continuous. Both results are due to Hyers and Ulam [16, 17]. I was able to treat the $L_p(0, 1)$, case for $p>1$ [4]. Moreover, I was able to demonstrate that a k existed for a uniformly convex space subject to two further conditions. Introduce the following notation: $\delta(\gamma)$ is the supremum of positive numbers compatible with $\|z_1+z_2\| \leqq 2(1-\delta(\gamma))$ max $(\|z_1\|, \|z_2\|)$ when $\|z_1-z_2\| \geqq \gamma$ max $(\|z_1\|, \|z_2\|)$ and $\psi(\delta)=\sup \{\gamma \mid \delta(\gamma) \leqq \inf (1, \delta)\}$. The conditions are

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$\sum \psi(2^{-n}C) < \infty, \tag{6}$$

$$L_{\lambda\to 0, \|z\|\to 0}((\|y_0 + z - \lambda y_1\| - \|y_0 + z\|)/\lambda) = 0, \tag{7}$$

for $\|y_0\|=1$ and $f_0(y_0)=1$ where $\|f_0\|=1$ and z and y are annihilated by f_0. We may remark that the ϵ isometry T can be shown to be a 36ϵ linear transformation and that U is the unique distributive operation satisfying $\overline{L}_{\|x\|\to\infty}(\|T(x)-U(x)\|/\|x\|) < \infty$. The consequences of a condition resembling (7) at least in the case $z=0$ have been recently investigated by Fortet [13]. In the case of $C(S)$ I was able to generalize the Hyers-Ulam result by demonstrating a similar theorem in which S was completely regular instead of a compactum and in which T was neither single-valued nor continuous [5]. No serious effort has been made in the cases successfully treated to determine the "best" value of k.

There are two parts to the problem. The first is the determination of U; the second, the verification of (4). In all cases studied $U(x)$ turns out to be $L_{n\to\infty}T(2^n x)/2^n$. In the first cases mentioned the existence of the limit is easy to establish; for instance, (6) guarantees this. The more difficult part is here the proof of (4), and relations of the form of (7) play a role. In $C(S)$ the order of the proof is reversed, and $U(x)$ is derived from the consideration of the known general form of an isometry on such a space in terms of a homeomorphism of S and an analysis of the T maps of certain classes of elements reminiscent of maximal ideals. It is only after (4) is established that the limit definition of U is verified. No general method has emerged as yet, and it is not known whether k exists for every Banach space. Indeed the problem is still open in cases such as l_1 where the form of an isometry is known.

If now X or Y or both are proper subsets of a Banach space, little is known. If, for instance, X is a subset of the Euclidean plane Y, and if we drop the restriction $T(X) \subset X$ even for compact X, then no k satisfying (4) need exist. I owe the counter example to Swain [35]. Take for X the triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, a)$. We define T by $T(x, y)=(x, y+a(|x|-1))$. Then for each value of k the choice $a=1/4k$ leads to $\epsilon \leqq 2a^2$ and may be shown to give an ϵ isometry of X and $T(X)$ for which (4) cannot be satisfied for any isometry U. (Swain has shown that it is true, however, that for a given k one can assign an ϵ so that (4) and (5) can be satisfied when X is a compactum in the plane.) Actually, however, the more natural problem is that of $X=Y$ where X is a compact subset of the plane or more generally of a Euclidean space. I believe that for each such X, k does exist; but I have no proof. If compactness is replaced by metric conditional compactness, there is the possible complication that an isometry may not be onto unless say a residue of some order vanishes [27]; and there seem to be no clear indications regarding the existence of k. The fact that compactness may be crucial is shown by the following example. Consider the set X of vertical segments in the plane $\{Y^{(n)} \mid Y^{(n)}=(x^n, y),\ 0 \leqq |y| \leqq h^n\}$. Then if $x^n \rightarrow \bar{x}$, $h^n \rightarrow 0$, there is a finite k; but one can determine a divergent sequence $\{x^n\}$ with $h^n \rightarrow 0$ for which no finite k is available.

2. **Isometries in groups.** When the underlying spaces have algebraic structure, one may ask whether the metric specification of a standard transformation, the isometry, implies algebraic properties. A well known result of this character is the Mazur-Ulam theorem [28] that an isometry of a Banach space which preserves the origin is a linear transformation. It is natural to consider then an isometry of a metric group with, say, left invariant metric which leaves the identity fixed. I point out here that the fundamental difficulties are in the non-Abelian case. The nub of the demonstration of the Mazur-Ulam theorem is the midpoint construction. This has general application and is nontrivial only when the unit sphere is flat. (Thus, the restriction to strictly convex spaces may be waived in some of Rothe's work [32] as observed to me by Charnes.) Aronszajn [2] has given an extension, but this again is for the Abelian situation alone. If one requires that every element be a square, then under weak subsidiary hypotheses the isometry U is a semi-automorphism, that is to say, $Usts = UsUtUs$. In general this is all one can expect. For the case of a field Hua [19] has shown that U is either a homomorphism or antihomomorphism, and Nakayama [29] has extended this to primary rings with a weak chain condition; but the proofs lean heavily on the existence of two operations. Of course, in the group case (or general ring case), for instance when the groups are product groups, the situation for T can be quite involved. It is not clear how to formulate non-artificial conditions that would guarantee our semi-automorphism is really an automorphism.

Eilenberg [12] showed that any member T of a compact group G of homeomorphisms of a metric space S is an isometry in some equivalent metric associated with the uniform topology. Is the conclusion still valid if the compactness of G is dropped and the other hypotheses are strengthened to read T is a topological automorphism of the compact metric group S? The answer is in the negative as shown by the following counter example I have recently constructed to settle this and other questions. We topologize the additive group of the reals as a compact group $G=\{x\}$, following Halmos [14], by considering it as the character group of the additive group of the rationals $R=\{r\}$ taken with discrete topology. Thus x is considered as $x(r)$, an element of P^R, where P is the set of reals mod 1 with the customary topology. The topology of G is then indicated by the compact open topology for P^R. Since the neighborhood basis in P at $p=0$ is denumerable and compact sets in R are finite, it follows that the neighborhood basis in P^R satisfies the first denumerability condition. Accordingly, since G is compact, it is metrizable. A sub-basis at $x=0$ is given by the neighborhoods $U(\epsilon, r)=\{x \mid |rx| < \epsilon \text{ mod } 1, r\in R\}$. Consider the automorphism A: $x\rightarrow 10x$. This is plainly 1-1. To show A is topological, it is here sufficient to demonstrate continuity at $x=0$. However, this follows from the observation that for small ϵ

$$A{:}\,U(\epsilon/10, r) = \{x \mid rx = 10N \pm \delta, \delta < \epsilon, N = 0, \pm 1, \cdots \} \subset U(\epsilon, r).$$

If A is an isometry, then $d(0, x)=d(0, 10x)=d(0, 10^n x)$, that is, $\{10^n x \mid n=0, \pm 1, \pm 2, \cdots \}$ is included in any metric sphere containing x. On the other hand consider $\mathfrak{N}=U(1/100, 1)$. For $x\neq 0$, we may as well assume that the first nonzero digit follows the decimal. Then either x or $10^{-1}x\notin\mathfrak{N}$; that is to say, $\mathfrak{N}$ cannot contain $\{10^n x\}$, and so A is not an isometry.

3. Approximate convexity.

A slight generalization of (1) involving dependence of $\bigcirc$ and $\oplus$ on a parameter is enough to comprehend the case of an affine or convex function on a convex subset of Euclidean n space. The results here were communicated to me by Hyers and Ulam [18]. An ϵ convex function is real-valued and satisfies

$$T(hx + (1-h)y) - (hT(x) + (1-h)T(y)) \leqq \epsilon \quad \text{for } 0 \leqq h \leqq 1.$$

Here U is convex; and the main theorem asserts that if X is a convex body or merely a convex subset in Y, the Euclidean n-dimensional space, and if T is continuous, then (4) can be satisfied with $k=n(n+3)/4(n+1)$.

4. Multiplicative transformations.

In view of the cardinal importance in Banach algebras of maximal ideals which are interpreted as linear multiplicative functionals, it seems worthwhile to consider purely multiplicative functionals and transformations, as standard transformations, in some detail. Specifically, suppose $X=C(S)$, S compact, and let Y be the real line, with $\bigcirc$ and $\oplus$ the obvious multiplications. We are interested in $Uxy=UxUy$, where U is continuous in the norm topology of $C(S)$. I have recently shown [6] that U is completely determined by the values of $x(s)$ on a denumerable closed set D in S which is independent of choice of x. This closed set, which is easily shown to be a compactum in the relative topology, is referred to as a determining set. Some immediate consequences have been noted in unpublished work by Charnes and me [9]. For instance, similar results are valid when Y is the complex field or when S is completely regular and $\beta(S)-S$ is the union of closed G_δ sets where $\beta(S)$ is the Čech compactification of S. Furthermore, we have been able to obtain the essential facts about the representation of continuous multiplicative transformations on $C(S_1)$ to $C(S_2)$ for S_1 and S_2 compact. For instance, if one requires the transformation to be completely continuous, then a denumerable compactum D in S_1 is again the determining set in the sense that $(Ux)(t)=\prod_{s_i\in D}|x(s_i)|^{\mu_i(t)}$. If the transformation is merely continuous, then the correspondence $t\rightarrow D(t)$ is upper semi-continuous. Although $\sum\mu_i(t)$ must be continuous, it is not true the individual $\mu_i(t)$'s need be. In this range of ideas there is an expected analogue of the Stone-Banach theorem. Indeed, we have shown that multiplicative isomorphism of the continuous multiplicative functionals on two compacta implies homeomorphism of the compacta.

Cardinal number aspects of multiplicative linear functionals seem in the spirit of the considerations above. For instance suppose S is completely regular and the minimal power of a neighborhood basis at every point is the same cardinal α. Let $\mathfrak{J}$ be a free ideal, in Hewitt's terminology, of $C_B(S)$, the Banach algebra of bounded continuous functions. Suppose the power of $\{\lambda|\Lambda\}$ is β, $\beta\leqq\alpha$, and $\{x^\lambda|\Lambda\}\subset\mathfrak{J}$. Then for some fixed collection $\{s^\omega|\omega\in\Omega$ a directed set$\}$ $\subset S$ of power less than or equal to $\mathfrak{B}$, we can assert $L_\omega x^\lambda(s^\omega)=0$ for each λ. An interesting consequence is that the free ideal $\mathfrak{J}$ cannot be maximal if it admits a basis of power $\leqq\alpha$.

Other directions of generalization I have in mind are to subrings of $C(S)$ or to the case of S no longer restricted to be compact and to function rings other than $C(S)$. For instance suppose G is a compact Abelian group and G' is its discrete character group; then $L_2(G)$ is a ring over the complex field K when multiplication is interpreted as convolution while $L_2(G')$ is a ring under *pointwise* multiplication.

The intervention of the Fourier transform yields the continuous multiplicative functionals on $L_2(G)$ in terms of those on $L_2(G')$. D is finite here. Multiplicative functionals yield the *one-dimensional case* of representations for the semi group parts of Banach algebras.

The bordering transformations now are what we may term approximately multiplicative. It must be confessed that, in order to preserve interesting properties, the approximately multiplicative transformations turn out to be either disguises of truly multiplicative ones or at best to satisfy somewhat artificial restrictions. For instance, we have with $\Delta xy = \|Txy - TxTy\|$, $|x|_i = \inf_{s \in S_i} |x(s)|$, the following alternative definitions:

(A) $\Delta xy \leqq \delta \max (\|x\| \, |y|_i, \|y\| \, |x|_i).$

(B) $\Delta xy \leqq \delta \min (\|x\| \, |y|_i, \|y\| \, |x|_i).$

(C) $\Delta xy \leqq \delta |x|_i |y|_i.$

(D) $\Delta xy = 0$ for $|xy|_i = 0.$

(E) $\Delta xy \leqq \delta |xy|_i + \epsilon.$

For more than one point in S_2, (C) implies multiplicativity. An example of (A) is $Tx = x + |x|_i$. As an example of (A) and (B) we have $T(x) = |x|_i (\text{sign } x(s))(\beta(x(s)/\|x\|)^2 - \alpha)$ with $0 < \alpha < \beta < 2\alpha$ and $2\alpha + 4\alpha^2 < \delta$. This is on $C(S)$ onto $C(S)$.

The multiplicative transformations or approximately multiplicative transformations yield analogues of the Stone-Banach theorem [**3, 34**]. I have given a rather general condition of the validity of such a theorem [**5**]. Indeed, let S_i, $i = 1, 2$, be compact, and refer to $\{x \mid |x(s_0)| \leqq \mu\}$ as a μ ideal. Then if there is a map of $C(S_1)$ onto $C(S_2)$ under which μ ideals in $C(S_1)$ go into subsets of $\mu + \rho$ ideals in $C(S_2)$ and conversely, S_1 and S_2 are homeomorphic. We can state, for instance, if T is 1-1 on $C(S_1)$ onto $C(S_2)$ and satisfies (C) and so does T^{-1}, and if $(\inf_s \sup_{\|X\|=1} (Tx)(s)) (\inf_t \sup_{\|X\|=1} (T^{-1}X)(t)) > 0$, then S_1 is homeomorphic to S_2.

In view of this last theorem and my theorem on the existence of a determining set, we can state: If T satisfies (D) and is 1-1 on $C(S_1)$ onto $C(S_2)$, then S_1 and S_2 are homeomorphic.

5. **Banach isomorphisms.** We now turn attention to some phases of the problem of the Banach isomorphism of two Banach spaces E_1 and E_2 [**3**]. There are interesting modern connotations because the isomorphism may be construed as merely a change of the multiplication definition in a Banach algebra in many cases. The general reference for the results in this section is [**9**]. Let $\sim$ indicate Banach

isomorphism. Evidently a multiplication can be introduced in the Banach space E under which it is algebraically isomorphic to $C(M)$, M compact, if and only if $E \sim C(M)$. That $C(M)$ represents either the Banach space or the Banach algebra will cause no confusion. If M and N are compact but not homeomorphic and $C(M) \sim C(N)$, then the multiplication in $C(N)$ can be used to define a new multiplication in $C(M)$. Since the "C norm" is the minimum one among all B isomorphs of $C(M)$ for which the identity of $C(M)$ has norm 1, it may be shown that if two multiplications exist such that $E \sim C(M)$ and $E \sim C(N)$, then N and M are homeomorphic if and only if $\overline{M} = \overline{N}$ where the bar denotes the $w^*(E^*)$ closure of the convex hull (when M and N are considered subsets of E^*). It is natural to ask when (α): $C(S) \sim C(S) \times C(S)$, S compact. A set of sufficient conditions is: S contains A, B, and $S' = A \cup B$, all homeomorphic to it, S' satisfies (α), and $A \cap B$ is a retract of S'. The proof depends on the observation that if L is a retract of the compact set S and if $C_L(S) = \{x | x | L = 0\}$, then $C(S) \sim C(L) \times C_L(S)$. The existence of proper subsets homeomorphic to S is not essential. Indeed (α) is true when S is the unit circle H and in fact $C(H) \sim C(I)$ where I is the unit interval $0 \leqq s \leqq 1$. We have no example in which $C(S)$ is not finite-dimensional and yet (α) is not satisfied. Among other results we have shown that if A_i, $i = 1, 2$, is compact and connected and convergent sequences s_i, $s_i = \{a_i^n\}$, $i = 1, 2$, exist in A_1 and A_2 respectively, and if B is the space $A_1 \cup A_2$ with a_1^n and a_2^n identified, then $C(B) \sim C(A_1) \times C(A_2)$. I can make a small remark on the open problem of whether $C(I \times I)$ is isomorphic to $C(I)$, namely, that $C(I \times s) \sim C(I)$ where $s = \{1/n\} \cup 0$.

6. **Orthogonal sequences.** We now turn to another aspect of isometric transformations and consider a problem that seems equivalent to the characterization of a class of unitary transformations. In the early 1930's in several lectures and conversations I raised the question as to whether for even or odd functions

$$\text{(I)} \qquad \frac{1}{\pi} \int_{-\pi}^{\pi} f(nx) f(xm) dx = \delta_{nm}, \qquad n, m = 1, 2, 3, \cdots,$$

was sufficient to guarantee periodicity of $f(x)$ even under restrictions to class C or C' functions, or the like. This question is still open (though I expect an answer in the negative), but more success has been achieved with the characterization of periodic odd functions satisfying (I). C. Mendel and I [8] demonstrated the existence of various types of solutions of (I) under the inessential restriction that $f(x)$ is odd. In a subsequent paper [7] I established some necessary and sufficient conditions besides the obvious one that

$$\text{(II)} \qquad \sum_{n=1}^{\infty} a_n a_{m/n} = \delta_{nm},$$

where $a_i = 0$ for nonintegral i and, for i a positive integer, is the ith coefficient in the Fourier sine series expansion of the odd function $f(x)$.

I call attention to merely one facet of the general problem, namely, the fact that if a_n vanishes except when $n = p_1^{i_1} \cdots p_N^{i_N}$ for fixed primes $p_1, \cdots, p_N$, the most general solution is obtained by finding the analytic functions of N complex variables which take on boundary values of unit modulus almost everywhere on the distinguished surface $|w|_i = 1$, $i = 1, \cdots, N$. Call this problem (III). Indeed, if a solution is $\sum c_{i_1 \cdots i_N} w_1^{i_1} \cdots w_N^{i_N}$, then the constants $c_{i_1 \cdots i_N} = a_j$, $j = p_1^{i_1} \cdots p_N^{i_N}$, satisfy (II), and conversely. In the case $N = 1$ the function theoretic problem admits the representation $B(w) \exp H(w)$ where $B(w)$ is a Blaschke product and $H(w)$ is the well known Herglotz factor [33]. For $N > 1$ no such representation theorem is known, though I have shown, for instance, that if $\{c_{i_1 \cdots i_n}\} \in l_1$, then the representative function must be a special type of rational function [7]. The converse would be the following: if $\{a_n\} \in l_1$, then there is a corresponding rational function of a finite number of complex variables satisfying (III); but I have no proof. A solution of (I) and (II) yields a solution of (III). For instance, there is a solution of (I) in terms of the properties of the Dirichlet expansion of a determining function $\phi(z)$. Another criterion is one involving equivalence with the unitary transformation in $L^2(0, \infty)$, $g(\lambda) = d\int_0^\infty H(l\lambda) f(l) l^{-1} dl/d\lambda$, where H is a self-reciprocal kernel defined by $H(\lambda) = \sum_{n > 1/\lambda} a_n n^{-1/2}$. Indeed, if

$$\text{(IV)} \qquad \int_0^\infty \frac{H(l\lambda) H(k\lambda)}{\lambda^2} d\lambda = \min(k, 1)$$

and $\{a_n\} \in l_1$, or satisfies even weaker convergence conditions, we have a solution of (I). I believe that if (IV) is satisfied, then $\{a_n\} \in l_2$ is sufficient to yield the general solution of (I). If this were established, we should have another tie-up with problem (III). On the other hand, in the general case it is clear we should get the generalization of (III) to holomorphic functions of an infinite number of complex variables.

7. Fixed point index. From now on the bond of association of standard and bordering transformation is the notion of homotopy.

It is commonplace in the modern literature of analysis and applied mathematics to base existence statements on the Leray-Schauder theorem [26]. This theorem is concerned with a homology type of invariant, namely, the index. Such an index can be defined in the case of a polyhedron in E_n even though no restriction on homogeneity in the dimension sense be imposed. Thus let a regular point of the polyhedron be one with a neighborhood in the complex homeomorphic to E_s for some $s \leq n$. A regular fixed point is then an isolated fixed point which is regular, and we assign it the dimension factor $(-1)^s$. (Non-regular fixed points are handled by approximating T by T_n with regular fixed points.) The index of such a fixed point x_0 is its order j under the transformation T with respect to the continuous cycle obtained by mapping some sufficiently small s sphere with x_0 as center by T. The index of an open set is then the algebraic sum of the indices of all fixed points contained in the open set under the proviso that none are on the boundary, that is to say, $\sum_{\alpha s}(-1)^s j_{\alpha s}$. When the open set is replaced by the polyhedron, this is simply the Lefschetz number.

We now signalize another formulation. The transformation T induces chain and cochain mappings, and Lefschetz has used these to define his now classical zero cycle termed the combinatorial graph of T. The Lefschetz number can be considered an intersection number or Kronecker index associated with the graphs of T and of the identity transformation [21]. It would appear that the index is bound to the concept of the Lefschetz number, and it seems likely it should be capable of definition as a Lefschetz number if relative chains and cochains are introduced. Part of the difficulty involved in a definition of the index in this way attaches to the point set assignments of carriers or supports, for there do not exist altogether satisfactory formulations of the notion of general position. Improvement and extension of the intersection theory for the relative homology case would permit corresponding extensions in the effective definition of the index.

The Leray-Schauder degree is an extension of the index beyond the finite complex or Euclidean space stage for the special situation that the transformation is "equivalent" to one restricted to a finite-dimensional linear subspace. The modern method of generalizing topological concepts from polyhedra to spaces is to replace the space and its mappings by a collection of nerves of coverings and to define the relevant notions by suitable limits. This suggests the following schematic plan for generalizing the index of an open set. Let $\mathfrak{N}^\beta$ be the nerve of the finite open cover U^β, and let m^β and n^β refer to the maps of X into $\mathfrak{N}^\beta$ and $\mathfrak{N}^\beta$ into X respectively. The maps are to be defined so as to guarantee continuity, and this condition can be met for, say, coverings admitting locally finite refinements as shown by Lefschetz [22] and Dowker [11]. We can then associate with T the map T^β on $\mathfrak{N}^\beta \to \mathfrak{N}^\beta$ such that $T^\beta = m^\beta T n^\beta$. The open set $O \subset X$ corresponds to $(n^\beta)^{-1}O$. Suppose i_β can be defined on $\mathfrak{N}^\beta$ for the subcomplex $(n^\beta)^{-1}O$. It is then plausible to define the index for O by the requirement that i_β remain constant for all $\beta > \beta_0$ so that i_β is in a sense determined by the complex $\mathfrak{N}^{\beta_1}$, $\beta_1 > \beta_0$. Thus, for example, a subclass of the compact spaces with homology groups those of a finite-dimensional complex forms a natural category. Both Leray [23, 24, 25] and later Browder [10] have defined an index for some spaces of this type. Browder's definition is essentially that given above for a compact absolute neighborhood retract which is uniformly locally equiconnected and is referred to below as a Q space. Leray treats a space he designates as convexoide, defined as one admitting a certain class of closed coverings, but no equivalent definition in terms of connectivity properties has been given. Very likely the convexoide spaces form a subclass of the Q spaces.

It is highly desirable that the restriction to compact spaces be waived. For locally compact Hausdorff spaces, I have little doubt that results will obtain of the same sort as in the compact case. However, since the usual infinite-dimensional metric linear spaces are not locally compact in the metric topology, further extension is called for. The schematic program I have given above suggests an extension in which locally finite covers replace finite covers. One must first define the index for locally finite complexes. I may remark that preliminary work indicates the feasibility of such extension of the index to certain paracompact, normal spaces and also extensions of the index in the case of certain general spaces when the transformations are restricted.

8. **Fixed point classes.** We should expect greater information and more precision by introducing homotopy invariants. The central problem for us here is: Given $T: S \subset S$, what is the minimum number of fixed points of T or, in slightly more general form, what is the minimum number of fixed points of a transformation arising from T by a homotopy? This is the natural question when one has to deal with branches of solutions. In the case of two-dimensional manifolds with or without boundaries, Nielsen [30] introduced the notion of fixed

point classes. This work was generalized in part by Wecken [36] to the case of polyhedra in Euclidean space and by Browder [10]. Let $\tilde{S}$ be the universal covering space of a compact, connected, locally connected, and locally simply connected space S. This consists of equivalence classes of paths with common initial point, and each class may be indicated by square brackets boxing a representation path. Suppose then T maps S into S. Let P denote the path, from the fixed initial point p_0 to p. Let C be a path from p_0 to $T(p_0)$. We define $\tilde{T}$ on $\tilde{S}$ to $\tilde{S}$ by $T[P]=[CT(P)]$. Evidently other nonequivalent definitions can be given for T differing in the choice of the paths C from p_0 to $T(p)$.

Let $\pi(S, p_0)$ be the fundamental group of S based at p_0 with elements the equivalence classes $[\ ]$ of closed loops. Then one shows easily that

$$\tilde{T}[\gamma P] = H(\gamma)T[P]$$

where

$$H(\gamma) = [CT(\gamma)C^{-1}].$$

A fixed point class is the set of the projections of the fixed points on $\tilde{S}$ of $[\gamma]^{-1}\tilde{T}[P]$. Under contractibility hypotheses on S one can assure that these fixed points are in the same class if and only if for some joining curve, Γ, $(T(p), p)\subset\mathfrak{N}$ for all p on Γ. The index is assigned to each fixed point class by taking an open set O containing the elements of some class and none on the boundary and using $i(O)$. Fixed point classes with nonzero indices remain fixed point classes of the same index under homotopic variation of T. When the index is 0 it is possible that a class may be homotopically correlated with one containing no fixed points.

Consider the homogeneous space derived from $\pi(S, p_0)$ by introducing the equivalence classes or cosets originally suggested by Reidemeister [31], namely, $\alpha\sim\beta$ if and only if for some $[\gamma]\in\pi(S, p_0)$, $H(\gamma)\alpha\gamma^{-1}=\beta$. Denote the coset containing α by (α). Then it was shown by Wecken [36, II] that each fixed point class corresponds to a coset (α). For spaces of the type (Q) it may be shown that the number of fixed point classes is finite. Under homotopy the number of classes with 0 index may change. If the number of cosets is finite, then obviously the number of fixed point classes has a finite upper bound. Otherwise it may be unbounded, though, of course, the number of classes with nonzero index stays fixed.

There are evidently at least as many fixed points as there are fixed point classes of nonzero index. Wecken has given an example where there is only one fixed point class, yet no transformation in the homotopy class has less than 2 fixed points.

Again preliminary analysis indicates that compactness may be replaced by σ local compactness without seriously affecting the main features of the results. However, further weakening such as, say, to paracompact spaces, runs into difficulties that suggest only restricted homotopies be used in view of Kakutani's result [20] that the Hilbert sphere $\{x \mid \|x\| = 1\}$ is homotopic to a point over itself.

I conclude with a few comments about sets of fixed points. In a general way it has been shown by Aronszajn [2] that when a transformation into a compactum can be uniformly approximated by transformations admitting unique solutions, then the fixed point set is an intersection of a decreasing sequence of absolute retracts and is referred to as an R_δ after the usual signification of δ. An R_δ set is like an absolute retract as regards its trivial fundamental and homology groups, but may have different local properties as indicated by the mathematician's *vade mecum*, the sin $1/x$ curve plus the vertical segment at $x=0$, which obviously is an R_δ. I mention in passing that it would be of interest to give an internal characterization of an R_δ. The precise statement of Aronszajn's result is: If T is completely continuous on the Banach space E into itself, and if there is a sequence of completely continuous transformations T_n such that (a) $\epsilon_n > \|Tx - T_n x\|$ for $x \in K$ a bounded closed set, with $\epsilon_n \downarrow 0$; (b) $w = H_n z = z - T_n z$ maps K in 1-1 fashion on a set containing a fixed sphere in E independent of n (this guarantees uniqueness in K of solution of $H_n z = w$ for w of sufficiently small norm); then the set of fixed points of T is an R_δ. There seem obvious applications of the index and fixed point class concepts in extending this result. Moreover, quite apart from the extension where compact replaces compact metric, another direction of generalization would be to the cases that the approximating transformations were not necessarily single-valued. There is reason to surmise then that the corresponding fixed point set would be an intersection of absolute neighborhood retracts.

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RIEMANN'S METHOD AND THE PROBLEM OF CAUCHY

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Introduction. The method devised by Riemann to solve the Problem of Cauchy applies to linear, hyperbolic, partial differential equations of second order for one unknown function u of two independent variables x, y. For a homogeneous equation the essential points in the method are:

(a) The introduction of the characteristics as coördinate lines,

(b) The construction of a line integral $I=\int\{Bdx-Ady\}$ which vanishes around closed paths, where:

(c) A and B are certain bilinear forms in u, u_x, u_y and v, v_x, v_y, and,

(d) v (Riemann's function) is a properly chosen two-parameter family of solutions of a second linear partial differential equation, the *adjoint equation.*

In this paper new bilinear forms are taken for A, B in (b), and the rôle of the adjoint equation in (d) is taken over by a partial differential equation termed the *associate equation* of the original equation. Each solution ϕ of the original equation leads to an associate equation and the Problem of Cauchy is then solved with the aid of a properly chosen two-parameter family of solutions of the associate equation called the *resolvent*, the analogue of Riemann's function.

This modification offers some hope of extending Riemann's method to the Problem of Cauchy for linear hyperbolic partial differential equations with more than two independent variables.[1] Such an extension to three independent variables is actually carried out in this paper for the equation of cylindrical waves.[2] In the treatment of this equation, following (a), characteristic coördinates are adopted. As an interesting corollary, it turns out that axially symmetric solutions are governed by one[3] of Euler's partial differential equations, namely,

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[1] The classic work on Cauchy's problem is the book by J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperbolique*, Paris, 1932, where references to the work of other authors are given.

[2] My colleague A. Weinstein has kindly pointed out that an extension of Riemann's method for this equation has been given by H. Lewy, *Verallgemeinerung der Riemannschen Methode auf mehr Dimensionen*, Nachr. Ges. Wiss. Göttingen (1928) pp. 118–123. His method differs from ours in that it employs three "Riemann functions," and neither makes use of characteristic coördinates, nor of an "associate equation."

[3] See, for example, G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, 2d ed., Paris, 1915, pp. 54–70.

$$(\alpha-\beta)u_{\alpha\beta}-(u_\alpha-u_\beta)/2=0.$$

1. **Riemann's method.** The solution of the problem of Cauchy given by Riemann[4] for the partial differential equation

$$u_{xy}-a(u_x-u_y)=0,\qquad a=a(x,y),\tag{1}$$

has been extended[5] to the general linear partial differential equation of second order in two independent variables x, y,

$$u_{xy}+au_x+bu_y+cu=f,\tag{2}$$

where a, b, c, f are given functions of x, y.

In this section first we set forth the essentials of this method, as usually presented, to prepare the way for a modification which permits extension to an equation with three independent variables, the equation for cylindrical waves

$$u_{xx}+u_{yy}-u_{tt}=0.\tag{3}$$

For simplicity we consider only equations of the type

$$L(u)=u_{xy}-au_x-bu_y=0,\qquad a=a(x,y),\quad b=b(x,y).\tag{4}$$

Actually, if a solution u_1 of (2) and a solution u_0 of the homogeneous equation ($f\equiv 0$) are known, equation (2) can always be reduced to one of type (4) for a new unknown function $\bar{u}$ defined by $u=u_1+u_0\bar{u}$.

The classical procedure associates with (4) its *adjoint equation*

$$M(v)=v_{xy}+(av)_x+(bv)_y=0,\tag{4'}$$

and is based[6] on the differential identity

$$A_x+B_y=vL(u)-uM(v),\qquad A=(vu_y-uv_y)/2-auv,$$
$$B=(vu_x-uv_x)/2-buv.\tag{5}$$

This identity implies that the line integral

$$I=\int\{Bdx-Ady\}$$

vanishes around closed paths lying in the interior of a domain D within which the function u, v are regular solutions of $L(u)=0$ and $M(v)=0$, respectively.

The fact that the line integral I vanishes around closed paths leads immediately to the solution of Cauchy's problem for (4). Let C be

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[4] B. Riemann, *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Göttingen Abhandlungen vol. 8 (1858–1859) pp. 245–264, or *Werke*, Leipzig, 1892, 2d ed., pp. 156–175.

[5] J. Hadamard, op. cit. pp. 93–95.

[6] J. Hadamard, op. cit., pp. 81–82.

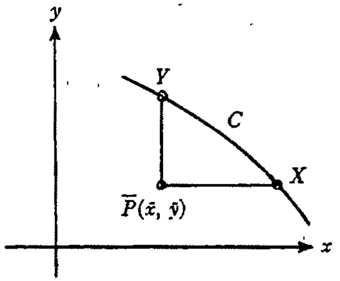

FIG. 1

an arc meeting any horizontal or vertical line in at most one point. Along C the values of u_x, u_y (*Cauchy's data*[7]) for the solution u of (4) are given. In order to calculate the value $u(\bar{P})$ of the solution at the point $\bar{P}$ in the figure, the line integral I is evaluated around the closed path $\bar{P}XY\bar{P}$ formed by the horizontal line segment $\bar{P}X$, the arc XY of C, and the vertical line segment $Y\bar{P}$. One finds

$$(6)\qquad \int_{\bar{P}}^{X} B(x, \bar{y})dx + \int_{X}^{Y} \{Bdx - Ady\} - \int_{Y}^{\bar{P}} A(\bar{x}, y)dy = 0.$$

If a solution $v = v(x, y; \bar{x}, \bar{y})$ of $M(v) = 0$ can be found for which

$$(7)\qquad \begin{gathered} v_x = -bv \text{ on } y = \bar{y}, \qquad v_y = -av \text{ on } x = \bar{x}, \\ v(\bar{x}, \bar{y}; \bar{x}, \bar{y}) = 1 \end{gathered}$$

hold on the characteristics $x = \bar{x}$, $y = \bar{y}$ through $\bar{P}$, we shall have

$$(8)\qquad A(\bar{x}, y) = (uv)_y/2, \qquad B(x, \bar{y}) = (uv)_x/2,$$

and consequently (6) will yield

$$(9)\qquad u(\bar{P}) = \frac{1}{2}\left[u(X)v(X) + u(Y)v(Y)\right] + \int_{X}^{Y} \{Bdx - Ady\}.$$

The function v is known as *Riemann's function*, and, once it has been determined, $u(\bar{P})$ may be calculated by (9) from a knowledge of the Cauchy data along C.

The first step in Riemann's method is the formulation of the line integral I so that it vanishes around closed paths. We note that the coefficients A, B of the differentials are bilinear forms in the two sets of variables u, u_x, u_y and v, v_x, v_y. Furthermore, the real reason for the introduction of the adjoint equation (4′) and its solution v is to make available the identity (5), which guarantees that I vanishes around closed paths.

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[7] In the usual terminology Cauchy data consist of the values u_0 of u on C and the values u_1 of its directional derivative in some direction not tangent to C. Assigning u_x, u_y on C determines u_1 exactly and u_0 up to an arbitrary additive constant.

In seeking an alternative approach to the solution of Cauchy's problem we might take A and B to be general bilinear forms in u, u_x, u_y and v, v_x, v_y with coefficients arbitrary[8] functions of x, y, and let the rôle of the adjoint equation be assigned to a partial differential equation,

$$(10)\quad \Lambda(v) = v_{xy} - \alpha v_x - \beta v_y = 0, \qquad \alpha = \alpha(x, y),\ \beta = \beta(x, y),$$

whose coefficients α, β are at our disposal.

With these general principles in mind, let us set $A = -\lambda^{-1}u_y v_y$, $B = \mu^{-1}u_x v_x$ and attempt to construct a line integral

$$I = \int \left\{\lambda^{-1}u_x v_x dx + \mu^{-1}u_y v_y dy\right\},$$

which vanishes around closed paths in the interior of a domain within which u, v are regular solutions of $L(u) = 0$, $\Lambda(v) = 0$ respectively, the functions λ, μ of x, y being, for the moment, perfectly arbitrary.[8]

From the identity

$$\begin{aligned}(\lambda^{-1}u_x v_x)_y - (\mu^{-1}u_y v_y)_x = {} & \lambda^{-1}\mu^{-1}\left[(\mu u_x - \lambda u_y)\Lambda(v) + (\mu v_x - \lambda v_y)L(u)\right] \\ & - \lambda^{-1}\mu^{-1}\left[\mu(\lambda^{-1}\lambda_y - a - \alpha)u_x v_x \right. \\ & + (a\lambda - \beta\mu)u_x v_y + (\alpha\lambda - b\mu)u_y v_x \\ & \left. - \lambda(\mu^{-1}\mu_x - b - \beta)u_y v_y\right],\end{aligned}$$

the analogue of (5), it is clear that I will vanish around closed paths for every pair of solutions u, v of $L(u) = 0$, $\Lambda(v) = 0$, provided the arbitrary functions α, β, λ, μ satisfy

$$(11)\qquad \begin{aligned} a\lambda - \beta\mu &= 0, & \lambda_y &= (a + \alpha)\lambda, \\ \alpha\lambda - b\mu &= 0, & \mu_x &= (b + \beta)\mu. \end{aligned}$$

When the equations in the first column are used to eliminate α, β from the equations in the second column, we find

$$\lambda_y = a\lambda + b\mu, \qquad \mu_x = a\lambda + b\mu.$$

Consequently $\lambda_y = \mu_x$ and a potential function $\phi = \phi(x, y)$ exists for which

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[8] Subject, of course, to the usual differentiability conditions.

$$\lambda = \phi_x, \qquad \mu = \phi_y, \qquad \text{with } L(\phi) = 0.$$

Moreover the first column in (11) yields

$$\alpha = \phi_x^{-1}\phi_y b, \qquad \beta = \phi_x \phi_y^{-1} a.$$

With this choice of α, β, λ, μ the identity reduces to

$$(\phi_x^{-1}u_xv_x)_y - (\phi_y^{-1}u_yv_y)_x = (\phi_x^{-1}u_x - \phi_y^{-1}u_y)\Lambda(v) + (\phi_x^{-1}v_x - \phi_y^{-1}v_y)L(u),$$

where

$$\Lambda(v) = v_{xy} - b\phi_x^{-1}\phi_y v_x - a\phi_x\phi_y^{-1}v_y. \tag{12}$$

The equation $\Lambda(v)=0$ will be termed an *associate equation* of $L(u)=0$. Each solution ϕ of $L(u)=0$ gives rise to such an associate equation, of which it is automatically a solution.

LEMMA. *The line integral*

$$I = \int \{\phi_x^{-1}u_xv_xdx + \phi_y^{-1}u_yv_ydy\}$$

vanishes around closed paths lying in a region D within which u is a regular solution of $L(u)=u_{xy}-au_x-bu_y=0$ and v is a regular solution of an associate equation $\Lambda(v)=v_{xy}-b\phi_x^{-1}\phi_yv_x-a\phi_x\phi_y^{-1}v_y=0$, the function ϕ being any solution of $L(u)=0$, regular in D, for which $\phi_x\neq 0$, $\phi_y\neq 0$.

Guided by the procedure in Riemann's method, we evaluate I around the closed path $\overline{P}XY\overline{P}$ in Fig. 1, obtaining, in place of (6),

$$\begin{aligned}\int_{\overline{P}}^{X}\phi_x^{-1}u_xv_xdx + \int_X^Y \{\phi_x^{-1}u_xv_xdx + \phi_y^{-1}u_yv_ydy\}& \\ + \int_Y^{\overline{P}}\phi_y^{-1}u_yv_ydy = 0.&\end{aligned} \tag{13}$$

Instead of the initial conditions (7) prescribed for v on the characteristics through $\overline{P}$, we now require

$$v_x = \phi_x \text{ on } y = \bar{y}, \qquad v_y = -\phi_y \text{ on } x = \bar{x}. \tag{14}$$

Once such a solution of the associate equation has been found, the integrands in the first and last integrals in (13) reduce to u_x, $-u_y$ respectively, and one finds quite readily that

$$u(\overline{P}) = \frac{1}{2}[u(X) + u(Y)] + \frac{1}{2}\int_X^Y \{\phi_x^{-1}u_xv_xdx + \phi_y^{-1}u_yv_ydy\}. \tag{15}$$

A solution $v=v(x, y; \bar{x}, \bar{y})$ of an associate equation $\Lambda(v)=0$ subject to the initial conditions (14) is termed a *resolvent* of $L(u)=0$ and, after it has been found, $u(\bar{P})$ can be calculated by (15) from the Cauchy data for u along C. Since each solution ϕ of $L(u)=0$ gives rise to an associate equation $\Lambda(v)=0$, there are, a priori, many such resolvent functions. In practice, one would, of course, choose for ϕ the simplest particular solution of $L(u)=0$ that is available.

THEOREM. *The solution to Cauchy's problem for the partial differential equation $L(u)=u_{xy}-au_x-bu_y=0$ with initial data prescribed along the arc C in Fig.* 1 *is given by*

$$u(\bar{P}) = \frac{1}{2}\left[u(X) + u(Y)\right] + \frac{1}{2}\int_X^Y \left\{\phi_x^{-1} u_x v_x dx + \phi_y^{-1} u_y v_y dy\right\},$$

where $\phi=\phi(x, y)$ is any regular particular solution of $L(u)=0$ for which $\phi_x\neq 0$, $\phi_y\neq 0$, and $v=v(x, y; \bar{x}, \bar{y})$, a resolvent of $L(u)=0$, is a two-parameter family of solutions of the associate equation

$$\Lambda(v) = v_{xy} - b\phi_x^{-1}\phi_y v_x - a\phi_x\phi_y^{-1} v_y = 0$$

subject to the initial conditions $v_x = \phi_x$ on $y = \bar{y}$, $v_y = -\phi_y$ on $x = \bar{x}$, upon the characteristics through P.

Let us apply this theorem to equation (1) treated by Riemann. For the particular solution ϕ we select $\phi=x+y$, whereupon the associate equation becomes $\Lambda(v)=v_{xy}+a(v_x-v_y)=0$, the *conjugate equation*[9] to $L(u)=0$, and (15) reduces to

$$(15') \quad u(\bar{P}) = \frac{1}{2}\left[u(X) + u(Y)\right] + \frac{1}{2}\int_X^Y \left\{u_x v_x dx + u_y v_y dy\right\}.$$

When $a=m(x-y)^{-1}$ the equation $L(u)=0$ is Euler's equation, and explicit formulas for its resolvent have been given[9] in terms of Appell's hypergeometric function F_1 of two variables.

2. The equation for cylindrical waves. Classical solution of Cauchy's problem.

It has been known for a long time that the solution to Cauchy's problem

$$u(x, y, 0) = u^0(x, y), \qquad u_t(x, y, 0) = u^1(x, y),$$

for the wave equation (3) is given by[10]

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[9] M. H. Martin, *The rectilinear motion of a gas*, Amer. J. Math. vol. 45 (1943) pp. 391–407.

[10] J. Hadamard, op. cit. p. 71.

$$(16)\qquad 2\pi u(\bar{x}, \bar{y}, \bar{t}) = F_{\bar{t}}(u^1) + \frac{\partial}{\partial \bar{t}} F_{\bar{t}}(u^0),$$

where the operator $F_r(f)$ operating on a function f is defined by

$$(17)\qquad F_r(f) = \int_0^{2\pi} \int_0^r \frac{f(\bar{x} + \rho \cos \phi, \bar{y} + \rho \sin \phi)}{(r^2 - \rho^2)^{1/2}} \rho d\rho d\phi,$$

or, on setting $\rho = r\tau$, by

$$(18)\qquad F_r(f) = r \int_0^{2\pi} \int_0^1 \frac{f(\bar{x} + \tau r \cos \phi, \bar{y} + \tau r \sin \phi)}{(1 - \tau^2)^{1/2}} \tau d\tau d\phi.$$

We shall give this classical solution another form so that it may more readily be compared with the solution to be obtained later by generalizing Riemann's method.

By differentiating (18) with respect to r under the integration sign, we obtain

$$\frac{\partial F_r(f)}{\partial r} = \frac{1}{r} \int_0^{2\pi} \int_0^r \frac{f\rho}{(r^2 - \rho^2)^{1/2}} d\rho d\phi + \frac{1}{r} \int_0^{2\pi} \int_0^r \frac{\rho^2 f_\rho}{(r^2 - \rho^2)^{1/2}} d\rho d\phi.$$

After an integration by parts in the first term, this reduces to

$$(19)\qquad \frac{\partial F_r(f)}{\partial r} = 2\pi f(\bar{x}, \bar{y}) + \int_0^{2\pi} \int_0^r \frac{r\bar{f}_\rho}{(r^2 - \rho^2)^{1/2}} d\rho d\phi,$$

and with the necessary modifications in (17), (19), it is easy to verify that (16) takes the form

$$(20)\qquad u(\bar{x}, \bar{y}, \bar{t}) = u^0(\bar{x}, \bar{y}) + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\bar{t}} \frac{\bar{t}\overset{0}{u}_\rho + \rho \overset{1}{u}}{(\bar{t}^2 - \rho^2)^{1/2}} d\rho d\phi.$$

This is the desired form for the solution to Cauchy's problem. With either (16) or (20), the value $u(\overline{P})$ of the solution at the vertex $\overline{P}(\bar{x}, \bar{y}, \bar{t})$ of the characteristic cone $(x-\bar{x})^2+(y-\bar{y})^2=(t-\bar{t})^2$ is computed from Cauchy's data given over the circular region $(x-\bar{x})^2+(y-\bar{y})^2 \leqq \bar{t}^2$ intercepted by the characteristic cone on the carrier of Cauchy's data, the plane $t=0$. We note in passing that (20) may be given a concise form

$$u(\bar{x}, \bar{y}, \bar{t}) = u^0(\bar{x}, \bar{y}) + \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\bar{t}} (l_{\bar{t}}\overset{0}{u}_\rho - l_\rho \overset{1}{u}) d\rho d\phi,$$

by the introduction of the length $l=(\bar{t}^2-\rho^2)^{1/2}$ of the tangent line drawn from the vertex $\bar{P}(\bar{x}, \bar{y}, \bar{t})$ of the characteristic cone to a sphere of radius ρ with center at the point $(\bar{x}, \bar{y}, 0)$ where the axis of the characteristic cone pierces the plane $t=0$.

3. The equation for cylindrical waves. Characteristic coördinates. Each point (x_0, y_0, t_0) of (x, y, t)-space is a vertex for a characteristic cone $(x-x_0)^2+(y-y_0)^2=(t-t_0)^2$. For the computation of the value of the solution u at a point $\bar{P}(\bar{x}, \bar{y}, \bar{t})$ we shall, for $\bar{t}>0$, be interested only in those points which lie on, or above, the plane $t=0$ and upon, or inside, the lower mantle of the characteristic cone with vertex at $\bar{P}$, that is, in the points of the conical region

$$C:\quad 0 \leqq t \leqq \bar{t}, \qquad (x-\bar{x})^2+(y-\bar{y})^2 \leqq (t-\bar{t})^2,$$

pictured in Fig. 2a. In this figure $\bar{P}\bar{P}_0$ is the axis of the characteristic cone with vertex at $\bar{P}$. We introduce curvilinear coördinates α, β, ϕ

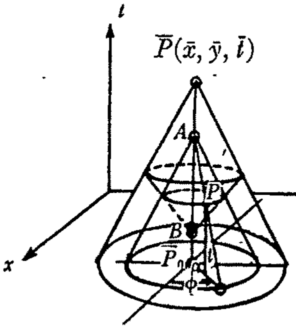

FIG. 2a. Conical region.

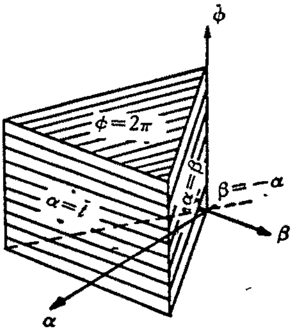

FIG. 2b. Wedge W.

to fix the position of a point $P(x, y, t)$ in C by taking points $A(\bar{x}, \bar{y}, \alpha)$, $B(\bar{x}, \bar{y}, \beta)$ on the axis so that the lower mantle of the characteristic cone with vertex A intersects the upper mantle of the characteristic cone with vertex B in a circle through P. The position of P on this circle is then fixed by the angle ϕ shown in Fig. 2a. Elementary geometrical considerations show that

$$(21)\qquad \begin{gathered} x=\bar{x}+\frac{1}{2}(\alpha-\beta)\cos\phi, \qquad y=\bar{y}+\frac{1}{2}(\alpha-\beta)\sin\phi, \\ t=\frac{1}{2}(\alpha+\beta), \end{gathered}$$

and consequently, on writing $\rho=((x-\bar{x})^2+(y-\bar{y})^2)^{1/2}$, we shall have

$$t = (\alpha + \beta)/2, \quad \rho = (\alpha - \beta)/2. \tag{22}$$

The coördinates α, β, ϕ of a point P will be termed *characteristic coördinates*, even though only two of the coördinate surfaces through P are mantles of characteristic cones, the third being a half-plane.

After the introduction of characteristic coördinates, equation (3) for cylindrical waves becomes

$$L(u) = u_{\alpha\beta} - \frac{1/2}{\alpha - \beta}(u_\alpha - u_\beta) - \frac{u_{\phi\phi}}{(\alpha - \beta)^2} = 0. \tag{23}$$

Obviously, we shall be interested only in those solutions of this equation which are periodic of period 2π in ϕ.

For solutions with axial symmetry about $x = \bar{x}$, $y = \bar{y}$ this equation reduces to Euler's equation

$$u_{\alpha\beta} - \frac{m}{\alpha - \beta}(u_\alpha - u_\beta) = 0, \qquad m = 1/2. \tag{24}$$

The transformation (21) carries the conical region C of (x, y, t)-space into the wedge

$$W: \ 0 \leqq \alpha \leqq \bar{t}, \qquad -\alpha \leqq \beta \leqq +\alpha, \qquad 0 \leqq \phi \leqq 2\pi,$$

in (α, β, ϕ)-space shown in Fig. 2b. From (22) the base of C becomes the vertical face $\beta = -\alpha$ in back of W; the conical mantle of C transforms into the vertical face $\alpha = \bar{t}$ of W; while the axis $\bar{P}\bar{P}_0$ inside C, from (21), appears as the vertical face $\beta = \alpha$ of W. Both horizontal faces $\phi = 0$, $\phi = 2\pi$ of W are transforms of the trace of C on the plane $y = \bar{y}$. The vertex $\bar{P}$ of C transforms into the edge $\alpha = \beta = \bar{t}$ of W; the periphery of the base of C becomes the edge $\alpha = \bar{t}$, $\beta = -\bar{t}$ of W; and the center $\bar{P}_0$ of the base of C appears as the edge $\alpha = \beta = 0$ of W.[11]

4. The equation for cylindrical waves. Solution of Cauchy's problem by Riemann's method. We begin by formulating Cauchy's problem in (α, β, ϕ)-space. The carrier $t = 0$ for Cauchy's data in (x, y, t)-space becomes the plane $\alpha + \beta = 0$ in (α, β, ϕ)-space. From (21) we find

$$u_\alpha = (u_\rho + u_t)/2, \qquad u_\beta = -(u_\rho - u_t)/2, \qquad u_\phi = u_\phi.$$

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[11] I am indebted to J. B. Diaz for drawing my attention to an intuitive process which carries the wedge W into the cone C. One shrinks the face $\beta = \alpha$ of W into the lower edge and brings the upper face of the resulting solid into coincidence with the lower face by revolution around this edge as axis.

In particular, for the carrier $\alpha+\beta=0$, this yields

$$(25) \qquad u_\alpha = (\overset{0}{u}_\rho + \overset{1}{u})/2, \qquad u_\beta = -(\overset{0}{u}_\rho - \overset{1}{u})/2, \qquad u_\phi = \overset{0}{u}_\phi,$$

so that Cauchy's data u_α, u_β, u_ϕ are prescribed on the carrier $\alpha+\beta=0$.

Thus the problem of computing the value $u(\bar{P})$ at the vertex $\bar{P}(\bar{x}, \bar{y}, \bar{t})$ of the characteristic cone in (x, y, t)-space, from a knowledge of Cauchy data given over the circular region intercepted by the characteristic cone on the plane $t=0$, becomes the problem of computing the value of a solution to (23) along the edge $\alpha=\beta=\bar{t}$ of W when Cauchy data are prescribed on the face $\beta=-\alpha$ of W.

We have pointed out in §1 that the first step in Riemann's method for two independent variables is the formulation of a properly chosen line integral I which vanishes around closed paths. In seeking a generalization to partial differential equations with three independent variables, it is natural to search for a surface integral I which vanishes when extended over closed surfaces.

For the partial differential equation (23) the proper surface integral is

$$(26) \qquad I = \int\int \left\{ u_\beta v_\beta d\beta d\phi - u_\alpha v_\alpha d\phi d\alpha + u_\phi \frac{v_\alpha - v_\beta}{(\alpha-\beta)^2} d\alpha d\beta \right\},$$

where $u=u(\alpha, \beta, \phi)$ is a solution of (23) and $v=v(\alpha, \beta)$ is a solution of the associate equation

$$(27) \qquad \Lambda(v) = v_{\alpha\beta} + \frac{1/2}{\alpha-\beta}(v_\alpha - v_\beta) = 0,$$

which, incidentally, is the conjugate equation to (24). That I actually vanishes over closed surfaces is easy to verify, for if we set $K=u_\beta v_\beta$, $L=-u_\alpha v_\alpha$, $M=u_\phi(v_\alpha-v_\beta)(\alpha-\beta)^{-2}$, a computation shows that

$$K_\alpha + L_\beta + M_\phi = 0.$$

Keeping in mind the procedure in §1, where we evaluated the line integral I of Lemma 1 around the closed path $\bar{P}XY\bar{P}$ in Fig. 1, we integrate I in (26) over the surface of the wedge W. We obtain

$$(28) \qquad \underset{\alpha=\bar{t}}{I} + \underset{\beta=-\alpha}{I} + \underset{\beta=\alpha}{I} + \underset{\phi=0}{I} + \underset{\phi=2\pi}{I} = 0.$$

Since u_ϕ has period 2π in ϕ and v_α, v_β are independent of ϕ, the last two terms cancel each other, because the projection of the top face on the bottom face of W has opposite orientation. When the integrations in the first three terms are written out explicitly, we obtain

$$\begin{aligned}(29)\qquad &\int_0^{2\pi}\int_{-\bar{t}}^{+\bar{t}}[u_\beta v_\beta]_{\alpha=\bar{t}}\,d\beta\,d\phi+\int_0^{2\pi}\int_0^{\bar{t}}[u_\alpha v_\alpha-u_\beta v_\beta]_{\beta=-\alpha}\,d\alpha\,d\phi\\ &\qquad-\int_0^{2\pi}\int_0^{\bar{t}}[u_\alpha v_\alpha+u_\beta v_\beta]_{\beta=\alpha}\,d\alpha\,d\phi=0.\end{aligned}$$

The next step in the procedure for two independent variables was to impose conditions (14) on the characteristics through $\overline{P}$ for the solution v of the associate equation (12). Corresponding to this, we require that the solution v of the associate equation (27) meet the conditions

$$(30)\qquad v_\beta=1 \text{ for } \alpha=\bar{t},\qquad v_\alpha=v_\beta=0 \text{ for } \beta=\alpha,$$

on the two faces of W intersecting in the edge $\alpha=\beta=\bar{t}$ of W representing P. The solution

$$v=\alpha+\beta+2((\bar{t}-\alpha)(\bar{t}-\beta))^{1/2}$$

of (27) actually meets these conditions, with

$$(31)\quad v_\alpha=1-\left(\frac{\bar{t}+\alpha}{\bar{t}-\alpha}\right)^{1/2},\quad v_\beta=1-\left(\frac{\bar{t}-\alpha}{\bar{t}+\alpha}\right)^{1/2}\quad \text{for } \beta=-\alpha,$$

When substitutions for v_α, v_β into (29) are made from (30), (31) we find that

$$\begin{aligned}(32)\qquad 2\pi u(\overline{P})&=\int_0^{2\pi}u\,\Big|_{\alpha=\bar{t},\beta=-\bar{t}}\,d\phi\\ &\quad-\int_0^{2\pi}\int_0^{\bar{t}}\left\{u_\alpha\left[1-\left(\frac{\bar{t}+\alpha}{\bar{t}-\alpha}\right)^{1/2}\right]\right.\\ &\qquad\left.-u_\beta\left[1-\left(\frac{\bar{t}-\alpha}{\bar{t}+\alpha}\right)^{1/2}\right]\right\}_{\beta=-\alpha}d\alpha\,d\phi,\end{aligned}$$

since $u=u(\overline{P})$ on the edge $\alpha=\beta=\bar{t}$ of W. This formula expresses the value $u(\overline{P})$ of the solution to (23) in terms of Cauchy data prescribed on the carrier plane $\beta=-\alpha$. The problem of Cauchy is solved in (α, β, ϕ)-space, and we shall now transform this solution back to (x, y, t)-space to obtain the classical result (20) as a check on our calculations.

If, in addition to (25), we observe that $u = u^0(\bar{t}, \phi)$ on the edge of $\alpha = \bar{t}$, $\beta = -\bar{t}$ of W, formula (32) reduces to

$$u(\overline{P}) = u^0(\bar{x}, \bar{y}) + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{t} \frac{\bar{t}\overset{0}{u}_\rho + \rho\overset{1}{u}}{(\bar{t}^2 - \rho^2)^{1/2}}\, d\rho d\phi,$$

that is, to (20), provided we note, from (22), that $\rho = \alpha$ when $\beta = -\alpha$.

University of Maryland

THE APRIL MEETING IN NEW ORLEANS

The four hundred sixty-seventh meeting of the American Mathematical Society was held at Tulane University, New Orleans, Louisiana, on Friday and Saturday, April 20–21, 1951. The total attendance was about 100 including the following 56 members of the Society:

H. P. Beard, R. G. Blake, R. C. Bullock, Eugenio Calabi, W. L. Duren, J. C. Eaves, D. O. Ellis, Trevor Evans, J. K. Feibleman, J. G. Horne, S. T. Hu, G. B. Huff, L. H. Kanter, H. T. Karnes, J. L. Kelley, R. J. Koch, G. B. Lang, F. A. Lewis, J. L. Loflin, C. W. McArthur, C. C. MacDuffee, J. D. Mancill, M. H. Martin, Benjamin Ernest Mitchell, D. R. Morrison, C. G. Mumford, H. M. Nahikian, H. V. Park, W. V. Parker, P. B. Patterson, W. D. Peeples, B. J. Pettis, A. S. Rayl, P. K. Rees, F. A. Rickey, J. K. Riess, N. E. Rutt, Judson Sanderson, W. C. Sangren, E. V. Schenkman, C. L. Seebeck, D. C. Spencer, J. M. Thomas, J. F. Thomson, R. M. Thrall, H. S. Thurston, G. J. Trammell, F. E. Ulrich, H. S. Vandiver, L. I. Wade, A. D. Wallace, H. C. Wang, L. E. Ward, Jr., M. J. Weiss, W. M. Whyburn, L. S. Winton.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor G. B. Huff of the University of Georgia and Professor M. H. Martin of the University of Maryland addressed the Society. Professor Huff's address, entitled *On the existence of plane curves with prescribed singularities*, was delivered at 11:00 a.m. on Saturday, with Professor W. V. Parker presiding. Professor Martin's address, entitled *Riemann's method and the problem of Cauchy*, was delivered at 8:30 P.M. Friday, with Professor W. L. Duren presiding.

Sessions for the presentation of contributed papers were held on Friday afternoon and Saturday morning. Presiding officers for these sessions were Professors F. A. Lewis and L. I. Wade. An informal business meeting was held just preceding the invited address on Saturday morning, at which the group voted to recommend to the Council that the invitation of the Alabama Polytechnic Institute to hold a meeting on its campus on November 23–24, 1951, be accepted.

Following the hour address on Friday evening, the members of the Department of Mathematics of Tulane University and their wives were hosts at a reception for visiting members and guests.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 246 was read by Mr. Burton, number 251 by Professor Calabi, number 256 by Professor Lang, and number 257 by Mr. McArthur. Mr. Novosad was introduced by Professor Pettis, Mr. Chehata by Dr. Neumann, Mr. Spencer by Professor Carlitz, Mr. Burton and Mr. Young by Professor Whyburn.

ALGEBRA AND THEORY OF NUMBERS

237*t*. A. T. Brauer: *On the irreducibility of polynomials with large third coefficient.* II.

Let $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ be a polynomial with integral rational coefficients, $a_n\neq 0$, and $t=|1+a_4|+|a_1+a_3|+|a_5|+\cdots+|a_n|$. Improving a theorem of Perron (J. Reine Angew. Math. vol. 132 (1907) pp. 288–307) it was shown in the first part of this paper (Amer. J. Math. vol. 70 (1948) pp. 423–432) that $f(x)$ is irreducible in the field of rational numbers if a_2 is sufficiently large. It is proved in this paper that $f(x)$ is irreducible if $a_2>t$ and one of the other coefficients is sufficiently large. (Received March 7, 1951.)

238*t*. C. G. Chehata: *An algebraically simple ordered group.*

Let F be a fully ordered field, and $G(F)$ the set of all monotone increasing one-valued functions $f(x)$ defined for $x\in F$ and taking their values in F with the property that for each f there exists an interval $[\lambda, \mu]$ in F on which f is piecewise linear and outside which $f(x)=x$. Then $G(F)$ is a group under substitution and can be linearly ordered so as to be ordinally simple in the sense that it has no proper convex normal subgroup. Also those elements of $G(F)$ for which λ, μ lie in a fixed closed interval I or a fixed open interval I' form a subgroup $G(I)$ or $G(I')$ respectively. All $G(I)$, for different closed intervals, are order-isomorphic. The derived group of $G([\alpha, \beta])$ is $G((\alpha, \beta))$. When F is the field of real numbers, then all $G(I')$ are order-isomorphic to $G(F)$. In this case the group $G([0, 1])$ was first considered by K. Iwasawa, *On linearly ordered groups*, Journal of the Mathematical Society of Japan vol. 1 (1948) pp. 1–9, example 8. The principal result is that $G(I')$ for any open interval I', and thus $G(F)$, is algebraically simple in the sense that it has no proper normal subgroup. (Received March 5, 1951.)

239. D. R. Morrison: *Quotient semigroups.*

A subset N of a semigroup S is called *normal* with respect to $M\subset S$ if $mN=Nm$ for each $m\in M$. If $N\subset S$ is a semigroup normal with respect to S, a *quotient semigroup* S/N is defined. If a semigroup $B\subset S$ has elements cancellable in S and if S is normal with respect to B, then B is contained in a unique minimal semigroup $C\subset S$ such that C is normal with respect to B. Then elements of C are cancellable in S, and C and S are normal with respect to C. A *cross product* $S\times C$ is defined, with diagonal D. The quotient $S\times C/D$ contains an isomorphic image $\theta(S)$ of S, and elements of $\theta(C)$ have inverses in $S\times C/D$. If S is a semigroup and $B\subset S$ is a semigroup whose elements are cancellable in S, then S can be embedded in a semigroup $G\supset\theta_2(S)$ in which every element of $\theta_2(B)$ has an inverse if and only if S can be embedded in a semigroup $T\supset\theta_1(S)$ in which elements of $\theta_1(B)$ are cancellable and T is normal with respect to $\theta_1(B)$. (Received March 15, 1951.)

240. W. V. Parker: *On quadratic matric equations.*

The matric equation $XAX+BX+XC+D=0$ is considered in connection with the unilateral equation $X^2P+X(Q-M)-N=0$ where the matrix with first column C, D and second column $-A, -B$ and the matrix with first column M, N and second column P, Q are commutative. The results obtained are extensions of results given by Roth in his study of the matric equation $X^2+AX+XB+C=0$ (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 586–589). It is shown that when A

and P are nonsingular, any solution U of one of the equations which is such that both A and B are commutative with $UP+Q$ is also a solution of the other equation. (Received February 26, 1951.)

241. W. D. Peeples: *On elliptic ternary cubics with an infinite number of rational solutions.* Preliminary report.

Although Mordell and others have studied the fundamental properties of the configuration of rational points on an elliptic ternary cubic, there exist no general conditions sufficient to assure that a given curve have an infinite number of rational points. In this paper the problem is studied from the point of view of valuation theory. It is shown that simple sufficient conditions exist and that these results include as special cases results given in some previous papers. (Received March 7, 1951.)

242t. J. A. Silva: *Representations of arithmetic functions in $GF[p^n, x]$ with values in an arbitrary field.*

By an arithmetic function is meant a single-valued function $f(A)$ defined for polynomials A over $GF[p^n, x]$ for which $\deg A \leqq r$ (where r is a fixed positive integer); the values of $f(A)$ lie in an arbitrary field F. (See Carlitz, *Representations of arithmetic functions in $GF[p^n, x]$*.) In this paper a set of p^{nr} linearly independent arithmetic functions is given by means of which an arbitrary arithmetic function $f(A)$ may be represented with coefficients from the field F. The results obtained for F of characteristic zero or $q \neq p$ are quite similar but differ markedly from the results for F of characteristic p. (Received February 19, 1951.)

243t. S. M. Spencer: *Transcendental numbers over certain function fields.*

The first part of the paper contains a proof of an analogue of a theorem due to G. Faber (Math. Ann. vol. 58 (1904) pp. 545–557). It is shown that certain entire functions $f(t)$ have the property that $f(\alpha)$ is transcendental for all algebraic nonzero α. The second part of the paper contains proofs of several theorems over fields of characteristic p, all of which are generalizations of or suggested by certain theorems due to L. I. Wade (Duke Math. J. vol. 8 (1941) pp. 701–720; vol. 11 (1944) pp. 755–758). A typical example of such a theorem is: The series $\sum_{k=0}^{\infty} G_k^{-g^{e_k}}$ is transcendental, where the G_k are polynomials such that $G_k \mid G_{k+1}$; $\deg G_0 \geqq 1$; $e_0 < e_1 < e_2 < \cdots$; $g \geqq 2$; and $p \nmid e_k$. The underlying field is $F(x_1, \cdots, x_w)$, where F is a field of arbitrary characteristic in the first part but $F = GF(p^n)$ in the second part. (Received February 24, 1951.)

244. H. S. Thurston: *Automorphism and anti-automorphisms of a quaternion algebra.* Preliminary report.

Under a biunique correspondence, $\alpha \leftrightarrow \alpha'$, of a quaternion algebra over a formally real field, let $i \leftrightarrow i'$, $j \leftrightarrow j'$, $k \leftrightarrow k'$. If v denote the vector $(1, i, j, k)$ and v' the vector $(1, i', j', k')$, let P be the matrix such that $vP = v'$. R and R' being the first regular matric representations of α and α', and S and S' their second regular representations, the following relations are proved: (a) if the correspondence is an automorphism, then $PRP^{-1} = R'$, $PSP^{-1} = S'$, (b) if the correspondence is an anti-automorphism, then $PRP^{-1} = S'^{T}$, the transpose of S'. (Received March 7, 1951.)

245t. A. D. Wallace: *Isomorphism theorems in group theory and cohomology.*

Fix a non-negative integer p. If the pair X, A is a group and normal subgroup, let (X, A) be X/A. If the pair X, A is a fully normal space and a closed set, let (X, A) be $H^p(X, A)$. For groups $A\cup B$ will mean the smallest group containing both A and B. Then (tacitly assuming the proper normality and closure conditions) the second isomorphism theorem and the excision theorem can both be written as $(A\cup B, B) \approx (A, A\cap B)$. Let $(X_1, A_1)\cap(X_2, A_2)$ be $(X_1\cap X_2, (A_1\cap X_2)\cup(X_1\cap A_2))$. The Zassenhaus isomorphism theorem asserts that $(X_1, A_1)\cap(A_1\cup X_2, A_1\cup A_2) \approx (X_1, A_1)\cap(X_2, A_2)$. With suitable closure conditions, this is a valid proposition for cohomology. It is interesting to observe that the cohomology group $(X_1, A_1)\cap(X_2, A_2)$ has arisen recently in a paper by N. E. Steenrod. (Received March 7, 1951.)

ANALYSIS

246. L. P. Burton and W. M. Whyburn. *Minimax solutions of ordinary differential systems.*

A system of n, nonlinear, differential equations of the first order is considered under conditions which insure the existence but not the uniqueness of a solution through a given point in $(n+1)$-dimensional real space. Perron, Kamke, and others have studied this system under hypotheses that insure the existence of solutious of maximum and minimum types through each point. Systems which do not have solutions of these types may have critical solutions of intermediate or "minimax" type provided they exhibit the monotonic character developed in the present paper. The paper establishes the existence and uniqueness of such solutions. It also shows that these solutions can be uniformly approximated by mixed over and under functions, these functions having the nature of the over and under functions used in the Perron theory of integration. (Received April 20, 1951.)

247t. J. D. Mancill: *On a proof of the Jacobi condition in the calculus of variations.* Preliminary report.

The proof of the Jacobi condition for simple problems of the calculus of variations by means of an envelope of a one-parameter family of extremals does not necessarily apply to all extremal arcs of the minimizing curve in the case when the minimizing curve may have arcs in common with the boundary of the region of admissible curves. So far as the author knows no use has been made of the second variation in this connection. The author has given a proof of this condition in two dimensions which is applicable to such problems (*The Jacobi condition for unilateral variations*, Duke Math. J. vol. 6 (1940) pp. 341–344). In the present paper this proof is extended to problems in space of three dimensions for the nonsingular case of the enveloping surface of a two-parameter family of extremals. (Received March 7, 1951.)

248t. K. S. Miller: *Self-adjoint differential systems.*

D. Jackson (Trans. Amer. Math. Soc. vol. 17 (1916) pp. 418–424) has given a criterion for an ordinary linear differential system to be self-adjoint. His results were later written in more explicit form by V. V. Latshaw (Bull. Amer. Math. Soc. vol. 39 (1933) pp. 969–978). In this paper a different criterion is derived which is more convenient to use in certain problems. If L is a linear differential operator of order n and $U_\alpha(u) = V_\alpha(u) + Z_\alpha(u)$, $\alpha = 1, 2, \cdots, n$, are two point boundary conditions in which all the terms involving the left end point of the interval are included in $V_\alpha(u)$ and all those involving the right end point are grouped in $Z_\alpha(u)$, then the differential system is self-adjoint if $L = (-1)^n L^+$ and the matrix $\Delta CD'$ is symmetric (skew-symmetric) if n is odd (even). Here L^+ is the formal adjoint of L. If V and Z are the matrices of the linear forms $V_\alpha(u)$, $Z_\alpha(u)$, then $D=V+Z$, $\Delta=V-Z$, and C is the matrix relating n linearly independent solutions of $Lu=0$ to the corresponding ones for $L^+u=0$. The proof makes use of the Green's function. (Received February 14, 1951.)

249t. B. J. Pettis: *On the continuity of parametric linear operations.*

Let H be an open semigroup in an additive second category topological group G, and let T_h be additive on H to the space $B(X)$ of all linear operations in a complex Banach space X. For each x_0 in X the function $\phi(h)=T_h(x_0)$ will be continuous on H to X if the following is true; for each x_0 there is a set $R(x_0)$ residual in H and a set $\Gamma(x_0)$ of complex linear functionals on X such that $\phi(R(x_0))$ is separable in X; $\Gamma(x_0)$ is total for $\phi(H)$, and $\gamma(\varphi(h))$ is continuous for each γ in $\Gamma(x_0)$. (Received February 13, 1951.)

250. J. M. Thomas: *Equations equivalent to a linear differential equation.*

If $c_1y_1+\cdots+c_ny_n$ is a general solution of a linear homogeneous ordinary differential equation $L(y)=0$ and $F(y_1, \cdots, y_n)$, where F is sufficiently often differentiable, is a general solution of an ordinary differential equation E, then E is to be called equivalent to $L(y)=0$. Any E equivalent to $L(y)=0$ can be given the form $L(y)=G$, where G is a rational function of y, $y_1, \cdots, y_n$, F, and their derivatives. If the solution of $L(y)=0$ is known, an equivalent equation can be had by arbitrarily assigning F. If merely the coefficients of $L(y)=0$ and the solutions of certain equations of lower order formed with them are assumed known, then F must satisfy a system of partial differential equations, particular solutions of which lead to categories of equations solvable in terms of linear equations. For $n=2$ there is found in this manner a category including the equation discussed by Pinney (Proceedings of the American Mathematical Society vol. 1 (1950) p. 681). (Received March 6, 1951).

GEOMETRY

251. Eugenio Calabi and D. C. Spencer: *Completely integrable almost complex manifolds.*

An almost complex manifold is defined as a real, differentiable manifold with a differentiable transformation H on its covariant vector bundle satisfying $H^2=-1$. Such a manifold is always even-dimensional. An almost complex manifold is called *completely integrable* if there exists in a neighborhood of each of its points an allowable coordinate system $(x^1, \cdots, x^{2k})$, satisfying (1) $Hd(x^\alpha+(-1)^{1/2}x^{k+\alpha})=(-1)^{1/2}d(x^\alpha+(-1)^{1/2}x^{k+\alpha})$ $(1\leqq\alpha\leqq k)$. Any two systems of complex-valued coordinates, $z^\alpha=x^\alpha+(-1)^{1/2}x^{k+\alpha}$ and $z'^\alpha=x'^\alpha+(-1)^{1/2}x'^{k+\alpha}$, associated with two systems satisfying (1) in a common domain, are complex analytically related, so that every completely integrable almost complex manifold corresponds to a complex manifold in a natural way. In an almost complex manifold one defines a tensor T^l_{ij}, called the *pseudo-conformal curvature tensor*, from the tensor h^i_j representing the transformation H by $T^l_{ij}=(1/4)(\delta^{ab}_{ij}-\delta^{rs}_{ij}\, h^a_r h^b_s)\partial h^l_b/\partial x^a$, which leads to the main theorem. *An almost complex manifold is completely integrable if and only if its pseudo-conformal curvature vanishes identically.* The necessity is proved by a simple computation of exterior differential forms; the sufficiency is proved by constructing locally a Riemannian metric satisfying Kähler's condition, from which one obtains the complex analytic coordinates by a variational method. (Received March 7, 1951.)

252t. H. S. M. Coxeter: *Group invariants and Poincaré polynomials.*

The product of the generators of a finite group generated by reflections is an n-dimensional orthogonal transformation whose period, h, has several interesting properties (Coxeter, Ann. of Math. vol. 35 (1934) pp. 606–617). In terms of ω, a primitive hth root of unity, the characteristic roots of this transformation are ω^m for n special integers m, which may be taken to lie between 0 and h. These are computed by means of a general formula (*Lösung der Aufgabe* 245, Jber. Deutschen Math. Verein. vol. 49 (1938) pp. 4–6). The point of interest is that the same integers occur in a different connection. Continuing the work of Klein (*Lectures on the icosahedron*, pp. 237, 242) and Burnside (*Theory of groups*, 2d ed., p. 362), Racah has found a set of n independent invariant forms for each group (Rendiconti della Accademia Nazionale dei Lincei (8) vol. 8 (1950) p. 112). These n invariants are a basic set, since the product of their degrees is equal to the order of the group. Actually, the degrees are just the n numbers $m+1$. In the case of the crystallographic groups, the Poincaré polynomial for the group manifold of the corresponding simple Lie group (Chevalley, Proceedings of the International Congress of Mathematicians, 1950) is $\prod(1+t^{2m+1})$. (Received March 6, 1951.)

253. D. O. Ellis: *On the metric characterization of metric lattices.*

We consider in this note a metric lattice in its metric topology and show that *each compact metric lattice has a first (and last) element.* Thus the result, *a metric space is congruent to a metric lattice with first element if and only if it is almost ordered,* of Smiley and Transue (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 280–287) yields: *A compactum is congruent to a metric lattice if and only if it is almost ordered.* We deduce from a study by Blumenthal and Ellis (Duke Math. J. vol. 16 (1949) pp. 585–590) that *no nonlinear subset of a Euclidean space or (real, separable) Hilbert space is congruent to a metric lattice,* and from a result in Birkhoff's book (Amer. Math. Soc. Colloquium Publications, vol. 25) that *any compact metric lattice is lattice complete.* (Received January 17, 1951.)

254. Benjamin Ernest Mitchell: *Vector in the complex plane, its representation.*

This paper endeavors to achieve an intuitive (real) conception of the imaginary vector in the complex plane. The result is an ordered vector pair in the real projective plane. An interpretation of the imaginary vector is revealed by representative vector pair including among the special cases the three mentioned by Eduard Study, to wit: The Kreuzlage, the Trapezlage, and the Isometrisch position. Equivalent representations in standard and normal positions (so called) are proposed. Also a planar representation of a real vector in a point space of four dimensions is offered. (Received March 5, 1951.)

Logic and Foundations

255. J. K. Feibleman: *The problem of the metapostulates.*

In setting up a mathematical system, there are assumptions which have never been brought out into the open. We do not start at the beginning when we set up undefined terms, called constants and variables, and unproved propositions, called axioms, and then proceed to draw theorems and other deductions from them. For the undefined terms, the unproved propositions, and the process of deduction each has its own set of unacknowledged presuppositions. These are stated in the proper statement of the metapostulates for any mathematical system. A tentative suggestion in this direction is proposed. There is a further problem engendered by this one, which will be here briefly indicated. The metapostulate-set, m_1, has its own set of presuppositions, which we shall call the meta-metapostulates, m_2. There are reasons for believing that the meta-metapostulate-set is equivalent to the metapostulate-set. The author seeks to devise a postulate-set, m_3, which allows for the repetition involved in the equivalence of m_1 and m_2, and which also is reflexive. (Received February 5, 1951.)

TOPOLOGY

256. D. O. Ellis and G. B. Lang: *The space of groupoids over a compactum.*

Let M be a compactum with distance function $\delta(x, y)$. Let $\mathfrak{S}$ be the set of all groupoid operations over the point set of M with distance function $\delta(\mathfrak{a}, \mathfrak{b}) = \sup \delta(x\mathfrak{a}y, x\mathfrak{b}y)$. We show: (1) $\mathfrak{S}$ is complete. (2) $\mathfrak{S}$ is distancial to M. (3) $\mathfrak{S}$ is convex when M is convex. (4) The following subsets of $\mathfrak{S}$ are closed: the continuous groupoids, continuous semigroups, continuous commutative groupoids, topological quasigroups, topological loops, and the topological groups. (5) An element of $\mathfrak{S}$ is continuous if and only if the sublattice of closed elements of the closure algebra of M is a subgroupoid of the closure algebra of M under the natural extension to this closure algebra of the given element of $\mathfrak{S}$. (Received January 17, 1951).

257. W. L. Gordon and C. W. McArthur: *On uniform Cauchy points.*

Let X be any topological space and ϕ any cardinal. A subset E of X is a I_ϕ-set if it is the union of ϕ nowhere dense sets; otherwise it is a II_ϕ-set. It is ϕ-residual if its complement $X-E$ is a I_ϕ-set. Let Y be any completely regular space, and of all uniformizations of the topology in Y let $\{V_\alpha | \alpha \in A\}$ be one having least cardinal $|A|$. For each element λ of a fixed directed set Λ let f_λ be a function on X to Y. A point x_0 in X is a *Cauchy point* of $\{f_\lambda\}$ if given any $\alpha \in A$ there is some $\lambda_\alpha \in \Lambda$ such that $f_\lambda(x_0) \in V_\alpha(f_{\lambda_\alpha}(x_0))$ for all $\lambda \geqq \lambda_\alpha$. x_0 is a *uniform Cauchy point* of $\{f_\lambda\}$ if for each $\alpha \in A$ there exists $\lambda_\alpha \in \Lambda$ and an open set G_α about x_0 such that $f_\lambda(x) \in V_\alpha(f_{\lambda_\alpha}(x))$ for all $\lambda \geqq \lambda_\alpha$ and all $x \in G_\alpha$. Let C be the Cauchy points and C_u be the uniform Cauchy points and $|\Lambda|$ be the least of the cardinals belonging to cofinal subsets of Λ. If each f_λ is continuous and $\phi \geqq \max(|A|, |\Lambda|)$, then $C-C_u$ is I_ϕ. Hence if C is a II_ϕ-set, C_u is also; if C is ϕ-residual, so is C_u. This includes some recent results of A. Alexiewicz, A. M. Ostrowski, and B. J. Pettis. (Received March 5, 1951.)

258*t*. S. T. Hu: *On products in homotopy groups.*

In 1947, a product analogous to those of J. H. C. Whitehead was introduced by the author (J. London Math. Soc. vol. 22) for the relative homotopy groups; namely, for any two elements $a \in \pi_{p+1}(X, A)$ and $b \in \pi_{q+1}(X, A)$, it associates with a unique element $[a, b] \in \pi_{p+q}(X, A)$. In the present paper, it is proved that $[a, b]$ depends only on the boundary elements $\partial a \in \pi_p(A)$ and $\partial b \in \pi_q(A)$. Let $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$ be any two given elements such that $i_*\alpha = 0 = i_*\beta$ and the Whitehead product $[\alpha, \beta] = 0$, where i_* denotes the homomorphisms induced by the identity map $i: A \subset X$.

Choose two elements a and b such that $\partial a = \alpha$ and $\partial b = \beta$. Then $[a, b]$ is an element in the image $j_* \pi_{p+q}(X)$ under the homomorphism $j_*: \pi_{p+q}(X) \rightarrow \pi_{p+q}(X, A)$ induced by the identity map $j: X \subset (X, A)$. By the exactness of the homotopy sequence, j_* induces an onto isomorphism $j_0: \pi_{p+q}(X)/i_* \pi_{p+q}(A) \approx j_* \pi_{p+q}(X)$. We denote by $\alpha \otimes_i \beta$ the element $j_0^{-1}[a, b]$ of the quotient group $\pi_{p+q}(X)/i_* \pi_{p+q}(A)$. By passing to the mapping cylinder of any given map $f: A \rightarrow B$, functional products are defined in a natural way. To each pair of elements $\alpha \in \pi_p(A)$, $\beta \in \pi_q(A)$, such that $f_*(\alpha) = 0 = f_*(\beta)$ and $[\alpha, \beta] = 0$, we associate a unique element $\alpha \otimes_f \beta$ of the quotient $\pi_{p+q}(B)/f_* \pi_{p+q}(A)$, which is an invariant of the homotopy class of the map f and is zero whenever f is homotopic to a constant. (Received March 5, 1951.)

259. J. L. Kelley: *Remarks on a theorem of Keesee.*

Keesee has shown that if X is a compact Hausdorff space the cohomology theory of Alexander-Kolmogoroff-Spanier-Wallace is isomorphic to the same theory with the restriction that each co-chain have finite range. The homology theory corresponding to this restricted co-chain theory can be easily described in terms of those contents (that is, finitely additive set functions) on X^{p+1} to a compact coefficient group g which vanish for sets whose closures fail to intersect the diagonal Δ^{p+1}. The connection with Kolmogoroff's original theory is then easy to see, as well as the fact that the contents considered may be restricted to interval functions of a rather special sort. (Received March 7, 1951.)

260. R. S. Novosad: *Groups of unit homomorphisms.*

Given a pair (X, x) consisting of a topological space X and a point x of X, a space Z can be constructed from a set of 1-spheres which are indexed by the elements of the fundamental group of (X, x). Z maps into X in a natural manner. Homotopies with the constant path, of paths in X which are images of paths in Z, form a subspace of the space of mappings of the unit square into X. The arcwise connected components of this subspace can be multiplied, and form a group which is denoted by $W_2(X, x)$. $W_2(X, x)$ is a central group extension of the second homotopy group of (X, x) by the group relations of the fundamental group of (X, x). $W_2(X, x)$ maps homomorphically onto the second singular homology group in a way similar to the mapping of the fundamental group onto the first homology group. (Received March 7, 1951.)

261*t*. B. J. Pettis: *A note on everywhere dense subgroups.*

Recently H. C. Wang has shown that any locally compact separable nondiscrete metric group must contain an uncountable everywhere dense proper subgroup. This can be extended as follows; in any second category nondiscrete separated group any everywhere dense proper subgroup lies in some uncountable everywhere dense proper subgroup. This in turn is a special case of the following conjecture; any proper subgroup of a second category topological group X lies in a second category proper subgroup. For abelian X the conjecture is verified for these two cases: (1) X is second category and x^k maps non-null open sets into somewhere dense sets for each positive integer k; and (2) X is locally euclidean. (Received January 19, 1951.)

262. B. J. Pettis: *Comments on open homomorphisms.*

This paper is concerned with certain refinements and extensions of recent results of the author (Ann. of Math. vol. 52 (1950) pp 293–308) and of J. Dieudonné (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 54–59) (Received March 19, 1951.)

263. A. D. Wallace: *Pseudo retraction and deformation.*

In this note, we define pseudo retraction and deformation in such a way that the main results connecting retraction and deformation with cohomology carry over to a more general situation. In this way, we can extend to spaces that are not locally connected theorems that were previously known to hold only in the locally connected case. (Received February 26, 1951.)

264. H. C. Wang: *On locally compact two-point homogeneous spaces.*

Let E be a metric space with metric d. It is called two-point homogeneous (or in short, a (*)-space) if, for any 4 points a_1, a_2, b_1, b_2 with $d(a_1, a_2) = d(b_1, b_2)$, there exists an isometry of E carrying a_1, a_2 to b_1, b_2 respectively. In a recent paper, the author has determined all the compact, connected (*)-spaces. This note concerns the locally compact (*)-spaces which are S.L. (roughly, any two points can be joined by a unique geodesic). We shall prove that such a space E must be a manifold, and if dim E is odd, then E is either the elliptic space, the euclidean space, or the hyperbolic space. (Received March 9, 1951.)

265. L. E. Ward, Jr.: *Relations between character groups of locally compact abelian T_0-groups.*

Let G be a topological group, and G^* its character group. Numerous computational theorems concerning the relations between G^*, the annihilator subgroups of G^*, and the character groups of closed subgroups and continuous relatively open images of G have been established by Pontrjagen, van Kampen, Freudenthal, and others in the cases where G is compact, or discrete, or locally compact and satisfying the first countability axiom. The author proves a number of these theorems as well as some new results for the case when G is a locally compact Abelian T_0-group, without countability restrictions. The purpose of these generalizations is to establish a set of computational theorems for homology theory. (Received March 7, 1951.)

266. C. T. Yang: *On Borsuk's problem.*

Borsuk has asked this question: If each of the topological spaces X and Y is homeomorphic with a retract of the other, are the cohomology groups of X and Y isomorphic? It is shown that this is so if X is compact Hausdorff and the coefficient group is of finite exponent. (Received April 20, 1951.)

W. M. Whyburn,
Associate Secretary

THE APRIL MEETING IN NEW YORK

The four hundred sixty-eighth meeting of the American Mathematical Society was held at Columbia University, New York City, on Friday and Saturday, April 27–28, 1951. The meeting was attended by about 250 persons, including the following 217 members of the Society:

C. R. Adams, E. J. Akutowicz, R. D. Anderson, R. L. Anderson, T. W. Anderson, R. G. Archibald, Helmut Aulbach, Maurice Auslander, M. C. Ayer, Joshua Barlaz, L. W. Baumhoff, E. G. Begle, A. A. Bennett, Stefan Bergman, Lipman Bers, Nicholas Bilotta, D. W. Blackett, E. K. Blum, Raoul Bott, Samuel Bourne, A. T. Brauer, Paul Brock, F. E. Browder, A. B. Brown, R. G. Brown, R. H. Brown, M. L. Browne, L. J. Burton, J. H. Bushey, Jewell H. Bushey, F. P. Callahan, W. R. Callahan, R. R. Christian, K. L. Chung, F. E. Clark, A. B. Clarke, F. M. Clarke, Eckford Cohen, L. W. Cohen, R. M. Cohn, T. F. Cope, Natalie Coplan, H. S. M. Coxeter, J. B. Crabtree, C. W. Curtis, P. H. Daus, J. C. E. Dekker, J. B. Diaz, J. A. Dieudonné, Jesse Douglas, Nelson Dunford, William H. Durfee, Aryeh Dvoretzky, B. M. Dwork, J. S. Dwork, M. E. Estill, R. M. Exner, W. H. Fagerstrom, Herbert Federer, J. M. Feld, William Feller, Werner Fenchel, D. T. Finkbeiner, R. S. Finn, Irwin Fischer, H. A. Forrester, R. M. Foster, J. S. Frame, Gerald Freilich, F. N. Frenkiel, Bernard Friedman, K. O. Friedrichs, R. F. Gabriel, G. N. Garrison, H. M. Gehman, W. M. Gilbert, B. P. Gill, Leonard Gillman, Sidney Glusman, H. E. Goheen, J. K. Goldhaber, Oscar Goldman, R. D. Gordon, Daniel Gorenstein, J. A. Greenwood, Laura Guggenbuhl, Carl Hammer, P. C. Hammer, G. A. Hedlund, M. H. Heins, Alex Heller, Aaron Herschfeld, E. H. C. Hildebrandt, T. H. Hildebrandt, Einar Hille, A. J. Hoffman, S. P. Hoffman, Banesh Hoffmann, L. A. Hostinsky, E. M. Hull, T. R. Humphreys, L. C. Hutchinson, B. M. Ingersoll, Nathan Jacobson, R. E. Johnson, R. V. Kadison, Shizuo Kakutani, Irving Kaplansky, Edward Kasner, M. E. Kellar, J. F. Kiefer, H. S. Kieval, J. R. Kline, E. R. Kolchin, B. O. Koopman, Saul Kravetz, Helen Kutman, M. K. Landers, V. V. Latshaw, Solomon Lefschetz, Benjamin Lepson, Howard Levene, M. E. Levenson, W. G. Lister, L. H. Loomis, E. R. Lorch, Eugene Lukacs, N. H. McCoy, L. A. MacColl, G. W. Mackey, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, Murray Mannos, W. T. Martin, F. I. Mautner, A. E. Meder, Jr., A. N. Milgram, K. S. Miller, W. H. Mills, M. L. Minsky, Don Mittleman, F. J. Murray, W. R. Murray, John von Neumann, C. V. Newsom, A. V. Newton, O. M. Nikodým, P. B. Norman, A. F. O'Neill, J. C. Oxtoby, G. W. Patterson, L. E. Payne, Anna Pell-Wheeler, R. S. Phillips, G. B. Price, M. H. Protter, Hans Rademacher, G. N. Raney, H. E. Rauch, H. W. Reddick, Helene Reschovsky, Moses Richardson, C. E. Rickart, R. E. Roberson, J. H. Roberts, H. P. Robertson, M. S. Robertson, G. B. Robison, I. H. Rose, Maxwell Rosenlicht, H. D. Ruderman, James Sanders, Arthur Sard, L. R. Sario, S. W. Saunders, I. R. Savage, R. D. Schafer, E. C. Schlesinger, Abraham Schwartz, C. H. W. Sedgewick, M. A. Shader, Dorothy Shaffer, James Singer, P. A. Smith, Andrew Sobczyk, J. J. Sopka, Fritz Steinhardt, R. W. Stokes, D. M. Stone, Walter Strodt, M. M. Sullivan, Olga Taussky, E. W. Titt, P. M. Treuenfels, A. W. Tucker, J. W. Tukey, J. R. Van Andel, H. E. Vansant, D. F. Votaw, R. J. Walker, J. L. Walsh, W. G. Warnock, Alan Wayne, J. V. Wehausen, Alexander Weinstein, Morris Weisfeld, Louis Weisner, J. G. Wendel, Franc Wertheimer, G. N. White, G. T. Whyburn, M. A. Woodbury, Arthur Wouk, L. A. Zadeh, Oscar Zariski, Arthur Zeichner.

At a general session on Friday Professor Herbert Federer of Brown University delivered an invited address on *Measure and area*, Professor G. B. Price presiding. An invited address on *σ-complete Boolean algebras* was delivered by Professor L. H. Loomis of Harvard University at a general session on Saturday, Professor E. R. Kolchin presiding.

The session for contributed papers on Friday was presided over by Professor R. S. Phillips. Professor J. S. Frame and Professor J. H. Roberts presided at the Saturday morning sessions.

The Council met at the Faculty Club at 8:00 P.M. on April 27.

The Secretary announced the election of the following forty-five persons to ordinary membership in the Society:

Assistant Professor Joseph Andrushkiw, Seton Hall University;
Mr. Lester Glenn Arnold, Teaching Assistant, University of Rochester;
Mr. Lester W. Baumhoff, Staff Member, Los Alamos Scientific Laboratory;
Mr. Aaron Besner, Physicist, Norden Laboratories Corporation, White Plains, N.Y.;
Mr. David Alvin Buchsbaum, Lecturer, Columbia University;
Mr. Francis Patrick Callahan, Jr., Mathematician and Physicist, Research Laboratories, Interchemical Corporation, New York, N.Y.;
Mr. Rene Felix Dennemeyer, Research Engineer, North American Aviation Company, Aerophysics Laboratory, Downey, Calif.;
Mr. James D. Esary, Teaching Assistant, University of California, Berkeley 4, Calif.;
Mr. Louis Samuel Gage, Instructor, Radar Department, Keesler Air Force Base, Biloxi, Miss.;
Mr. Deba Brata Ghosh, Assistant, Reserve Bank of India, Calcutta, India;
Mr. Raef K. Haddad, Teaching Assistant, Psychology Department, New York University;
Mr. Bradford Frank Hadnot, Instructor, North Georgia College, Dahlonega, Ga.;
Mr. Lawrence Harris, Research Engineer, Norden Laboratories Corporation, White Plains, N.Y.;
Mr. Joseph Owens Harrison, Jr., Assistant Supervisor, Computational Analysis Laboratory, Eckert-Mauchly Computer Corporation, Philadelphia, Pa.;
Mr. Herbert Milton Heughan, Instructor, Agricultural and Technical College, Greensboro, N.C.
Miss Gladys Naxiene Hurt, Instructor, Morehouse College;
Mr. Shikao Ikehara, Staff Member, Research Laboratory of Electronics, Massachusetts Institute of Technology;
Dr. Salem Hanna Khamis, Statistician, Statistical Office, United Nations, New York, N.Y.;
Mr. V. V. Lavroff, Comptroller, Atlanta Division, University of Georgia, Atlanta, Ga.;
Sister Catherine Marie Lee, Instructor, Trinity College, Washington, D.C.;
Miss Louise L. Leon, Actuarial Department, Guardian Life Insurance Company, New York, N.Y.;
Sister Mary Ferrer McFarland, Instructor, St. Xavier College, Chicago 15, Ill.;
Mr. John Patterson Mayberry, Research Assistant, Princeton University;
Mr. Henry I. Meyer, Mathematician, United Gas Corp., Shreveport, La.

Associate Professor Robbie Lee Beaven Morris (Mrs. Andrew F.), Cumberland University, Lebanon, Tenn.;
Associate Professor Harry Ernest Nelson, Augustana College, Rock Island, Ill.;
Mr. Frederick Esser Nemmers, Milwaukee, Wis.;
Assistant Professor Charles Jefferson Pipes, Jr., Southern Methodist University;
Mr. Harold L. Platzer, Columbia University;
Mr. Solomon Leonard Pollack, Statistical Draftsman, Office of Surgeon General, Washington, D.C.;
Mr. John Francis Ries, Instructor, Kings College, Kings College, Del.;
Mr. George Schaeffer Ritchie, University of Michigan;
Mr. Paul George Rooney, Assistant, California Institute of Technology;
Dr. Edward Walter Samson, Chief, Communications Laboratory, Air Force Cambridge Research Laboratories, Cambridge, Mass.;
Mr. Earl Justin Schweppe, University of Illinois;
Mr. Raymond Sedney, Teaching Assistant, Carnegie Institute of Technology;
Mr. Frederico Mariano Sioson, Pantihan No. 1, Naval Street, Malabon, Rizal, Philippines;
Dr. Roy C. Spencer, Chief, Antenna Laboratory, Air Force Cambridge Research Laboratories, Cambridge, Mass.;
Mr. Robert Delmer Stalley, Instructor, University of Arizona;
Mr. Carl Worth Steeg, Jr., Instructor, Massachusetts Institute of Technology;
Mr. Marvin Stern, Consultant, Republic Aviation Corporation, Farmingdale, N.Y.;
Mr. Carl Henry Tross, Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.;
Mr. Michael Anthony Tschupp, Testing Engineer, Electric Specialty Company, South Street, Stamford, Conn.;
Mr. Richard Joseph Turyn, Harvard University;
Mr. Howard Raymond Wright, Jr., Aeronautical Engineer, Latin-American Division, Pan-American Airways, Miami, Fla.;
Mr. Herbert S. Wolf, Assistant, University of Illinois;

It was reported that the following seventy persons had been elected as nominees of institutional members as indicated:

University of Alabama: Mr. Thomas Curtis Yarbrough.
University of British Columbia: Mr. Ronald Alvin Macauley.
California Institute of Technology: Messrs. Douglas Robert Anderson and Richard Scott Pierce.
University of Chicago: Messrs. George Edward Backus, William Ralph Ballard, Joseph Bram, Arlen H. Brown, Ernest Lyle Griffin, Jr., Frank Douglas Quigley, Andrew Hugh Wallace, and Oswald Wyler.
University of Cincinnati: Messrs. Hugo Heermann and Frank Levin.
City College of New York: Messrs. Barry Bernstein, Jacob Leon Bricker, Herman Hanisch, and Harold Widom.
Columbia University: Messrs. Paul Otokar Backer, Enrique Bayó, Roberto Joseph Beeber, Lawrence Goldman, and Katsumi Nomizu.
Cornell University: Messrs. John Wells Brace, Tseng Yeh Chow, and Jean-Pierre G. Meyer.
Harvard University: Messrs. Kurt Bing and Zdziskaw Alexander Melzak.
Institute for Advanced Study: Professor Rudolf Inzinger.
Kenyon College: Mr. Lane Schofield Hart, IV.

Michigan State College: Messrs. John William Coy, Rubin Kuritsky, and Richard Zindler.

New York University: Messrs. Peter Ceike, Richard Goldberg, and Leon Weliczker.

Northwestern University: Messrs. Thomas Cleveland Littlejohn and James Sparks Thale.

Oklahoma Agricultural and Mechanical College: Mr. Allan Kellerman Jennings.

State University of Iowa: Mr. Howard Henry Wicke.

Syracuse University: Dr. Jim Christoph Edmond Dekker, Miss Margaret A. Finn, and Mr. Melvin Aaron Shader.

University of Illinois: Mr. Thomas Blaney Elfe, Jr.

University of Michigan: Mr. Walter Ferdinand Bauer.

University of Minnesota: Mr. William O. S. Moser.

University of Missouri: Mr. Steven J. Bryant.

University of Texas: Messrs. David Fleming Dawson, Bennie J. Pearson, and John Marshall Slye.

University of Toronto: Messrs. Jean Eugene LeBel, Gordon Marshall Petersen, and Richard James Semple.

University of Virginia: Messrs. Julian Brody, Ralph Warren Jollensten, Wendell Endicott Malbon, and Robert Lee Plunkett.

University of Washington: Mr. Calvin Abraham Rogers.

University of Wisconsin: Messrs. Henry Beiman, Wilbur Eugene Deskins, Carl Edwin Diesen, Trevor Evans, Philip Wayne Gillett, Adrien Leroy Hess, Lucien Wolf Neustadt, Marshall Flynn Ruchte, Robert Louis San Soucie, and Bernard Witt.

Vanderbilt University: Mr. Thomas Lee Jordan.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Mr. James Alexander Edmond Black, Research Laboratories, General Electric Company, Stanmore, Middlesex, England; Professor Jacob Lionel Bakst Cooper, University College, Cardiff, Wales; Société Mathématique de France: Professor André Jean-Marie Combes, Versailles Lycée, Mantes-Gassicourt, Seine at Oise, France; Professor Abolghassen Ghaffari, Teheran University, Teheran, Iran; Professor Sophie Piccard, University of Neuchâtel, Neuchâtel, Switzerland; Unione Matematica Italiana: Professor Francesco Cecioni, Institute of Mathematics, University of Pisa, Pisa, Italy; Dr. Carmelo Roberto Fusa, National Institute of Higher Mathematics, Rome, Italy; Wiskundig Genootschap te Amsterdam: Dr. Leonard Lips, Lecturer, University of Indonesia, Bandung, Java, Indonesia.

Queens College, New York, N. Y. was elected to institutional membership.

The following actions taken by mail vote of the Council were reported: election of Professors W. Feller and M. H. Heins as members of the Executive Committee of the Council for a period of two years beginning January, 1951; approval of the recommendation of the

Policy Committee that adherence to the International Mathematical Union be through the National Research Council; the election of North Texas State Teachers College to institutional membership.

The following appointment of a representative of the Society was reported: Mary Ann Lee at inauguration of John Rutherford Everett as President of Hollins College on April 16, 1951.

The following additional appointments by the President were reported: T. F. Cope and B. P. Gill as auditors of the Society's accounts for 1951; E. Lukacs (Chairman), Leon Cohen, E. J. Cox, F. E. Johnston, O. J. Ramler, F. D. Rigby, and F. W. Cohon as a committee on arrangements for Washington, D. C. meeting October 27, 1951; A. E. Meder, Jr. (Chairman), H. M. MacNeille, and P. A. Smith as a committee to make recommendations on policy on bequests; C. R. Adams (Chairman), E. R. Lorch, and D. V. Widder as a committee to consider dues of members who are wives of members; J. R. Kline (Chairman), Oswald Veblen, and G. T. Whyburn as a committee for consultation on the Duke Mathematical Journal; G. A. Hedlund (Chairman), William Feller, M. R. Hestenes, F. B. Jones, and S. B. Myers as a committee to nominate officers and members of the Council for 1952; A. W. Tucker (Chairman), Einar Hille, J. R. Kline, and P. A. Smith as a committee to study problems of publication.

The following items were reported for the information of the Council: selection of G. T. Whyburn as Managing Editor of Transactions and Memoirs for 1951; A. W. Tucker as Chairman of Mathematical Surveys Editorial Committee for 1951; G. B. Price as Chairman of Bulletin Editorial Committee for 1951; Nathan Jacobson as Chairman of Proceedings Editorial Committee for 1951; R. L. Wilder as Chairman of Colloquium Editorial Committee for 1951; William Feller as Chairman of Mathematical Reviews Editorial Committee for 1951; as new members of the Policy Committee for Mathematics: E. H. C. Hildebrandt and Saunders MacLane (The Committee now consists of: Einar Hille, J. R. Kline (Chairman), W. T. Martin, and Marston Morse for the American Mathematical Society; H. M. Gehman, Saunders MacLane, and C. V. Newsom for the Mathematical Association of America; A. E. Meder for the Association of Symbolic Logic; Henry Scheffé for the Institute of Mathematical Statistics; and E. H. C. Hildebrandt for the National Council of Teachers of Mathematics); acceptance by Alfred Tarski of invitation to deliver the Colloquium Lectures at the Summer meeting of 1952; acceptance by Kurt Gödel of invitation to deliver the 25th Josiah Willard Gibbs Lecture at the Annual meeting, 1951; Committees to Select Hour

Speakers have invited Ivan Niven, June meeting, Washington State College, and G. W. Whitehead, Summer meeting, Minneapolis to deliver addresses.

The following resolution on the death of Professor G. A. Miller was read and adopted by the Council:

> The Council of the American Mathematical Society notes with regret the passing of George Abram Miller. Fifty-nine years ago last October Professor Miller was elected to membership in the New York Mathematical Society; of the present members of the American Mathematical Society only two preceded him. His membership was interrupted; he became a member again in 1898 and a life member in 1900. He was one of the founders of the San Francisco Section in 1902 and was its Secretary until 1906. He was a Vice President of the Society in 1908. Having returned to the Middle West, he was Chairman of the Chicago Section in 1908 and 1909.
>
> During the years when he seems not to have been officially connected with the Society, he was carrying on the work to which the Society is dedicated. From 1893 to 1895 he was instructor at the University of Michigan. He lived in the home of F. N. Cole who was for so many years Secretary of the Society and who in 1891 was Secretary of the Mathematical Society of the University of Michigan. It was Cole's influence that started him on the study of finite groups which he followed throughout his life. From 1895 to 1897 he was at Leipzig and Paris. In Paris he became associated with C. Jordan who sponsored all his communications to the Comptes Rendus of the Société Mathématique de France.
>
> Miller's first publications were two articles in volume 3 of the Bulletin of the New York Mathematical Society. By the end of the year of his return from Europe, he had published twenty-eight articles, fourteen of them in the Bulletin; by the end of 1915 he had contributed seventy-five articles to the Bulletin and the Transactions. He continued to contribute to the Bulletin until 1931 and to the Transactions until 1934. His last contribution to group theory, published in the Proceedings of the National Academy of Science, appeared in 1946 when he was eighty-three years old.
>
> The Council salutes a man who was a pioneer in American mathematics, who brought prestige to the Society, and who for years enough was universally acknowledged as the leader in one branch of mathematics.

The Executive Director reported for the information of the Council on some of the problems arising in the central office. The late arrival of programs of meetings is being investigated.

The Executive Committee reported to the Council for its information that six bids had been received for the library and that the high bid of $66,000, made by the University of Georgia, was the one best calculated to further the objectives of the Society, and that the Executive Committee had recommended to the Board of Trustees that this bid be accepted. (At their meeting on April 28, the Board of Trustees accepted this recommendation.) The Executive Committee also reported that after consideration of a number of possibilities it had recommended to the Board of Trustees that the Society lease a house from Brown University and move all its permanent offices there. (The Board of Trustees accepted this recommendation also.)

Professor J. R. Kline reported for the Policy Committee and announced that the four governing boards had approved the addition of the National Council of Teachers of Mathematics to the Policy Committee; that a new Division of Mathematics has been created in the National Research Council with Marston Morse as chairman and M. H. Stone as vice chairman; that the United States will be represented in the International Mathematical Union by a U. S. National Committee for Mathematics with five members nominated by the Policy Committee and four by the Division of Mathematics.

The Council voted to grant a leave of absence to Professor J. W. Green as Associate Secretary for the academic year 1951–52 and appointed Professor W. T. Puckett to fill this position during Professor Green's absence.

The Council voted to approve the request of the Board of Trustees that the Associate Secretaries resume the arranging of programs of meetings.

The Council voted to authorize the President to appoint a committee to advise the Council in its reconsideration of its actions with regard to the University of California and to study the implications of these actions for the general policies of the Society.

The Council voted to invite Professor M. H. Stone to serve as the Society's Visiting Lecturer for 1951–1952.

The Council received the final report of the Birkhoff Memorial Finance Committee, which indicates that this project will approximately break even in the long run.

A meeting of the Southeast Section was set for November 23–24 at the Alabama Polytechnic Institute, Auburn, Alabama.

Abstracts of the papers presented are listed below, those with a "*t*" after their numbers having been read by title. The joint papers numbered 268, 285, 291, 313, 316 were presented by Professor Brauer, Mr. Brock, Dr. Roberts, Dr. Hammer, and Dr. Chung, respectively. Dr. Arf was introduced by Professor Alexander Weinstein, Mr. Cherlin by Professor M. S. Robertson, and Mr. Ludford by Dr. Stefan Bergman.

ALGEBRA AND THEORY OF NUMBERS

267t. R. C. Bose and K. A. Bush: *Upper bounds for orthogonal arrays.*

Given an orthogonal array of strength t and index λ, the question arises as to the maximum number of rows that can be constructed without destroying the orthogonality of the array. A number of such bounds are obtained which represent improvements over the bounds previously established by C. R. Rao. The improvements made are in general predicated upon certain questions of divisibility. Examples are given showing that these bounds may actually be attained in a number of cases. (Received March 12, 1951.)

268. A. T. Brauer and T. L. Reynolds: *On a theorem of Aubry-Thue.*

L. Aubry (Mathesis (4) vol. 3 (1913) pp. 33–35) and A. Thue (Det Kongelige Norske Videnskabers Selskab Skrifter (1915) no. 3 and Archiv for Mathematik og Naturvidenskab vol. 34 (1917) no. 15) proved independently that the Diophantine equation $ax-by\equiv 0 \pmod{m}$ has solutions for which $0<x\leqq m^{1/2}$, $|y|\leqq m^{1/2}$. Vinogradov (Trans. Amer. Math. Soc. vol. 29 (1927) pp. 209–217) showed the existence of a solution for which $0<x\leqq k$, $0\leqq|y|\leqq m/k$ for any positive k. In this paper it is proved that a system of r linear homogeneous congruences in s unknowns $x_1, x_2, \cdots, x_s$ always has a nontrivial solution $|x_\sigma|\leqq m^{r/s}$ $(\sigma=1, 2, \cdots, s)$. Similarly Vinogradov's theorem is generalized. The theorem of Aubry-Thue was used in particular for a simple proof of the representation of the primes of form $4n+1$ as sum of two squares and for the proof that the least kth power nonresidue mod p with $p\equiv 1 \pmod{k}$ is less than $p^{1/2}$ for odd k. Correspondingly the generalization gives a simple proof for the four-square theorem. Moreover, it follows that each of the $k-1$ classes of kth power non-residues contains elements less than $p^{(k-1)/k}$. (Received March 14, 1951.)

269*t*. K. A. Bush: *Orthogonal arrays of strength 2.*

An orthogonal array of strength 2 is a rectangular array with the properties: i. Each row consists of the integers $0, 1, 2, \cdots, s-1$ each repeated s times. ii. Each of the s^2 possible ordered pairs that can be formed from the integers $0, 1, 2, \cdots, s-1$ occurs exactly λ times when all pairs of the form (a_{ik}, a_{jk}) are constructed, where a_{ik} is the element in the ith row and kth column of the array and where k ranges over the s^2 columns of the array. This property holds for all choices of i and j, $i\neq j$. The construction of such arrays is effected by use of Galois fields where $s=p^u$ and $\lambda=p^v$ and p is a prime. This generalizes certain earlier results such as the case when $v=0$ (identical with the problem of constructing a complete set of orthogonal Latin squares) and a more recent result due to R. A. Fisher. (Received March 12, 1951.)

270. K. A. Bush: *Orthogonal arrays of index unity and strength t.*

Generalizing the ideas of the preceding abstract, we may consider ordered t-plets by selecting t rows from the orthogonal array. There are s^t such t-plets. If independently of the t rows selected from the array, every t-plet occurs exactly once, we say we have an orthogonal array of index unity, that is, $\lambda=1$, and strength t. In case t exceeds s, it is possible to construct $t+1$ such rows. In the nontrivial case where s exceeds t, it is possible to construct $s+1$ orthogonal rows using polynomials in a finite geometry if s is a prime or a power of a prime. For the particular case $t=3$, $s=2^m$ it is possible to improve this result, and here $s+2$ orthogonal rows can be constructed. This anomaly is due to the fact that a certain matrix of Vandermonde type is singular for this case. (Received March 12, 1951.)

271*t*. Leonard Carlitz: *A conjecture of Dickson.*

Let p be an odd prime. Dickson (Trans. Amer. Math. Soc. vol. 10 (1909) pp. 109–122) considered binary forms $Q(x, y)$ with coefficients in $GF(p^n)$ such that $Q(a, b)$ is a nonzero square in $GF(p^n)$ for all (a, b) except $(0, 0)$ in the field. He conjectured that every such form of degree $2r$, $r>1$, is the square of a form provided p^n exceeds a certain bound N_r. In a previous note (Duke Math. J. vol. 14 (1947) pp. 1139–1140) the writer showed that this conjecture is a consequence of A. Weil's theorem on the zeros of the Artin zeta function. In the present note it is shown that the result can be obtained without the use of Weil's theorem. It suffices to make use of a theorem of Davenport (Acta Math. vol. 71 (1939) pp. 99–121, Theorem 4); however this method leads to a larger bound N_r. (Received March 7, 1951.)

272. A. B. Clarke: *Directed sums of cardinal algebras.*

A definition of a directed sum of cardinal algebras (C.A.'s) is given, analogous to the definition for partially ordered sets. Necessary and sufficient conditions are given for such a directed sum to be a C.A., and it is shown that the only essentially nontrivial case is when the index algebra is idemmultiple and linearly ordered. Every C.A. has a unique representation as such a linear directed sum of C.A.'s which are indecomposable in this way. A C.A. is called *primary* if $(\infty a < \infty b \rightarrow a < b)$. By application of the preceding theorem it is shown that every primary C.A. can be represented as a directed sum of C.A.'s which are idemmultiple or simple. An operation on C.A.'s is defined which is a generalization of finite direct product, and the class of all C.A.'s that can be obtained from primary C.A.'s by this operation is characterized. To round out the discussion it is shown that a necessary and sufficient condition for a simple C.A. to be a subalgebra of the non-negative reals is ($a \neq 0$, b finite $\rightarrow b \leq na$ for some $n < \infty$). (Received March 13, 1951.)

273. C. W. Curtis: *Ideal theory in noncommutative rings.* Preliminary report.

Generalizations of primal and primary ideals are given for two-sided ideals in a noncommutative ring R with an identity, and applied, in case R satisfies the maximum condition for ideals, to discuss the additive ideal theory in R, completing and extending in some respects the work of Fuchs (Proceedings of the American Mathematical Society vol. 1 (1950) pp. 1–6) and Krull (Math. Zeit. vol. 28 (1928) pp. 481–503). The existence of representations of ideals as intersections of primal ideals is proved for arbitrary rings, and the uniqueness is investigated when R satisfies the maximum condition. The existence of primary representations is proved for certain classes of rings. For example, one class consists of all polynomial (or power series) rings in n commuting indeterminates with coefficients in a central simple algebra; another class consists of all rings with maximum condition having the property that every ideal is two-sided. The theory of primary ideals is used to study the powers of an ideal. It follows from this discussion and a result of Jacobson (Amer. J. Math. vol. 67 (1945) pp. 300–320) that if every ideal in R is an intersection of primary ideals, and if every ideal in R has a finite basis as a right ideal, then the intersection of the powers of the Jacobson radical of R is zero. (Received March 9, 1951.)

274. Jesse Douglas: *On the existence of a basis for every finite abelian group.*

Every element x of a finite abelian group $G=a, b, c, \cdots, z$ (of order g) can be expressed as a product of powers of a properly chosen subset of elements, called a *basis*: for example, $x=b^{\beta}a^{\delta}l^{\lambda}r^{\rho}$, the exponents being uniquely determined modulo the corresponding periods β_0, δ_0, λ_0, ρ_0 $(g=\beta_0\delta_0\lambda_0\rho_0)$. The proofs of this fundamental theorem of abelian groups given in the literature are fairly complicated, involving: H. Weber (1899), mathematical induction; R. Fricke (1924), cosets and quotient groups; L. Pontrjagin (1946), matrices. The present paper offers a proof superior in simplicity and perspicuity. Supposing first G of prime power order, $g=p^h$, let us arrange the elements of G in a nonascending order of their periods: $p^m, p^{m-1}, p^{m-2}, \cdots, p, 1 (m \leqq h)$. Then, running through the elements in this order, remove successively each element equal to a power product of those yet remaining, distinct from itself. What is left will be a basis of G. The extension to the general case—where $g=p^hq^k \cdots r^l$, $p, q, \cdots, r$ being different primes—is easy, resting on the standard fact that the abelian group G is then the direct product of abelian groups of the respective prime power orders: $p^h, q^k, \cdots, r^l$. The complete paper will appear in an early issue of Proc. Nat Acad. Sci. U.S.A. (Received April 28, 1951.)

275*t*. Gerald Freilich: *Ideals in rings of transformations.*

A complete characterization of two-sided ideals in the ring $\mathcal{L}$ of all linear transformations of a vector space V into itself is given. If F is a two-sided ideal in $\mathcal{L}$, then it is proved that $F=\{\alpha \in \mathcal{L} \mid \dim [\alpha(V)] < \aleph\}$ for some cardinal $\aleph$. There is further established a correspondence between one-sided ideals in $\mathcal{L}$ and sets of subspaces of V. By considering retractions in dense subrings of $\mathcal{L}$, a theory for left ideals similar to Jacobson's theory of right ideals is established (see N. Jacobson, *Structure theory of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228–245). Finally, an example is given of a dense ring of linear transformations containing no nontrivial finite-valued transformation. (Received February 6, 1951.)

276. R. E. Johnson: *The extended centralizer of a ring over a module.*

For the ring R and the right R-module M, let $\mathfrak{M}$ denote the set of all submodules N of M having the property that $N \cap N' \neq 0$ for all nonzero submodules N' of M. An R-homomorphism of any submodule N of M into M is called a semi-endomorphism of M. In the set $\mathfrak{K}$ of all semi-endomorphisms of M operating on elements of $\mathfrak{M}$, operations of addition and multiplication are introduced. Relative to these, $\mathfrak{K}$ has ring-like properties. The extended centralizer of R over M is $\mathfrak{K}/\mathfrak{H}$, where $\mathfrak{H}$ is the set of all semi-endomorphisms of M having annihilator in $\mathfrak{M}$. The extended centralizer of R over M is shown to be a regular ring. By letting $M=R$, necessary and sufficient conditions are obtained for a ring R to have a regular quotient ring. (Received March 12, 1951.)

277*t*. Jakob Levitski: *Remarks on the determinant* $[x_1, x_2, \cdots, x_r]$.

Consider the "determinant" $[x_1, x_2, \cdots, x_r] = \sum_{(i)} x_{i_1} x_{i_2} \cdots x_{i_r}$, where (i) is an arbitrary permutation of r letters, and the sign is positive for even permutations and negative for odd permutations. Let A denote the total metric algebra F_n of the n by n matrices over a field F. Denote further by A_0 the linear space over F of all matrices having a zero trace, and by M_r the linear space over F spanned by the totality of the determinants $[a_1, a_2, \cdots, a_r]$ formed for all possible sequences of r matrices $a_1, a_2, \cdots, a_r$. It has been recently shown (A. S. Amitsur and J. Levitzki, Proceedings of the American Mathematical Society vol. 1 (1950)) that $M_r = 0$ if and only if $r \geqq 2n$. In the present note we determine M_r for $r < 2n$. Evidently $M_1 = A$ and it is known that $M_2 = A_0$. We generalize these facts by showing that if $2m < n$ then $M_{2m} = A_0$, while for $2m+1 < 2n$ and if the characteristic of F is zero we have $M_{2m+1} = A$. This equality holds also in case F has a positive characteristic p which is not a factor of $2m+1$. In case p is a factor of $2m+1$ we have the equality $M_{2m+1} = A_0$. These results remain valid for general central simple algebras of finite dimension. (Received February 16, 1951.)

278*t*. R. C. Lyndon: *Identities in groups, semigroups, and algebras.*

An identity in an abstract algebra is an equation in indeterminates that holds true under every substitution of elements from the algebra. *Results*: (1) All identities in a nilpotent group of class n are derivable from a finite set of identities involving no more than n indeterminates; (2) All identities in a two-element algebra are derivable from some finite set of identities; (3) There exists an infinite semigroup (without cancellation) whose identities are not derivable from any finite set. *Problem*: Does every finite abstract algebra have a finite set of identities from which all others are derivable? (Received March 15, 1951.)

279. Maxwell Rosenlicht: *Equivalence relations on algebraic curves.*

The structure of an arbitrary subring of an algebraic function field of one variable over an arbtirary ground field is analyzed in detail. If, in particular, one deals with a subring that is semilocal in the sense of Chevalley, then the analysis amounts to the study of the singularities of a suitable projective model of the field. In this case an equivalence relation among divisors of the field can be defined by setting equivalent to the unit divisor those divisors that are divisors of units in the semilocal ring; this generalizes the classical linear equivalence of divisors. A generalized Riemann-Roch theorem is then proved, in which corresponding to the ordinary differentials of the first kind appear differentials having certain types of singularities at certain places of the field. In the case where the ground field is the complex numbers, one obtains generalizations of Abel's theorem and the Jacobi inversion theorem; in the abstract case one has new Jacobian varieties. The theory affords a very simple proof of the theorem of Gorenstein on the conductor of the local ring of a point on a plane curve. (Received March 6, 1951.)

280. R. D. Schafer: *Representations of alternative algebras.*

Eilenberg's definition of representation (Annales de la Société Polonaise de Mathématique vol. 21 (1948) pp. 125–134) is applied to alternative algebras. A generalization to alternative algebras of characteristic 0 of the Whitehead first lemma for associative algebras is obtained as a corollary of Jacobson's analogous result for Jordan algebras. However, a stronger form of this lemma is required in the proof of the generalization to alternative algebras of Malcev's theorem on the strict conjugacy of semisimple components in Wedderburn decompositions. A related theorem states that an alternative algebra (of characteristic 0) is semisimple if and only if its derivation algebra is semisimple or $\{0\}$. (Received February 26, 1951.)

281*t*. J. A. Silva: *A theorem on cyclic determinants.*

In this paper the following generalization of a well known result on cyclic determinants is given. Let A be a cyclic matrix of order n^2 given by $A=\|a_{j-i+1}\|$, $i, j=1, 2, \cdots, n$, $a_r=a_s$ for $r\equiv s \pmod{n}$. Let p be a rational prime such that $n=p^t m$; $p\nmid m$. Then $d(A)\equiv \prod_{i=1}^{m}(\sum_{j=1}^{n}\alpha_i^{j-1}a_j)^{p^t} \pmod{p}$, where the α_i run through the m distinct nth roots of unity (mod p). (Received March 12, 1951.)

ANALYSIS

282. Cahit Arf: *On Rayleigh-Ritz-Weinstein's method.*

The author considers a self-adjoint continuous operator Q which is such that there exists a number ζ and a continuous self-adjoint operator R_ζ for which $R_\zeta'(Q-I\zeta)$

$=(Q-I\zeta)R'_\xi=\overline{P}_\zeta$ is a projector which satisfies the condition $(Q-I\zeta)(I-\overline{P}_\zeta)=0$. The main known results about Weinstein's determinant are extended to those operators which satisfy the conditions stated above. The subspaces considered in this extension do not necessarily differ from the whole space by a finite space. (Received April 19, 1951.)

283*t*. Stefan Bergman: *On the solution of the initial value problem for equations of mixed type.*

For the differential equation $\psi_{\mathrm{HH}}+l(\mathrm{H})\psi_{\theta\theta}=0$, $l(\mathrm{H})>0$ for $\mathrm{H}<0$, $l(\mathrm{H})<0$ for $\mathrm{H}>0$, the solution of the initial value problem in the small is considered for data given on the transition line. This is done by means of the set of particular solutions introduced in a paper by the author (Bergman, Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) p. 276), $\psi_{n,1}(\mathrm{H},\ \theta)=1!C_{n,1}\theta^{n-1}\int_0^{\mathrm{H}}d\mathrm{H}_1-3!\,C_{n,3}\theta^{n-3}\int_0^{\mathrm{H}}d\mathrm{H}_3\int_0^{\mathrm{H}_3}l(\mathrm{H}_2)d\mathrm{H}_2\int_0^{\mathrm{H}_2}d\mathrm{H}_1$, $\psi_{n,2}(\mathrm{H},\ \theta)=\theta^n-2!C_{n,2}\theta^{n-2}\int_0^{\mathrm{H}}d\mathrm{H}_2\int_0^{\mathrm{H}_2}l(\mathrm{H}_1)d\mathrm{H}_1+\cdots$ $(n=1,\ 2,\cdots)$. The following theorem holds. Let the functions $\chi_k(\theta)$ $(k=1,\ 2)$ possess expansions of the form $\chi_k(\theta)=\sum_{\nu=0}^{\infty}A^{(k)}\theta^\nu$, $|\theta|\leqq\theta_1$. $\psi(\mathrm{H},\theta)=\sum_{n=1}^{\infty}n^{-1}A_{n+1}^{(2)}\psi_{n,1}(\mathrm{H},\theta)+\sum_{n=0}^{\infty}A_n^{(1)}\psi_{n,2}(\mathrm{H},\theta)$ then converges in $|\theta|+\sigma|\mathrm{H}|<\theta_1$, for some $\sigma>0$, and represents a solution of the above differential equation for which $\psi(0,\theta)=\chi_1(\theta)$, $\psi_{\mathrm{H}}(0,\theta)=\chi_2(\theta)$. (Received March 19, 1951.)

284*t*. R. H. Breusch: *On the distribution of the roots of a polynomial with integral coefficients.*

D. H. Lehmer (Ann. of Math. vol. 34 (1933) p. 476) has proposed the following problem: to find irreducible, noncyclotomic polynomials $f(z)\equiv z^r+a_1z^{r-1}+\cdots\pm 1$ with integral coefficients, such that $\Omega(f)$, the product of the absolute values of those roots of $f(z)$ which lie outside the unit circle, shall be as small as possible. In the present note it is shown that for nonreciprocal polynomials $\Omega(f)>1.179$. The proof makes use of the fact that $R(f,g)$, the resultant of $f(z)$ and $g(z)\equiv z^r\cdot f(1/z)$, is absolutely not less than 1. (Received March 9, 1951.)

285. Paul Brock and F. J. Murray: *The use of exponential sums in step by step integration.*

R. E. Greenwood has suggested the use of exponential sums instead of polynomials in the step by step integration of a system of differential equations (Ann. Math. Statist. vol. 20 (1949) p. 608–611). In the present paper it is shown that if an "open" method of integration is precise for a step h and exponents $v_1,\cdots,v_n$, then the error in integrating the function $e^{\lambda t}$ for a t step of length h is in the form $h\epsilon(\lambda h)e^{\lambda t}$ where $\epsilon(\lambda h)$ is given by the following. Let $y_j=e^{-v_jh}-1$, $y=e^{-\lambda h}-1$, let S_k denote the sum of the monomials of degree k in $y, y_1,\cdots,y_n$, and let A_l be defined by the equation $y(1+y)^{-1}(\log\ (1+y))^{-1}=\sum_l A_l y^l$. Then $\epsilon(\lambda h)=(-1)^{n+1}\prod_{j=1}^{n}(y-y_j)(\sum_{l=0}^{\infty}A_{n+l}S_l)$. For λh and v_jh small this behaves like $-h^n\pi(\lambda-v_j)$ and thus the step by step error goes to zero like h^{n+1}. In order to take care of the case in which the integrand is equal to a linear combination of exponential functions and a small residual function $\sigma(t)$, the error for an arbitrary integrand, $\sigma(t)$, is also studied. (Received March 12, 1951.)

286*t*. Felix Browder: *On isolated singularities of second-order linear elliptic partial differential equations.*

Let $L(u)=Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G$ be the general second-order linear elliptic partial differential equation whose coefficients are defined in a neighborhood N of a point P in E^2 and such that A, B, C are three-times continuously differentiable in N, D and E twice continuously differentiable, F continuously differentiable and G continuous in N. ($AC-B^2>0$.) Suppose u is a regular solution of $L(u)=0$ in $N-\{P\}$ such that $\int_N(u_x^2+u_y^2)dA<\infty$. Then the singularity of u at P is removable. More generally let $L(u)=\Delta u+\sum_{j=1}^n b^j u_{x^j}+cu=d$ be a second-order linear partial differential equation of elliptic type in a neighborhood N of a point P in E^n, the n-dimensional Euclidean space, where $b^j\in C^2(N)$, $c\in C^1(N)$, $d\in C^0(N)$ and Δ is the Laplace operator $\sum_{i=1}^n\partial^2/\partial x_i^2$. Suppose u is a regular solution of $L(u)=0$ in $N-\{P\}$ such that $\int_N(\sum_{j=1}^n u_{x_j}^2)dA<\infty$. Then the singularity of u at P is removable. Weaker conditions in terms of the rate of growth of the Dirichlet integral near P are formulated which insure the removability of the singularity when the latter integral is divergent. (Received March 15, 1951.)

287*t*. K. A. Bush: *Continuous functions possessing no derivative.*

A class of continuous functions is constructed with the interesting property that $f(x)-k=0$ possesses a nondenumerable number of solutions where $f(x)$ is a member of the class and k is any constant lying between the infimum and supremum of $f(x)$. These solutions form a set of measure zero, and these functions fail to have a derivative anywhere. (Received March 12, 1951.)

288*t*. G. Y. Cherlin: *Bounds for the coefficients of some classes of odd multivalent functions.*

Let $f(z)=\sum_0^\infty a_{2n+1}z^{2n+1}$, an odd function, be regular for $|z|<1$ with the property that for a given positive integer p (which must be odd) and a given complex number α the imaginary part of $\alpha f(z)$ has exactly $2p$ changes of sign on $|z|=r$ for a range $1-\epsilon<r<1$. Then a bound for all the coefficients is given in terms of the first $(p+1)/2$ coefficients. The proof is by induction following a modification of the method of A. W. Goodman and M. S. Robertson [Trans. Amer. Math. Soc. vol. 70 (1951) pp. 127–136]. Application is made to a coefficient problem of multivalent odd functions of order p. (Received March 12, 1951.)

289*t*. V. F. Cowling and George Piranian: *On the summability of ordinary Dirichlet series by Taylor methods.*

Let V_α be the series-to-series transformation $V_\alpha\sum x_n=\sum y_n$ represented by the upper-triangular matrix (v_{nk}) with $v_{nk}=(1-\alpha)^n C_{k,n}\alpha^{k-n}$ for $k\geqq n$. The authors establish the following result: if the series $\sum a_n(n+1)^{-s}$ is summable by V_α at s_0, and the parameter α is suitably restricted, then the series is absolutely summable by V_α in the half-plane $\Re s>1+\Re s_0$; if the summability at s_0 is absolute, the series is absolutely summable in the half-plane $\Re s>\Re s_0$. (Received April 9, 1951.)

290. J. B. Diaz: *A remark on the method of Trefftz and on its generalizations.*

It is shown that a lower bound for the Dirichlet integral $D(u)=\int_R(u_x^2+u_y^2)dxdy$ of the solution u of Dirichlet's problem: $\Delta u=0$, on R, $u=$a given function, on C (where R is a plane domain and C its boundary), is given by $(*)$: $D(\alpha w)\leqq D(u)$, where w is an *arbitrarily chosen nonconstant function* (possibly multiple-valued, but such that w_x and w_y are single-valued) and $\alpha=(\int_C u(\partial w/\partial s ds)(D(w))^{-1}$. The inequality (*) generalizes a classical result of E. Trefftz (Proceedings of the Second International Congress of Applied Mechanics, Zürich, 1927, p. 131), which states that $D(\beta h) \leqq D(u)$, where h is required to be a nonconstant harmonic function and $\beta = (\int_C u(\partial h/\partial n)ds)(D(h))^{-1}$. The inequality (*) corresponds to the lower bound for the Dirichlet integral in Neumann's problem, which was recently given, also in terms of an arbitrary nonconstant function (J. B. Diaz and A. Weinstein, Journal of Mathematics and Physics vol. 26 (1947) p. 133, and Amer. J. Math. vol. 70 (1948) p. 107; see also the article by A. Weinstein, Proceedings of Symposia in Applied Mathematics vol. 3, 1950, p. 141). Similar results hold for any number of dimensions. (Received January 18, 1951.)

291. J. B. Diaz and R. C. Roberts: *On the numerical solution of the Dirichlet problem.*

The convergence of several computational methods for the solution of the difference equation form of the classical Dirichlet problem in a bounded domain is considered. In particular, it is shown that the precise analogues of Poincaré's "méthode de balayage" and of its extension by O. D. Kellogg (in his book, *Potential theory*, Berlin, 1929) yield sequences of functions which converge to the solution of the difference boundary value problem. In the usual proofs of the convergence of successive approximations to the solution of the difference boundary value problem (see, for example, P. Frank and R. v. Mises, *Differential und Integralgleichungen der Mechanik und Physik*, New York, 1943, p. 734, and I. G. Petrovskii, *Lectures on partial differential equations*, Moscow, 1950, p. 282) a definite ordering of the points of the domain is essential. By the present method, such a preliminary ordering of the points is seen to be entirely unnecessary. The relationship between the "methode de balayage" and the relaxation method of numerical approximation is also clarified. (Received March 14, 1951.)

292. J. A. Dieudonné: *Sequences of convergent Radon measures.*

Let (f_n) be a sequence of Lebesgue integrable functions on the closed interval (0, 1). It is well known that if $\int_A f_n(x)dx$ tends to a limit for every Borel set A in the interval, then there is an integrable function f such that, for every bounded and measurable function g, $\int f_n(x)g(x)dx$ tends to $\int f(x)g(x)dx$. It is proved that the same conclusion is true if it is assumed only that $\int_A f_n(x)dx$ tends to a limit for every *closed* set A in the interval. The result can be extended to any Radon measure on a compact space. (Received March 12, 1951.)

293*t*. Herbert Federer: *Essential continua of surfaces.* I.

Suppose f is a map of the closed k-cell X into n-space with finite Lebesgue area $L(f)$. The class Z of all maximal continua of constancy of f is metrized so that the distance between A and B in Z is the infimum of the diameters of the f images of those connected subsets of X which meet both A and B. To this metric corresponds the k-dimensional Hausdorff measure H over Z. Let $f = lm$ be the monotone-light factorization of f with middle space Z. For every positive number r, each continuum A in Z is a subset of a unique component $U(A, r)$ of the f counterimage of the closed sphere in n-space with center $l(A)$ and radius r. $L^*(A)$ and $L_*(A)$ are the upper and lower limits, as r approaches zero, of $L(f| U(A, r))/(wr^k)$, where w is the volume of a unit sphere in k-space. Let H' be the ordinary k-dimensional Hausdorff measure over n-space. It is shown that $L(f) \geqq \int L^*(A)dHA \geqq \int \sum_{l(A)=y} L^*(A)dH'y$. (Received February 5, 1951.)

294*t*. Herbert Federer: *Essential continua of surfaces.* II.

Let $s(g)$ be the k-dimensional Lebesgue measure of the set of all stable values of any mapping g into k-space. An orthogonal projection p of n-space onto k-space is essential at a point A of Z—and A is an essential continuum of f—if and only if $\lim_{r\to 0} s(pf|U(A, r))/(wr^k)=1$. This is shown to imply $\lim_{r\to 0} s(pf|T\cap U(A,r))/(wr^k)=1$ whenever $t<1$ and $T=\{x|t\cdot|f(x)-l(A)|\leqq|pf(x)-pl(A)|\}$. The number of projections which are essential at A does not exceed $L_*(A)$, and the set E of all essential continua of f has the finite measure $H(E)\leqq\int L_*(A)dHA$. If $L(f)=\int L^*(A)dHA$ and $H(\{A|L^*(A)>0\})$ is finite, then H almost all points of E possess unique essential projections, $l(E)$ is an (H', k) rectifiable subset of n-space (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 114–192), whose H' approximate tangent planes correspond to the essential projections, and $H(E)=\int N(y)dH'y$, where $N(y)$ is the number (possibly ∞) of points A in E for which $l(A)=y$. If $k=2$, all this holds with $L(f)=H(E)$, the projection properties of f are those of the set $l(E)$ which H' almost lies on countably many continuously differentiable surfaces, and surface integrals over f are computable by applying the classical formalism to $l(E)$. All objects considered in this paper are proved to be Fréchet covariants of f. (Received February 5, 1951.)

295. R. S. Finn: *On sequences of solutions of quasi-linear elliptic equations.* Preliminary report.

Quasi-linear elliptic partial differential equations of the form $(*)$ $(\rho\phi_x)_x+(\rho\phi_y)_y=0$ are considered. The function $\rho=f(x, y, \phi_x^2+\phi_y^2)$ is to satisfy the conditions: (a) $A_1\omega^{-\theta}<f(x, y, \omega^2)<A_2$, (b) $f(x, y, \omega^2)+2\omega^2\partial f(x, y, \omega^2)/\partial\omega^2>\omega^{-m}$, for $0<\theta<1$, $A_1>0$, $A_2>0$, $m>0$, and ω sufficiently large. It is assumed that $f(x, y, \omega^2)$ satisfies certain regularity conditions. The class of equations thus defined includes the minimal surface equation. Let $\{\phi_n(x, y)\}$ be a sequence of solutions of $(*)$ regular in a domain D and continuous in $D+D'$, which converge uniformly on D'. Then (1) $\{\phi_n(x, y)\}$ converges uniformly in $D+D'$ to a continuous function $\phi_0(x, y)$, (2) there exists in D an open set U equal in measure to D and a subsequence $\{\phi_{n_k}\}$ whose derivatives up to the second order converge uniformly to those of $\phi_0(x, y)$ on every closed subset of U. It follows [see Morrey (Trans. Amer. Math. Soc. vol. 43 (1938) pp. 126–166), also Schauder (Math. Zeit. vol. 37 (1933) pp. 623–634)] that if D is convex and if $\rho=f(\phi_x^2+\phi_y^2)$, then there exists a continuous function $\phi(x, y)$ which is a solution of $(*)$ throughout D except on a set E whose closure $\bar{E}$ is of measure zero, and which assumes on D' arbitrarily prescribed continuous values. It is shown that the set $\bar{E}$ is composed of continua connected to D'. (Received March 15, 1951.)

296. B. M. Ingersoll: *On a problem of Cauchy type for linear hyperbolic equations.* Preliminary report.

The equation $L(u)=u_{xy}+au_x+bu_y+cu=f$ is considered. Let C be a smooth arc which is nowhere tangent to a characteristic and R the Riemann rectangle determined by C. Two mixed derivatives of arbitrary orders are assigned on C, and values of a solution at N points of R, where the integer N depends on the orders of the derivatives. The coefficients of $L(u)$ are assumed to have certain regularity properties which, again, depend on the orders of the derivatives. It is shown that the above problem is correctly set, in general. The choice of the N points is not entirely arbitrary. This gives rise to certain geometrical configurations, among which are curves which correspond to conjugate points in N point problems for ordinary linear differential equations of Nth order. Exceptional cases, which arise from relations among the orders of the assigned derivatives, reveal certain phenomena. Among these is the existence of new characteristics, which depend only on a and b, and along which the prescribing of two derivatives does not determine a solution—or, at least, not uniquely. Moreover, in certain cases, apparently C must satisfy higher order differentiability conditions. (Received February 23, 1951.)

297t. P. D. Lax: *Completely continuous transformations.* Preliminary report.

Let B be a Banach space within which a continuous scalar product (x, y) is defined, that is, (x, y) is a bilinear skew-symmetric positive definite functional, with $(x, x)^{1/2} \leqq \text{const} \, \|x\|$. Let T be a transformation in B which is symmetric with respect to this scalar product. Then: if T is bounded in B, it is bounded with respect to the metric $(x, x)^{1/2}$, and if T is completely continuous in B, it is completely continuous in the metric $(x, x)^{1/2}$ and the eigenelements of T in B with nonzero eigenvalue span the image of B under T in the sense of the metric $(x, x)^{1/2}$. This result is applied to formal self-adjoint second order elliptic operators M in m dimensions by noting that $v = Tu$, defined as the solution of $Mv = u$, $v = 0$ on boundary, is completely continuous in the sense of the maximum norm, and symmetric with respect to an appropriate scalar product. (Received January 31, 1951.)

298. Benjamin Lepson: *On exceptional sets of regular hyperdirichlet series.* Preliminary report.

Certain necessary measure theoretic properties of these sets have been given previously by the author. It is shown here, using a known unpublished analog for capacity of Egoroff's theorem, that the sets are of logarithmic capacity zero. The sufficiency question is attacked, and a partial result is contained in the following theorem. *Let S be a closed bounded set of transfinite diameter zero. Then there exists a regular hyperdirichlet series which is M-convergent on S but whose half-plane of termwise boundedness does not intersect S.* (Received March 14, 1951.)

299t. Albert Novikoff: *Integral representations for Legendre polynomials.*

Integral representations, analogous to the formulae of Laplace and Dirichlet-Mehler for Legendre polynomials, are obtained for the polynomials of Pollaczek-Szegö, defined by the generating function $(1-ze^{i\theta})^{\lambda_1+i\phi(\theta)}(1-ze^{-i\theta})^{\lambda_2-i\phi(\theta)} = \sum_0^\infty z^n R_n^{(\lambda_1,\lambda_2)}(\cos\theta; a, b)$, $\phi(\theta) = (a\cos\theta + b)/2\sin\theta$, $a \geqq \pm b$. Using the saddle-point method, the Laplace representation is made to yield an estimate of $R_n(\cos t/n^{1/2})$, for large n. This estimate reveals a fundamental disparity between the Pollaczek-Szegö polynomials and the customary generalizations of the Legrendre polynomials with algebraic weight functions. It is shown that the zeros of $R_n(\cos t/n^{1/2})$ all cluster at $t = (a+b)^{1/2}$ for n large, and hence the associated limiting function $\lim_{n\to\infty} n^{1/4} \cdot e^{((a+b)/2)(1+n^{1/2})} R_n(\cos t/n^{1/2}; a, b) = S(t; a, b)$, which would play the role of the Bessel functions for Jacobi polynomials, is seen to have an essential singularity at $(a+b)^{1/2}$. (Received March 14, 1951.)

300t. R. S. Phillips: *Spectral theory for semi-groups of linear operators.*

A given semi-group of operators $T(t)$ on a Banach space X, strongly continuous for $t > 0$, is embedded in a commutative Banach algebra R. According to the Gelfand theory, each $B \in R$ can be represented as a continuous complex-valued function $B(m)$

over the maximal ideals M of R. A semi-group $T(t)$ is uniformly continuous at zero if and only if $T(t)(m)$ is of the form $\exp\ [a(m)t]$ on M. If the spectrum $\sigma[T(t)]$ for some interval $0 \leqq a < t < b$ contains no point of a connected open set containing 0 and ∞, then $T(t)$ has a bounded infinitesimal generator A. For a measure function α, we define $\Theta(\alpha)x = \int_0^\infty T(t)x d\alpha$ and $f_\alpha(\lambda) = \int_0^\infty \exp\ (-\lambda t) d\alpha$. If $T(t)$ is uniformly continuous for $t \geqq c > 0$, then $\sigma[\Theta(\alpha)] = f_\alpha[\sigma(A)] \cup \alpha(0)$ and this correspondence holds as well for the detailed spectra. If $T(\zeta)$ is also analytic sufficiently far out in some sector, $\phi_1 < \arg \zeta < \phi_2$, and if we define $\sigma(\phi) = \lim_{r \to \infty} r^{-1} \log \| T[r \exp\ (i\phi)] \|$, then $\sigma(\phi)$ is the function of support for the closed convex extension of the complex conjugate to $\sigma(A)$. The method is also applied to obtain a new and direct proof of the Stone theorem for locally compact groups of unitary transformations on a Hilbert space. (Received March 7, 1951.)

301*t*. M. S. Robertson: *A coefficient problem for functions regular in an annulus.*

Let $w = f(z) = \sum_{-\infty}^{\infty} a_n z^n$ be regular and single-valued in the annulus $0 \leqq \rho \leqq |z| < 1$. For each $|z| = r$ in the range $\rho < r < 1$ let the image of $|z| = r$ through the mapping $w = f(z)$ cross a given straight line l precisely $2p$ times (p fixed). Sharp bounds are obtained for the complex coefficients a_n, $|n| > p$, in terms of a_k, $\pm k = 0, 1, 2, \cdots, p$, and the parameters locating l. This is an extension of recent results by A. W. Goodman and M. S. Robertson [Trans. Amer. Math. Soc. vol. 70 (1951) pp. 127–136] where $a_{-n} = 0$ for $n \geqq 0$, a_n real for $n \geqq 1$, and where l was the real axis. An application is made of the extension to prove that the Goodman conjecture for the coefficients of multivalent functions $F(z) = \sum_1^\infty a_n z^n$ of order p in $|z| < 1$ is correct at least for those p-valent functions $w = F(z)$ for which $\partial \Re F(re^{i\theta})/\partial\theta$ changes sign $2p$ times on each circle $|z| = r$, $\rho < r < 1$. (Received March 12, 1951.)

302. James Sanders: *An existence theorem for some partial differential equations of the fourth order.* Preliminary report.

Equations of the form (1) $L(u) \equiv u_{xx} + (\tau_1(y)\sigma_2(x)/\tau_2(y)\sigma_1(x))u_{yy} + (\sigma_1'/\sigma_1)u_x + (\sigma_2\tau_1/\sigma_1)(1/\tau_2)' u_y = 0$, a generalization of Laplace's equation, were considered by Bers and Gelbart (Trans. Amer. Math. Soc. vol. 56 (1944)). In this paper, equations of the form (2) $LL(\phi) = 0$, generalization of the biharmonic equation, are considered. By an application of Green's theorem, the uniqueness of the solution of the boundary-value problem (3) $LL(\phi) = 0$, $\phi = f(s)$, $\partial\phi/\partial n = g(s)$, on C, is proved. C is a simple, closed, analytic curve bounding the simply connected domain D. Existence of a solution of (3) is proved by first reducing (3) to a system of second order equations and then reducing the second order system to a system of integral equations. (Received March 8, 1951.)

303. L. R. Sario: *Alternating method for analytic functions.*

The alternating method for harmonic functions, created by Schwarz for surfaces with finite genus, can be generalized for arbitrary Riemann surfaces (C. R. Acad. Sci. Paris vol. 229 (1949) pp. 1293–1295; vol. 230 (1950) pp. 42–44 and 168–170). An analogous method for analytic functions can be developed and furnishes the following theorem. Consider on an arbitrary Riemann surface R a finite set γ of analytic Jordan curves, relatively bounding a domain R_1, compact or not. In the complement $R_2 = R - R_1$ let z be a complex function (in general multi-valued) which is analytic in a neighborhood of γ. *If* $\int_\gamma dz = 0$, *then there exists on* R *a function* w *satisfying the fol-*

owing conditions: 1° *w is analytic and Dirichlet bounded in R_1*, 2° *$w-z$ is analytic and Dirichlet bounded in R_2*. The generalized alternating methods have applications to existence and mapping problems on Riemann surfaces (cf. problems in Annales Academiae Scientiarum Fennicae Ser. A. I. 50 (1948) pp. 1–79). In certain cases they allow a simpler approach than the Dirichlet principle or Hilbert space methods. (Received March 14, 1951.)

304t. L. R. Sario: *Linear operators on Riemann surfaces.*

On an arbitrary Riemann surface R consider two compact domains D and F ($\bar{D}\subset F$), bounded by finite sets a and b respectively of analytic Jordan curves. Let u be a harmonic function in $\bar{F}-D$ such that $u=0$ on a, $\int_a d\bar{u}=0$, and denote by v the harmonic function in F with boundary values u on b. The harmonic function in $R-D$ which minimizes the Dirichlet integral among functions with values v on a is a linear operator $L(u)$ of u. *The sequence $\{h_n=\sum_{i=1}^{i=n}L^i(u)\}$ converges uniformly in $R-D$ towards a harmonic function h, bounded and Dirichlet bounded. The harmonic continuation of $u+h$ in D is single-valued and regular.* (Received March 14, 1951.)

305t. Albert Schild: *On a problem in conformal mapping of schlicht functions.*

Let S^* be the class of functions $w=f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, regular and schlicht in the unit circle, map the unit circle into a region D^* in the w-plane, starshaped with respect to $w=0$. (The region D^* is called starshaped with respect to $w=0$ if every point of D^* can be joined to $w=0$ by a straight line segment lying entirely in D^*.) Let d^* be the distance of the closest point of the boundary of D^* from $w=0$, and let d_0 be the distance of the closest point of the map of $|z|=r_0$—where r_0 is the "Rundungs Schranke" (bound of convexity) of $w=f(z)$—from $w=0$. It has been conjectured that $d_0/d^*\geqq 2/3$ for all functions of the class S^*, the lower limit being attained by the function $w=f(z)=z(1+z)^{-2}$. The author has demonstrated the truth of this conjecture for certain subclasses of S^*, while for the general function of class S^* certain lower bounds for d_0/d^* have been obtained. The methods employed are of elementary nature. (Received February 21, 1951.)

306. E. W. Titt: *Leibnitz rule for differentiating multiple integrals over arbitrary variable regions; extended to cases where the classical rule yields divergent integrals.*

This paper presents rules for differentiating multiple integrals over arbitrary regions which make successive application of Leibnitz rule for single integrals unnecessary. By the new rules the derivative of an n-tuple integral becomes an n-tuple integral of the derived integrand plus an $(n-1)$-tuple integral (taken over the boundary of the region) of the original integrand multiplied by a determinant involving derivatives of the representation. The discovery of the rules was enhanced by a theory of multiple integration (by one of the authors) which inductively defines an n-tuple integral over an arbitrary region in terms of an $(n-1)$-tuple integral over the boundary. These rules have been extended so as to enable one to differentiate the integral of a scalar product where the classical rule yields divergent integrals. These latter rules perform three operations at one time: (1) an integration by parts, integrating one vector and differentiating the other; (2) the application of the aforementioned rule for differentiating multiple integrals; (3) reverse integrations by parts and simplification. The result is an easily applied rule involving n, $(n-1)$, and $(n-2)$-tuple integrals. (Received March 12, 1951.)

307. Alexander Weinstein: *On the fundamental solutions of the Tricomi equation.*

In a previous paper (A. Weinstein, Bull. Amer. Math. Soc. vol. 55 (1949) p. 520) two fundamental solutions of Tricomi's equation have been given in closed form for all values of the variables in the elliptic half-plane. It is shown now that both solutions can be extended to the hyperbolic half-plane where they are given in closed form by means of formulas derived from classical results on the equations of Euler-Poisson (G. Darboux, *Leçons sur la théorie générale des surfaces*, vol. 2, Paris, 1914, p. 68). (Received March 14, 1951.)

308*t*. D. V. Widder: *The Weierstrass transform of positive functions.*

Necessary and sufficient conditions are obtained in order that $f(x)$ should be of the form $\int_{-\infty}^{\infty} \exp\,[-(x-y)^2/4]d\alpha(y)$, where $\alpha(y)$ is a nondecreasing function. They are: (1) $f(x)$ is a real entire function such that $f(x+iy)=O\,[\exp\,(-y^2/4)]$ as $|y|\rightarrow\infty$, uniformly in every vertical strip; (2) $\exp\,(-tD^2)f(x)\geqq 0$ for $0<t<1$, $-\infty<x<\infty$. The symbolic operator (2) is interpreted as $(4\pi t)^{-1/2}\int_{-\infty}^{\infty}\exp\,[-y^2/4t]\{\cos yD\}f(x)dy$, where $\{\cos yD\}f(x)$ is the obvious infinite series. The result is the analogue of the familiar Bernstein theorem concerning the representation of completely monotonic functions as Laplace transforms of positive functions. The chief tool is the author's result concerning positive temperatures on an infinite rod (Trans. Amer. Math. Soc. vol. 55 (1944)). (Received April 5, 1951.)

309*t*. Y. K. Wong: *Some inequalities of determinants of Minkowski type.*

Consider a matrix a of order n with non-negative elements. Let $M(a)$ be the supremum of $|x'ay|$ for all x, y on a unit Euclidean sphere. Let D_n be the determinant of $I-a$ and D_{n-1} the determinant of the minor $I-a_{n-1}$ which consists of the first $n-1$ rows and columns of $I-a$. If $M(a)<1$, then D_n is bounded above by $[1-a(n, n)-\sum_{k=0}^{m}\sum_{i,j<n}a(n, i)a_{n-1}^{k}(i, j)a(j, n)]D_{n-1}$ for $m=1, 2, \cdots$. The equality holds if $m=\infty$. If $M(a)<1$ is replaced by the condition that for every $j=1, \cdots, n$, $\sum_i a(i, j)\leqq 1$ (summing for all $i\neq j$), $a(j, j)<1$, and D_{n-1} is nonvanishing, the inequality holds for $m=1$. Under proper restriction, if a term $p\sum a(n, i)a_{n-1}^{m}(i, j)a(j, n)D_{n-1}$ is subtracted from the above expression, one gets a very good lower bound for D_n. The values for p are determined under various conditions. (Received March 20, 1951.)

APPLIED MATHEMATICS

310*t*. G. S. S. Ludford: *The behavior at infinity of the potential function of a two-dimensional subsonic compressible flow.*

The well known expansion $w(z)=Az+B\log z+\sum_{n=1}^{\infty}C_n/z^n$ at infinity for the complex potential function of an incompressible fluid is transformed into the logarithmic plane, where it is seen to have a certain set of singularities at the image of the point $z=\infty$ (in the physical plane). By constructing a similar set of singularities in the pseudo-logarithmic plane, where the potential function of a compressible fluid satisfies a linear elliptic differential quation in normal form (S. Bergman, Trans. Amer.

Math. Soc. vol. 62 (1947) p. 452), and then transforming back to the physical plane, the expansion $\phi = ar \cos(\theta-\theta_0)+b(\theta)\log r+c(\theta)+\sum_{m,n=0}^{\infty}(d_{mn}(\theta)/r^n)((\log r)/r)^m$ with d_{00} constant is obtained, where (r, θ) are polar coordinates, and the series converges uniformly and absolutely for r large enough. It is further shown that if $b(\theta)$, $c(\theta)\equiv 0$, that is, there is no source or circulation in the flow, then $d_{mn}(\theta)\equiv 0$ for $m\neq 0$, so that the logarithmic terms disappear. On the basis of these expansions the force and moment formulae for an incompressible flow may be extended to subsonic compressible flow. The reference above gives an alternative derivation of these formulae. (Received March 12, 1951.)

311t. L. A. Zadeh: *Theory of filtering of signals in a λ-domain.*

An integral representation of a signal $u(t)$ in the form $u(t)=\int_C k(t;\lambda)U(\lambda)d\lambda$ may be interpreted as the resolution of $u(t)$ in terms of $k(t;\lambda)$, with $U(\lambda)$ playing the role of a weighting function. $U(\lambda)$ is related to $u(t)$ by $U(\lambda)=\int_{-\infty}^{\infty}k^{-1}(t;\lambda)u(t)dt$ where $k^{-1}(\lambda;t)$ is the inverse of $k(t;\lambda)$. Let $u(t)$ be the input to a linear system N and let $v(t)$ be the response of N to $u(t)$. $v(t)$ is given by $v(t)=\int_C K(t;\lambda)U(\lambda)d\lambda$ where $K(t;\lambda)$ is the response of N to $k(t;\lambda)$ and is called a *characteristic function* of N. For the particular case where $k(t;\lambda)$ is a delta function $\delta(t-\xi)$, $\lambda=\xi$, the characteristic function becomes the impulsive response $W(t,\xi)$ or Green's function of N. $W(t,\xi)$ is related to $K(t;\lambda)$ by $W(t,\xi)=\int_C K(t;\lambda)k^{-1}(\xi;\lambda)d\lambda$. A system N is said to be an *ideal filter* if $K(t;\lambda)=k(t-\beta;\lambda)$ (β = a constant) for λ in a region R of the λ-plane, and $K(t;\lambda)=0$ for λ not in R. Two signals $u_1(t)$ and $u_2(t)$ are called *separable* if in some λ-domain (that is, for some $k(t;\lambda)$) $U_1(\lambda)=0$ for λ in R and $U_2(\lambda)=0$ for λ not in R. Such signals can be physically separated from each other by using an ideal filter in this domain. (Received March 9, 1951.)

GEOMETRY

312. H. S. M. Coxeter: *Interlocked rings of spheres.*

Dupin's cyclide is the common envelope of two systems of spheres [A. Mannheim, Nouvelles Annales de Mathematiques vol. 19 (1860) p. 71]. Any sphere of either system belongs to a sequence of spheres in that system, each touching the next (and all touching every sphere of the other system). By means of an excursion into four-dimensional space it is proved that, if one system contains a *closed* sequence (the last sphere touching the first), then the other system likewise contains a closed sequence. The figure consists of a cycle of n spheres interlocked with a cycle of m, where either $m=2n$ with n odd, or $2m=n$ with m odd, or $m=n\equiv 0 \pmod 4$. The case when m and n are 3 and 6 was noticed by F. Soddy [Nature vol. 138 (1936) p. 958]. (Received March 14, 1951.)

313. P. C. Hammer and Andrew Sobczyk: *Planar line families.* I.

Let F be a family of lines in the plane which simply covers all points, including points at infinity, exterior to some circle. Then there exists a convex body C the essential diameters of which extend to give precisely all lines in F. Moreover, C may be taken as a body with constant diameter length and with a smooth boundary curve. It is demonstrated how all planar convex bodies of constant diameter length may be constructed given all such families of lines. Given all continuous increasing functions mapping the closed interval [0, 1] biuniquely into [0, 1], all such families of lines are constructed. (Received February 14, 1951.)

314t. A. J. Hoffman: *On chains.*

Let E be a field admitting an involutoric automorphism having F as fixed field, let m be the projective line over E and $n \subset m$ the projective line over F. Define a chain in m as the image of n under any projective transformation of m. In the classical case, E is the complex numbers, and the chains can be regarded as the circles of the real inversion plane. C. T. Yang (Duke Math J. (1948)) has shown that many of the properties of chains in the classical case hold also if E is $GF(p^{2k})$. The purpose of this note is to point out that these properties hold in the general case. In fact, stronger results, such as Miquel's theorem, are valid in general. (Received March 13, 1951.)

315t. Ira Rosenbaum: *A new system of completely independent postulates for betweenness.*

Huntington and Kline (Trans. Amer. Math. Soc. (1917)) considered 12 postulates for betweenness, from which 11 sets of independent postulates were selected. Eight of the eleven sets of independent postulates contained 6 postulates each, three contained 7 each. Huntington (Trans. Amer. Math. Soc. (1924)) obtained a set of 5 completely independent postulates for betweenness by adding to the original 12 postulates a 13th. The present paper indicates that a set of 5 completely independent postulates for betweenness may be selected from Huntington and Kline's original set of twelve, by slightly modifying two of the postulates, without bringing in any essentially novel postulates as Huntington did. The postulates A, B, D of Huntington and Kline's paper are retained; their postulates 1 and 2 are modified by removing the hypothesis that A, B, X, Y are distinct, and thereafter cited as 1′ and 2′. Mr. Walter E. Roop, a student of the author, noted that from A, B, D and the author's 1′ and 2′, $\sim(AXY \,.\, AYX)$ is derivable. It follows easily that Huntington and Kline's postulate C is derivable; moreover 1′ implies 1, and 2′ implies 2. Hence A, B, D, 1′, and 2′ imply A, B, C, D, 1, and 2 (Huntington and Kline's first set of 6 independent postulates for betweenness). Interpretations showing the postulates A, B, D, 1′, and 2′ to be completely independent, in the sense of E. H. Moore, are given. (Received January 18, 1951.)

STATISTICS AND PROBABILITY

316. K. L. Chung and Jacob Wolfowitz: *On a limit theorem in renewal theory.*

Let $X_1, X_2, \cdots$ be independent, identically distributed, integral-valued chance variables. Let d be the absolute value of the greatest common divisor of the values which each variable can assume with positive probability. Let $m=E(X_n)$ be the mathematical expectation of each X_n and let $S_n=\sum_1^n X_k$. Suppose $0<m\leqq\infty$. The authors prove that $\lim \sum_{n=1}^{n=\infty} P(S_n=xd)=d/m$ or 0, according as x approaches $+\infty$ or $-\infty$ through integral values. In the case where the X_n's assume only non-negative integral values this was proved by Kolmogorov and Erdös, Feller, and Pollard. (Received February 26, 1951.)

317. L. A. Zadeh: *On stochastic Heaviside operators.*

The result of operation with a time-dependent Heaviside operator $H(p;t)$ on $u(t)$ is given by $H(p;t)u(t)=(1/2\pi)\int_{-\infty}^{\infty}H(j\omega;t)U(j\omega)e^{j\omega t}d\omega$ where $U(j\omega)$ is the Fourier transform of $u(t)$. An $H(p;t)$ which is a random function of t involving p as a fixed parameter is called a stochastic Heaviside operator. Let $u(t)$ and $H(p;t)$ be independent and stationary in the wide sense, and let $v(t)=H(p;\, t)u(t)$. Furthermore, let $\psi_u(\tau)=u(t)u(t+\tau)$, $\psi_v(\tau)=v(t)v(t+\tau)$, and $\psi_H(p;\, \tau)=H(p;\, t)H(-p;\, t+\tau)$ be the correlation functions of $u(t)$, $v(t)$, and $H(p;\, t)$. It can readily be shown (Bull. Amer. Math. Soc. Abstract 56-4-329) that these correlation functions are related to each other by the operational relation $\psi_v(\tau)=\psi_H(p;\, \tau)\psi_u(\tau)$. Consider two independent stochastic operators H_1 and H_2 and let H_3 be their product $H_3=H_1 * H_1$ (the product of H_1 and H_2 is given by the operational relation $H_3(s;\, t)=H_1(p+s;\, t)H_2(s;\, t)$ where s is a parameter). Let ψ_1, ψ_2, and ψ_3 be the correlation functions of H_1, H_2, and H_3. It is shown that $\psi_3=\psi_1 * \psi_2$. This result is useful in the determination of the correlation functions of complex waveforms. (Received March 13, 1951.)

TOPOLOGY

318. Raoul Bott: *Nondegenerate critical manifolds.* Preliminary report.

Let M be a real n-dimensional compact manifold with a three times differentiable structure. Let f be a real-valued function of a class C on M. A submanifold N of M is said to be a nondegenerate critical manifold of f if (1) $\partial f/\partial x^i=0$ $(i=1, \cdots, n)$. (2) rank $(\partial^2 f/\partial x^i \partial x^j)=\dim M-\dim N$, on N. It is shown that the Morse type numbers M of this critical set are given by $M^i_N=R^{i-\lambda}(N)$, where λ is the number of negative characteristic values of $(\partial^2 f/\partial x^i \partial x^j)$ on N, and $R^s(N)$ is the s-dimensional Betti number mod 2 of N. When N is a point P this reduces to the early result of Morse: $M^i_P=\delta^i_\lambda$. Applications of this lemma are made to fibre bundles and a generalized form of the billiard ball problem. (Received January 15, 1951.)

319*t*. Felix Browder: *On the fixed-point index.* I.

If X is a compact neighborhood retract of a Tychonoff cube, a fixed-point index, a non-negative integer $i(O, f)$, is defined for all continuous mappings of $\bar{O}$ into X where O is open in X and f has no fixed-points on the boundary of O. This index is shown to have the basic properties of the classical fixed-point index defined for self-mappings of finite polytopes and is a proper generalization of the latter as well as coinciding with the Leray index for any neighborhood retract of a Tychonoff cube which is convexoid. By a simple extension to the case in which only the image under f needs to be compact, the definition of this paper becomes a proper generalization of the Schauder-Leray index obtained from the degree. The definition is carried through by the use of approximating mappings of the nerves of finite open coverings of X which are obtained from the given mapping by applying the Alexandroff mapping theorem and homotopy local connectedness properties of X. A generalization is given also of the Nielsen Reidemeister-Wecken theory of fixed-point classes. (Received March 15, 1951.)

320*t*. Felix Browder: *On the fixed-point index.* II.

If X is a compact, not necessarily metric, HLC* space, a fixed-point index $i(O, f)$ is defined for all continuous mappings of $\bar{O}$ into X where O is an open subset of X and f has no fixed points on the boundary of O. The index is defined by the use of a chain derivation similar to, but not identical with, that defined for quasi-complexes in Lefschetz' *Algebraic topology.* Using linear endomorphisms, induced by the given mapping, of the rational chain groups of subcomplexes of the nerves of finite open coverings of X, the definition and proof of the properties of the fixed-point index make no use of properties of the classical fixed-point index for finite polytopes. The index as defined for compact HLC* spaces constitutes a proper generalization of all previous definitions of the fixed-point index. (Received March 15, 1951.)

321. Mary E. Estill: *Concerning separability.*

Let axiom 1.3 denote the axiom resulting from the omission of condition 4 in the statement of R. L. Moore's *Foundations of point set theory.* In this paper there is shown the existence of a connected and connected im kleinen space satisfying axiom 1.3 which is not separable and which does not contain uncountably many mutually exclusive domains. There is also shown the existence of a connected space satisfying axiom 1.3 which is not separable and which does not contain uncountably many mutually exclusive domains such that, if A is a point of a region R and B is a point distinct from A, there exists a point lying in R separating A from B. However, there does not exist a connected im kleinen space satisfying axiom 1.3 which is not separable and does not contain uncountably many mutually exclusive domains such that, if A is a point of a region R and B is a point distinct from A, there exists either a separable point set lying in R separating A from B or a finite point set separating A from B. (Received March 15, 1951.)

322*t*. W. S. Massey: *Applications of exact couples in algebraic topology.*

Exact couples of groups have many applications to algebraic topology, of which three will be mentioned here. The first application is obtained by considering relative and absolute homotopy groups of the various skeletons of a cell complex. The exact couple thus obtained is not a topological invariant of the complex, but the successive derived couples are invariants. The first derived couple contains among other things an exact sequence recently considered by J. H. C. Whitehead [Ann. of Math. vol. 52 (1950) pp. 51–110]. The second application is obtained by considering the Spanier-Borsuk cohomotopy groups of the various skeletons of a cell complex. The first derived couple is again a topological invariant, and it contains the cohomotopy groups of the complex, and various cohomology groups. The third application is to the problem of obtaining relations between the homology groups of the base space, total space, and fiber in a fiber bundle. The geometrical basis for this application is to be found in recent work of Spanier and Chern [Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 248–255]. (Received March 15, 1951.)

323*t*. W. S. Massey: *The algebra of exact couples of groups.*

An *exact couple* of groups consists of two abelian groups, A and C, homomorphisms $g: A \to C$ and $h: C \to A$, and an endomorphism $f: A \to A$. These groups and homomorphisms are required to satisfy the following "exactness" conditions: image f=kernel g, image g=kernel h, and image h=kernel f. Define an endomorphism $d: C \to C$ by $d=gh$. Then d is a differential operator, that is, $d^2=0$. Let C' denote the derived group of C with respect to this differential operator and let $A'=f(A)$. Then f defines an endomorphism $f': A' \to A'$, h defines a homomorphism $h': C' \to A'$, and gf^{-1} defines a homomorphism $g': A' \to C'$. The groups A' and C' together with the homomorphisms f', g', h' satisfy the exactness condition mentioned above, and hence constitute an exact couple, called the *first derived* exact couple. This process can be repeated to obtain the second derived couple, and so on. In applications to topology, the groups A and C usually also have a bi-graded structure, and the homomorphisms

f, g, and h are homogeneous. In an analogous manner, one can consider exact couples of vector spaces, modules, rings, or algebras. (Received March 15, 1951.)

324t. Philip Rabinowitz: *Normal coverings and uniform spaces.*

The structure of uniform spaces is investigated using graphs of normal coverings. The following two theorems are the main results of this investigation: (1) The collection of graphs of all locally-finite normal coverings of a uniformizable space X forms a base of the universal uniformity of X. (2) Every fully normal space is complete relative to its universal uniformity. Since Stone (Bull. Amer. Math. Soc. vol. 54 (1948)) has proven the equivalence of paracompactness and full normality for Hausdorff spaces, the latter theorem answers in the affirmative the problem posed by Dieudonné (J. Math. Pures Appl. vol. 23 (1944)). (Received February 16, 1951.)

L. W. COHEN,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The four hundred sixty-ninth meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 27–28, 1951. There were about 225 registrations including the following 205 members of the Society:

W. R. Allen, Nachman Aronszajn, W. L. Ayres, G. E. Backus, Reinhold Baer, W. R. Ballard, E. W. Banhagel, W. E. Barnes, R. G. Bartle, P. T. Bateman, A. F. Bausch, J. H. Bell, L. D. Berkovitz, Gerald Berman, S. F. Bibb, R. H. Bing, K. E. Bisshopp, H. D. Block, R. P. Boas, D. G. Bourgin, G. F. Bradfield, Joseph Bram, A. H Brown, R. C. Buck, P. B. Burcham, R. S. Burington, Eugenio Calabi, A. P. Calderón, R. E. Carr, A. B. Carson, Lamberto Cesari, S. S. Chern, Joshua Chover, H. M. Clark, R. A. Clark, M. D. Clement, B. H. Colvin, E. G. H. Comfort, Max Coral, J. J. Corliss, D. S. Curry, M. L. Curtis, Allen Devinatz, A. H. Diamond, Flora Dinkines, J. L. Doob, W. F. Eberlein, M. H. M. Esser, H. P. Evans, R. L. Evans, Trevor Evans, H. S. Everett, G. M. Ewing, Chester Feldman, W. H. Fleming, L. R. Ford, Abraham Franck, J. C. Freeman, C. G. Fry, R. E. Fullerton, J. W. Gaddum, M. P. Gaffney, B. A. Galler, Murray Gerstenhaber, Michael Golomb, S. H. Gould, L. M. Graves, R. E. Graves, V. G. Grove, M. M. Gutterman, Franklin Haimo, P. R. Halmos, Frank Harary, W. L. Hart, Charles Hatfield, L. L. Helms, R. G. Helsel, Melvin Henriksen, I. N. Herstein, D. G. Higman, J. G. Hocking, Carl Holtom, T. C. Holyoke, W. A. Howard, H. K. Hughes, W. E. Jenner, A. K. Jennings, Meyer Jerison, R. N. Johanson, L. W. Johnson, G. K. Kalisch, Wilfred Kaplan, William Karush, L. M. Kelly, D. E. Kibbey, S. C. Kleene, Erwin Kleinfeld, J. C. Koken, Jacob Korevaar, M. Z. Krzywoblocki, R. G. Kuller, H. G. Landau, E. P. Lane, R. E. Langer, Leo Lapidus, J. R. Lee, O. I. Litoff, Lee Lorch, S. W. McCuskey, A. W. McGaughey, R. W. MacDowell, Saunders MacLane, H. B. Mann, H. G. Mazurkiewicz, D. M. Merriell, B. E. Meserve, H. L. Meyer, E. A. Michael, C. E. Miller, J. M. Miller, J. M. Mitchell, C. W. Moran, E. J. Moulton, E. D. Nering, E. A. Nordhaus, R. Z. Norman, C. O. Oakley, Rufus Oldenburger, H. W. Oliver, E. H. Ostrow, R. R. Otter, G. K. Overholtzer, Gordon Pall, S. T. Parker, M. H. Payne, M. M. Peixoto, George Piranan, D. H. Potts, A. L. Putnam, Gustave Rabson, L. I. Rebhun, O. W. Rechard, R. F. Reeves, W. T. Reid, Haim Reingold, Daniel Resch, R. A. Roberts, Alex Rosenberg, Murray Rosenblatt, Arthur Rosenthal, E. H. Rothe, M. F. Ruchte, R. G. Sanger, L. J. Savage, O. F. G. Schilling, Lowell Schoenfeld, A. L. Schurrer, W. R. Scott, W. T. Scott, I. E. Segal, R. J. Semple, D. H. Shaftman, Daniel Shanks, M. E. Shanks, Edward Silverman, R. J. Silverman, Annette Sinclair, M. L. Slater, D. M. Smiley, M. F. Smiley, A. H. Smith, Albert Soglin, E. S. Sokolnikoff, E. H. Spanier, E. J. Specht, G. L. Spencer, W. L. Stamey, R. G. Stanton, H. E. Stelson, B. M. Stewart, M. H. Stone, L. W. Swanson, Otto Szász, C. T. Taam, T. T. Tanimoto, H. P. Thielman, G. H. M. Thomas, G. L. Thompson, C. J. Titus, E. A. Trabant, E. F. Trombley, H. L. Turrittin, E. H. Umberger, G. L. Walker, D. R. Waterman, George Whaples, J. J. Wheeler, L. R. Wilcox, V. M. Wolontis, Oswald Wyler, L. C. Young, P. M. Young, J. W. T. Youngs, R. K. Zeigler, Daniel Zelinsky.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor I. E. Segal spoke on *Noncommuta-*

tive integration and measurable operators at 2:00 P.M. Friday. Professor M. H. Stone presided at the session.

Sessions for the presentation of contributed papers were held at 10:30 A.M. and 3:15 P.M., Friday, and 10:30 A.M. on Saturday. Presiding officers at these sessions were Professors Reinhold Baer, G. M. Ewing, Saunders MacLane, E. H. Spanier, and D. G. Bourgin. Relative to contributed papers, a happy note is that the time limit of ten minutes is coming more to be accepted as an upper bound, rather than a lower bound as has so frequently been the case in the past.

The list of papers presented follows. Those abstracts indicated by the letter "*t*" were presented by title. Mr. George Klein was introduced by Professor Antoni Zygmund.

ALGEBRA AND THEORY OF NUMBERS

325. Melvin Henriksen: *Maximal ideals of the ring of entire functions and their residue class fields.* Preliminary report.

Let R denote the ring of entire functions, M any maximal ideal. The latter are classified and it is shown that R/M is always isomorphic to the complex field K (as a ring but not as an algebra). No topological restrictions are placed on the maximal ideals M. The method of proof is to show that (i) R/M is algebraically closed, (ii) R/M has the cardinal number of the continuum. Together with a theorem of Steinitz, (i) and (ii) imply that R/M is isomorphic to K. This result can be extended to other function rings. (Received April 10, 1951.)

326. I. N. Herstein: *A generalization of a theorem of Jacobson.*

The result obtained in this paper is as follows: Let R be a ring with a unit element; and center Z. If there exists an integer $n>1$ so that $x^n-x\in Z$ for all $x\in R$, then R is commutative. The method of proof is to first prove the result for division rings and primitive rings, and so obtain it for all semi-simple rings. In the case that the ring has a radical, this enables us to locate precisely the commutators and the radical. From this information the result can be obtained for subdirectly irreducible rings, and so for all the rings under consideration. It is conjectured that the result remains true if n is not fixed but $n=n(x)$, and even if the ring does not have a unit element. (Received February 23, 1951.)

327. D. G. Higman: *Lattice homomorphisms induced by group homomorphisms.*

A single-valued mapping ϕ of the lattice $L(G)$ of all subgroups of a group G onto the lattice $L(H)$ of all subgroups of a group H will be called a *lattice homomorphism of G onto H* if (1) $(\bigcup_\nu S_\nu)\phi=\bigcup_\nu(S_\nu\phi)$ and (2) $(\bigcap_\nu S_\nu)\phi=\bigcap_\nu(S_\nu\phi)$ for every system of subgroups S_ν of G. ϕ will be called *proper* if it is neither 1-1 nor trivial. Every homomorphism of a group G onto a group H induces a single-valued mapping of $L(G)$ onto $L(H)$ which satisfies (1). The main result of this note is a characterization of those groups which admit *homomorphisms* which induce *proper lattice homomorphisms*. Our results contain those obtained by G. Zappa for finite groups (Giornale di Matematiche di Battaglini (4) vol. 78 (1949) pp. 182–192). (Received March 5, 1951.)

328t. Erwin Kleinfeld: *On the structure of simple alternative rings without chain conditions.* Preliminary report.

Let the alternative ring R be simple, not associative, without nilpotent elements and of characteristic not 2. Then it follows that R is a Cayley-Dickson division algebra. Refinements of methods from a paper by R. H. Bruck and the author (See Bull. Amer. Math. Soc. Abstracts 56-5-446 and 56-6-453) are employed. (Received February 5, 1951.)

329. E. D. Nering: *On the extensions of the field of constants of an algebraic function field.*

Let k be any field whatsoever, and let K be an algebraic function field of one variable of genus p over k. If Λ is transcendental over K, then $K(\Lambda)$ is an algebraic function field of genus p over $k(\Lambda)$. If λ is algebraic over k, let $K(\lambda)$ be the Kronecker product of K and $k(\lambda)$, and express $K(\lambda)$ as a direct sum of completely primary rings, $K(\lambda)=K(\lambda)_1+\cdots+K(\lambda)_s$. Let Γ_i be the radical in $K(\lambda)_i$, and let the radical degree of $K(\lambda)_i$ over $K_i=K(\lambda)_i/\Gamma_i$ be e_i. If p_i is the genus of the algebraic function field K_i over (the isomorphic image of) $k(\lambda)$, then $p-1=\sum_{i=1}^{s}e_i(p_i-1)+\rho$ where $\rho\geqq 0$ is one-half the order of the norm of a conductor in $K(\lambda)$. (Received February 19, 1951.)

330. Alex Rosenberg: *Subrings of simple rings with minimal ideals.*

The following theorems are proved: (1) If a π-regular ring is atomic modulo its radical, so is every π-regular subring. (2) Let $\mathfrak{A}$ be a simple ring with a minimal left ideal (S.M.I. ring). If $\mathfrak{B}$ is a π-regular simple subring not satisfying the minimum condition, $\mathfrak{B}$ is again S.M.I. and $\mathfrak{A}(\mathfrak{B})$, the commutator of $\mathfrak{B}$ in $\mathfrak{A}$, is the two-sided annihilator of $\mathfrak{B}$ in $\mathfrak{A}$. $\mathfrak{A}(\mathfrak{B})$ may have a radical which is a zero ring, and $\mathfrak{A}(\mathfrak{B})-\mathfrak{R}$ is again S.M.I. (3) Let $\mathfrak{A}$ be a dense algebra of finite-valued linear transformations in a vector space $\mathfrak{M}$ over a field. Let $\mathfrak{B}$ and $\mathfrak{C}$ be simple, isomorphic, π-regular subalgebras. Let $\mathfrak{M}\mathfrak{B}$ and $\mathfrak{T}_{\mathfrak{B}}$ denote the range and null space of $\mathfrak{B}$ in $\mathfrak{M}$ respectively; $\mathfrak{M}\mathfrak{B}\cap\mathfrak{T}_{\mathfrak{B}}=0$. Then the isomorphism between $\mathfrak{B}$ and $\mathfrak{C}$ can be extended to an inner automorphism of the entire algebra of linear transformations on $\mathfrak{M}$ if (i) $\mathfrak{A}(\mathfrak{B})$ is isomorphic to $\mathfrak{A}(\mathfrak{C})$, (ii) $\mathfrak{M}\mathfrak{B}$ and $\mathfrak{M}\mathfrak{C}$ break up into equipotent families of irreducible $\mathfrak{B}$ and $\mathfrak{C}$ modules, (iii) $\mathfrak{M}=\mathfrak{M}\mathfrak{B}\oplus\mathfrak{T}_{\mathfrak{B}}=\mathfrak{M}\mathfrak{C}\oplus\mathfrak{T}_{\mathfrak{C}}$. (Received March 14, 1951.)

331. W. R. Scott: *Groups and cardinal numbers.*

Let G be an infinite group with identity e. For $x\in G$, let $E(x)$ be the set of $g\in G$ such that the equation $g^n=x$ has no solutions for n. Let K be the set of $k\in G$ such that $o(E(k))<o(G)$ where $o(S)$ is the cardinal number of elements in S. Let D be the intersection of all subgroups G_α of G with $o(G_\alpha)=o(G)$. It is shown that $K\subseteq D$ and that K is a central, fully characteristic subgroup of G, which is either a cyclic group of order p^n or a p^∞ group. For Abelian groups G, $K=D=H$ if $G=H\times F$ where H is a p^∞ group and F is finite, and $K=D=e$ otherwise. A layer $L(n)$ $(L(\infty))$ of G is the set of elements of order n (∞). Several results are obtained concerning the size of layers of infinite groups. In particular, if G is not periodic, then $o(L(\infty))=o(G)$. If G is Abelian and nondenumerable, then there are $2^{o(G)}$ subgroups G_α of G with $o(G_\alpha)=o(G)$. (Received January 18, 1951.)

332. Daniel Zelinsky: *Complete fields without valuations.*

There exists complete topological fields other than those complete in a valuation. A class of such fields is obtained as follows. Let R be any integral domain not a field,

F its quotient field. The R-topology on F is defined by designating the nonzero ideals in R as the neighborhoods of zero [Bull. Amer. Math. Soc. vol. 54 (1948) pp. 1145–1150]. Usually F is neither complete nor completable. However, if R is a local ring complete in the local topology, then F is complete in the R-topology. If R is any complete local ring with dimension greater than one and no divisors of zero (for example, the ring of power series in n variables, $n>1$), its quotient field, in the R-topology, is a complete field whose topology is not given by a valuation. (Received February 1, 1951.)

ANALYSIS

333. Nachman Aronszajn: *A characterization of commutative types of order.*

The following theorem is proved: For two types of order α and β, a necessary and sufficient condition in order that $\alpha+\beta=\beta+\alpha$ is that at least one of the following representations be valid. (i) $\alpha=\beta\omega+\gamma+\beta\omega^*$, with some type of order γ, (i′) $\beta=\alpha\omega+\gamma+\alpha\omega^*$, with some type of order γ, (ii) $\alpha=\sum_{0\leqq\xi<\xi_0}\gamma(\xi)$, $\beta=\sum_{\xi_0\leqq\xi<1}\gamma(\xi)$, where ξ_0 is some number greater than 0 and less than 1; $\gamma(\xi)$ is a function defined for $0\leqq\xi<1$, having for values types of order and satisfying the condition $\gamma(\xi_1)=\gamma(\xi_2)$ for any ξ_1, ξ_2 for which there exist integers m, m_0, such that $\xi_1-\xi_2=m+m_0\xi_0$. (Received March 12, 1951.)

334. Nachman Aronszajn and A. K. Jennings: *A new approximation method for partial differential eigenvalue problems.* Preliminary report.

The method approximates the eigenvalues by upper bounds. The approximations are obtained as eigenvalues of systems of ordinary differential equations. (Received March 12, 1951.)

335. R. C. Buck: *Operator algebras and dual spaces.*

Let X be a translation invariant linear space of functions on a group G, and $B(X)$ be its associated algebra of linear operators. Let G_0 be the group of left translation operators, induced by G on X. Then, under suitable conditions, the centralizer B^0 of G_0 in $B(X)$ is isomorphic as a linear space to the dual space X' of X. The operator multiplication then induces a convolution product in X'. This has applications to the study of group algebras, and to the theory of distributions (Schwartz, Actualités Scientifiques et Industrielles, Paris, 1950). (Received March 12, 1951.)

336. R. H. Cameron and Charles Hatfield: *On the summability of certain series for unbounded nonlinear functionals.*

In a previous paper (*On the summability of certain orthogonal developments of nonlinear functionals*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 130–145) the authors showed that the Fourier-Hermite expansion (Ann. of Math. vol. 48 (1947) pp. 385–392) of a bounded Wiener measurable (nonlinear or linear) functional is Abel summable (in an appropriate infinite-dimensional sense) to the value of the functional at each point where the functional is continuous in the Hilbert topology. In the present paper the authors remove the condition of boundedness on the functional and assume a functional $F(x)$ such that $|F(x)|\leqq B\exp\left\{A\int_0^1[x(t)]^2dt\right\}$. (Received March 13, 1951.)

337*t*. Harvey Cohn: *A conformal mapping property of the Eisenstein series*. Preliminary report.

The upper half z-plane is mapped on the upper half w-plane with slits running from $w=p/q$ to $w=p/q+i\epsilon/q^n$. The mapping function is given for small ϵ approximately by $w=z+\epsilon^2\phi_n(z)$ where $d^{(2n-1)}\phi_n(z)/dz^{2n-1}$ is (essentially) the Eisenstein series $\sum(pz+q)^{-2n}$. For $n=1$ a slight modification of the limit ($\epsilon\rightarrow 0$) produces the partition function. The method used is a perturbation method, which brings out a type of infinitesimal invariance in the mapping function a priori. (Received March 6, 1951.)

338. R. L. Evans: *Solution in the large of linear differential equations*.

W. B. Ford's method of solution in the large is extended to equations whose rank at an irregular singular point exceeds unity and can first be reduced by Poincaré's method. Also, the treatment of Ford's case 2 of type III is simplified and made to give more explicit results by a change of the dependent variable that simultaneously replaces the solution of a certain second order difference equation with that of a first order difference equation. As a different development, a simple but apparently new method of solution in the large is presented which utilized convergent solutions about an irregular singular point, usually at infinity. These involve series of inverse factorials and were supplied for many differential equations by J. Horn and W. J. Trjitzinsky. Another method of getting such solutions and a new criterion for their convergence are described. These suggest a change of the independent variable that makes possible the solution in the large of some previously unsolvable linear differential equations. Several examples are given, including one previously unsolvable and two previously unsolved equations. (Received March 14, 1951.)

339. R. P. Gosselin and L. D. Berkovitz: *On the theory of localization for double trigonometric series*.

In this paper the authors consider the problem of localization for double trigonometric series $\sum a_{mn} \exp i(mx+ny)$ summed by the restricted method. Let the coefficients a_{mn} be $o(|m|^\gamma |n|^\delta)$, $\gamma>-1$, $\delta>-1$, and set $\beta=\gamma+\delta+1$. If the series is formally integrated with respect to x and y sufficiently many times to produce uniform convergence to a function $F(x, y)$, and if $\lambda(x, y)$ is a sufficiently good localizing function associated with the rectangles R' and $\bar{R}$, $R'\subset R$, then $\Delta_{M,N}(x, y)=\sum a_{m,n}e^{i(mx+my)} - ((-1)^{K+L}/\pi^2)\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}F(u, v)\lambda(u, v)D_M^K(x-u)D_N^L(y-v)du dv$ is (C, β, β) restrictedly summable to zero, uniformly for (x, y) in R', where $D_M^K(u)$ denotes the kth derivative of the Dirichlet kernel of order M. From the above Riemann formula one obtains a localization principle involving only ordinary neighborhoods for double trigonometric series summed restrictedly. The method of proof involves the use of formal multiplication of double trigonometric series in a manner analogous to that used by Rajchman and Zygmund for the one variable case. An extension to two variables of Fatou's theorem on power series is obtained as a by-product of the above theory. (Received March 15, 1951.)

340. Wilfred Kaplan: *On Gross's star theorem, schlicht functions, logarithmic potentials, and Fourier series*.

Gross's star theorem asserts that each element of the inverse of a meromorphic function $g(z)$ can be continued to infinity along "almost all rays" from the center of the element. This theorem is generalized and connections with the theory of schlicht functions, logarithmic potentials, and Fourier series are shown: 1. The Gross theorem remains valid if the family of rays is replaced by any conformal image of a family of parallel lines. 2. Gross's theorem and 1. remain valid if $g(z)$ is replaced by any function $G(z)$ which satisfies an equation $f(G(z)) = h(z)$, where f and h are meromorphic for finite z. 3. Let $\mathcal{E}$ be the class of all closed sets E on $|z| = 1$ for which there exists a schlicht function $f(z)$ in $|z| < 1$ such that f becomes infinite for arbitrary approach to each point of E. Then $\mathcal{E}$ coincides with the class of closed sets of logarithmic capacity 0 on $|z| = 1$. 4. The sum of the conjugate trigonometrical series $\bar{S}[f]$ of a function $f(\theta)$ of bounded variation can be represented as a logarithmic potential of a mass distribution of variable sign on $|z| = 1$. 5. The class of closed sets E on $|z| = 1$ for which there exists an $f(\theta)$ of bounded variation such that $\bar{S}[f]$ diverges for $\exp(i\theta)$ in E is the class $\mathcal{E}$. (Received March 7, 1951.)

341. George Klein: *On the divergence of the Lagrange polynomials interpolating analytic functions.*

Let B be the Banach space of functions analytic interior to and continuous on the closed unit disc in the complex plane, with the norm of an element f of B defined as $\|f\| = \sup_{|z|=1} |f(z)|$. Let B_n be the subspace of B consisting of polynomials in z of degree at most n. Let I_n be the linear operator acting on B which maps an element f of B into the polynomial of degree n which agrees with f at $n+1$ points equally distributed on the circumference of the unit circle. The following two theorems are established: I. *Suppose that, for each n, L_n is a linear operator mapping B_n into B. Then for an arbitrary such sequence, the sequence L_nI_n of linear operators mapping B into B does not converge strongly to the identity mapping on B.* II. *Suppose that, for each n, L_n is the linear operator defined on B which maps each function $f(z)$ in B into the function $f_n(z) = ((n+1)/2\pi)\int_{-\pi/(n+1)}^{\pi/(n+1)} f(ze^{it})dt$. Then the sequence of composite linear operators I_nL_n defined on B converges strongly to the identity operator acting on B.* (Received April 27, 1951.)

342. Lee Lorch: *On derivatives of infinite order.*

Some of the results derived by Boas and Chandrasekharan concerning $\lim f^{(n)}(x)$ as n becomes infinite (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 523–526, 1191) are restudied with "lim" taken to mean summation by Borel's exponential means instead of convergence. Additional remark: Following their notation, it is pointed out that if the lim inf occurring in their theorem 3(ii) (loc. cit. p. 525) is not the same as lim sup (that is, if the limit does not exist), then $g(x) \equiv 0$ in $a \leqq x \leqq b$. (Received March 14, 1951.)

343. Josephine M. Mitchell: *The Bergman kernel function in the geometry of matrices.*

We consider the domain D defined by $Z\bar{Z}' < I$, where Z is a rectangular matrix of complex numbers z_{jk} $(j = 1, \cdots, m;\ k = 1, \cdots, n)$, $\bar{Z}'$ its conjugate transpose, and I the identity matrix. It is proved by means of a minimal problem that the function $K(Z, \bar{W}') = V^{-1}[\det(I - Z\bar{W}')]^{-m-n}$, where $Z, \bar{W}' \in D$ and V is the Euclidean volume of D, is the Bergman kernel function of the domain D. From this follows the symmetry property $K(Z, \bar{W}') = \overline{K(W, \bar{Z}')}$ and the reproducing formula $f(Z) = \int \cdots \int_D K(Z, \bar{W}')f(W)dV$ $(Z \in D)$ for any function $f(Z)$, analytic in $z_{11}, \cdots, z_{mn}$, such that $\int \cdots \int_D |f(Z)|^2 dV$ is finite. Similar results hold for the cases where Z is a symmetric or skew-symmetric matrix. (Received March 12, 1951.)

344. George Piranian: *Comparison of the collective Hausdorff method with the method of essential Hausdorff cores.*

A sequence s is evaluated to the finite value p by the collective Hausdorff method $\mathcal{H}$ [Agnew, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 217–237] provided there exists a regular Hausdorff matrix A such that the sequence As converges to p. If s is bounded, the core $C(s)$ is defined as the least convex set containing all limit points of s; the definition has been extended in an appropriate manner to deal with unbounded sequences. The essential Hausdorff core of s is the set H^*s consisting of the intersection of all cores $C(As)$, where the symbol A ranges over all completely regular Hausdorff matrices. The author shows that if s is evaluated to p by $\mathcal{H}$, then the set H^*s is either unbounded or consists of the single point p. On the other hand, he exhibits a bounded sequence s such that the set H^*s consists of the origin while the method $\mathcal{H}$ does not evaluate s. (Received March 13, 1951.)

345. O. W. Rechard: *A note on the summability of infinite series by sequence to sequence and series to sequence transformations.*

If the infinite matrix $A=(a_{ij})$ defines a regular sequence to sequence method of summability, it is well known that the matrix $B=(b_{ij})$ obtained by defining $b_{ij}=\sum_{n=j}^{\infty}a_{in}$ defines a regular series to sequence method of summability. Moreover, a series with bounded partial sums is summed by A to s if and only if it is summed by B to s. However, if $B=(b_{ij})$ determines a regular series to sequence method of summability and $A=(a_{ij})$ is defined by $a_{ij}=b_{ij}-b_{i,j+1}$, then the sequence to sequence method of summability determined by A need not even be regular. It is the purpose of this note to point out that this apparent superiority of series to sequence methods of summability over sequence to sequence methods is really illusory. In fact, if $B=(b_{ij})$ determines a regular series to sequence method of summability which sums some divergent series with bounded partial sums, then the corresponding sequence to sequence method A, as defined above, is regular, and a series with bounded partial sums is summed by B to s if and only if it is summed by A to s. (Received March 9, 1951.)

346. R. F. Reeves: *A center of gravity problem.* Preliminary report.

Let r be the position vector in a three-dimensional Euclidean space, and let $mF(r)$ be the force exerted on a particle of mass m placed at r. Assume $F(r)$ to be continuous and to admit of centers of gravity in the following sense. For every finite system of particles of mass $m_1, m_2, \cdots, m_n$ located at $r_1, r_2, \cdots, r_n$ respectively, there exists a point $r_c(r_1, r_2, \cdots, r_n)$, called the center of gravity of the system, which satisfies the following conditions: (a) $\sum_{i=1}^{n}m_iF(ri)=(\sum_{i=1}^{n}mi)F(r_c)$, (b) $r_c=\sum_{i=1}^{n}\alpha_ir_i$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are real numbers, (c) $r_c(Or_1+b, Or_2+b, \cdots, Or_n+b)=Or_c(r_1, r_2, \cdots, r_n)+b$, where O is an arbitrary orthogonal matrix and b any constant vector. Then $\sum_{i=1}^{n}\alpha_i=1$, and $F(r)=Ar+c$, A being an arbitrary constant matrix and c being any fixed vector. In the paper *On fields of force in which centers of gravity can be defined* by John Aczél, and Stephen Fenyö, Hungarica Acta Mathematica vol. 1 (1948), a similar problem was considered, conditions (b) and (c) being different. (Received March 9, 1951.)

347. Edward Silverman: *Set functions associated with Lebesgue area.*

If x is a continuous function defined on a square Q into E_n, let $P(x)$, $L(x)$, and $x(Q)$ denote the Peano area, the Lebesgue area, and the range of x, respectively. There is an analytic procedure, independent of the notion of Lebesgue area, for constructing a monotone function y on Q into m, the space of bounded sequences, such that $L(x)=L(y)$. Then there are defined two, possibly different, non-negative set functions λ and μ, with domain the subsets of m, which give the elementary area for elementary configurations and satisfy the following statements: (i) If $P(x)=L(x)$, then $\lambda(y(Q))=L(x)$, and (ii) $\mu(y(Q))=L(x)$. (Received March 12, 1951.)

348. Annette Sinclair: *Approximations and the Weierstrass-factor and Mittag-Leffler theorems.*

In this paper the existence of a function is shown which—in addition to satisfying the conditions required of the function constructed in the Wierstrass-factor (or Mittag-Leffler) Theorem—satisfies a preassigned approximation condition in a certain neighborhood of each zero (or pole). Let $M(z)$ be any function which is analytic and not identically zero in a neighborhood of each point of an isolated set J and which vanishes at each point of J. It is proved that there exists a function $h(z)$ such that: (1) $h(z)$ is analytic in the entire plane except at limit points of J; (2) $h(z)$ has zeros at precisely the points of J of the same orders as those of $M(z)$; (3) $h(z)$ approximates $M(z)$ arbitrarily closely in a certain preassigned neighborhood of each point of J. An analogous generalization of the Mittag-Leffler Theorem is given in which an approximation condition is preassigned in a certain deleted neighborhood of each pole. In either generalization the closeness of approximation may be preassigned independently for neighborhoods of different points of the isolated set on which zeros (or poles) are required. (Received March 9, 1951.)

349. Otto Szász: *On products of summability methods.*

Suppose that A and B are two regular summability methods for sequences $\{S_n\}$ or series $\sum a_n$. Denote by AB the iteration product which associates with a given sequence the A transform of its B transform. Particular cases of the problem are discussed: when does summability A imply summability (AB)? The answer is shown to be in the affirmative in the following cases: (a) B is (C, α) and A is either Abel or Borel summability; (b) B is Riesz summability and A is the Laplace transform; (c) B is Euler- and A is Borel-summability. The special case $\alpha=1$ has been discussed before. (Received March 7, 1951.)

350. H. L. Turrittin: *Asymptotic expansions of solutions of systems of ordinary linear differential equations containing a parameter.*

W. J. Trjitzinsky has foreseen that the system of differential equations (1), $\epsilon^h dX/dt=A(t, \epsilon)X(t)$, where h is a positive integer; ϵ is a small parameter; X is a column vector with components $x_1(t), \cdots, x_N(t)$; and where the square matrix $A(t, \epsilon)$ has a convergent expansion $A=\sum_{i=0}^{\infty}\epsilon^i A_i(t)$ in the domain $|\epsilon|\leqq\epsilon_1$, $t_1\leqq t\leqq t_2$, possesses N formal vector solutions each of the form $X=\sum_{j=0}^{\infty}\epsilon^{j/r}X_j(t)\cdot\exp\{\int_{t_1}^{t}\epsilon^{-h}\rho(\tau, \epsilon)d\tau\}$ where the X_j are vectors; r is a suitable positive integer; and $\rho(t, \epsilon)=\sum_{j=0}^{hr-1}\epsilon^{j/r}\rho_j(t)$ where the $\rho_j(t)$ are suitably chosen functions of t (see Acta Math. vol. 67 (1936) pp. 1–50). In this reference Trjitzinsky also proves under suitable hypotheses that, *if such formal solutions exist*, they represent asymptotic solutions of (1). In the present paper a straightforward method for evaluating the $\rho_j(t)$'s, r, and $X_j(t)$'s in the formal solutions is presented, based on a sequence of non-singular matrix transformations of four types: $X=P(t)Y$; $X=Y\exp\{\int_{t_0}^{t}\epsilon^{-h}\rho(\tau)d\tau$;

$X=[I+\epsilon^k Q_k(t)]Y$ where I is the unit matrix; and $x_i(t)=\epsilon^{(N-i)k}y_i(t)$ $(i=1,\cdots,N)$, where k is a positive integer or fraction. (Received February 6, 1951.)

Applied Mathematics

351. M. R. Hestenes and William Karush: *The solutions of $Ax=\lambda Bx$.*

Let A, B be Hermitian matrices of order n with B positive definite. An iterative gradient method is studied for determining the characteristic values and vectors of the equation $Ax=\lambda Bx$. The method stems from the fact that the characteristic vectors are the critical points of the "Rayleigh quotient" $\mu(x)=(x, Ax)/(x, Bx)$. One passes from x to the next approximation x' by means of the formula $x'=x-\alpha\eta$ where $\alpha>0$ and $G\eta=Ax-\mu x$ (G is an arbitrary fixed positive definite matrix with known inverse). It is shown that with appropriate choice of α the iteration converges to a characteristic vector, and, in some cases, to the vector with least characteristic value. The procedure can be adapted to yield all the characteristic vectors; it has the advantage of avoiding transformation of the problem and of being mechanizable. Finally it is shown that a general problem $Cx=\lambda Dx$ (C, D arbitrary matrices with $|C-\lambda D|\neq 0$) is equivalent to one of the above type if and only if there exists a positive definite Hermitian matrix P such that CPD^* (or C^*PD) is Hermitian ($D^*=$conjugate transpose of D). (Received March 5, 1951.)

352. M. Z. Krzywoblocki: *Extension of Kampé de Fériet's formula on kinetic energy in simply connected domain to compressible fluid.*

Leray and subsequently Kampé de Fériet derived and proved the exponential form of the function expressing the decrease of kinetic energy of an incompressible viscous fluid flow in a simply connected domain without any external forces to resist or to excite the motion. In the present note the author derives such a form for a compressible viscous fluid flow and shows that the formulae obtained by previous authors are only particular cases of the more general forms derived by him. (Received February 23, 1951.)

353t. Daniel Resch: *Temperature bounds on the infinite rod.*

It is desired to know the bounds on the positive temperature, $u(x, t)$, of an infinite rod at any time $(t>0)$ and at any point of the rod when the temperature is known at certain points and at certain times. While the problem is treated by P. C. Rosenbloom, *Quelques classes de problèmes extrèmaux* (to appear in Bull. Soc. Math. France, 1951), the author employs simple methods to show that if the temperature is known at one point and time, $u(x_0, t_0)=c$, then $u(x, t)\leqq cH(x, t)$ for $0<t\leqq t_0$ and the inequality reversed for $0<t_0\leqq t$ where $H(x, t)=(t_0/t)^{1/2}\exp[(x-x_0)^2/4(t_0-t)]$. With the aid of the first inequality, the author proves an analog of Harnack's second theorem; namely, that a monotone increasing sequence of solutions of the heat equation bounded at one point (x_0, t_0) converges uniformly in the domain $|x-x_0|\leqq A(t_0-t)^{1/2}$, $t\geqq\epsilon>0$ (A, ϵ constants). (Received March 13, 1951.)

Geometry

354. Eugenio Calabi: *Isometric, complex analytic imbedding of complex manifolds.*

The problem considered is whether a given complex manifold M^k with any metric admits an isometric, complex analytic imbedding in a Fubini-Study space of dimension $N(\leqq \infty)$ and constant curvature b on holomorphic sections (see Bochner, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 179–195). A first necessary condition is that the metric on M^k be an analytic, Kähler metric. The latter gives rise by integration to a real analytic function of pairs of points, called their *diastasis*, which is asymptotically (as the points become close) the square of their geodesic distance. The diastasis, in contrast with geodesic distance, has the property that, if the two points lie on a complex submanifold, their diastasis with respect to the metric induced on the submanifold is the same as that obtained from the ambient space. This implies that any isometric, complex analytic imbedding is determined up to within a motion in the ambient space. The necessary and sufficient conditions for the existence of an isometric, complex analytic imbedding locally are then deduced in terms of the coefficients of the power series expansion of the diastasis; the imbedding is global if the diastasis is single-valued. (Received March 14, 1951.)

355. Mary Dean Clement: *A criterion for determining the space of immersion of a variety of arbitrary dimensionality.*

This paper establishes a necessary and sufficient condition that a proper analytic variety V_m: $x=x(u_1, \cdots, u_m)$, defined in a projective linear space S_n, be immersed in a linear subspace S_k. Relative to a given variety V_m let the set of matrices $M_m^{(h)}$ $(h=0, 1, 2, \cdots)$ be defined recursively: $M_m^{(0)}\equiv(x)$; $M_m^{(h)}(h\geqq 1)$ is formed from $M_m^{(h-1)}$ by adjoining as columns all partial derivatives of x of order h. Consideration of properties of this set of matrices and consideration of geometric properties of V_m lead to the conclusion: a necessary and sufficient condition that a proper analytic variety V_m be immersed in a projective linear space S_k is that the matrix $M_m^{(k-m+2)}$ be of rank $k+1$. Interpretation of this result in terms of partial differential equations yields the equivalent result: a necessary and sufficient condition that V_m be immersed in S_k is that the coordinates of an ordinary point on V_m satisfy a completely integrable system of $C_{k+2,m}-(k+1)$ linearly independent linear homogeneous partial differential equations of order $k-m+2$ or less, of which precisely $C_{k+1,m-1}$ equations are actually of order $k-m+2$. Specialization of the latter condition when $m=1$ gives the classical criterion for the space of immersion of a proper analytic curve. (Received March 14, 1951.)

356. J. W. Gaddum: *Spread of an arc in a metric space.*

Let A be an arc in a metric space, with $p=f(x)$, $p\in A$, $x\in[0, 1]$, f a homeomorphism between A and the unit interval. If $\lim_{x,y\to t} f(x)f(y)/xy$ exists finite it is denoted $f'(t)$ and called the spread of f at t. $A(a, b)$ will denote the subarc of A from $f(a)$ to $f(b)$. If it is rectifiable its length will be denoted $s(a, b)$. The following theorems can be proved: (1) if $f'(t)$ exists on a set E, it is continuous on E; (2) if $f'(t)$ exists at each point of a, b, $A(a, b)$ is rectifiable and $s(a, b)=\int_a^b f'(t)dt$; (3) if $f'(t)$ exists at each point of $[a, b]$ and $f'(t_0)\neq 0$, $t_0\in[a, b]$, then $\lim_{x,y\to t_0} f(x)f(y)/s(x, y)=1$; (4) if A is rectifiable and f is the arc length parameterization and the chord arc ratio approaches 1 at t, then $f'(t)=s$. It may happen that $f'(t)$ fails to exist even though $\lim_{x\to t} f(x)f(t)/xt$ exists and is continuous at each point of $[0, 1]$. (Received March 12, 1951.)

STATISTICS AND PROBABILITY

357. Murray Rosenblatt: *On the oscillation of sums of random variables.*

$X_1, X_2, \cdots$ are independent, identically distributed random variables with distribution function $F(x)$. $S_n = \sum_1^n X_j$. The sequence S_n is said to oscillate if $\Pr(S_n > 0$ infinitely often$) = \Pr(S_n \leqq 0$ infinitely often$) = 1$. If $\Pr(X_j \neq 0) > 0$ and $E(|X_j|) < \infty$, oscillation takes place if and only if $E(X_j) = 0$. $\limsup_{n\to\infty} \Pr(S_n > 0)$, $\limsup_{n\to\infty} \Pr(S_n \leqq 0) > 0$ is a sufficient condition for oscillation. When $E(|X_j|) = \infty$, it yields the following criterion for oscillation: If there are positive monotone sequences of integers $n_k \uparrow \infty$ and numbers $a_k \uparrow \infty$ such that (1) $\limsup_{k\to\infty} n_k/a_k < \infty$, (2) for some $\epsilon > 0$ $\limsup_{k\to\infty}(n_k^2/a_k^2) \cdot \mathrm{Var}\ (X^2 \| X| < a_k\epsilon) < \infty$, (3) $\lim_{k\to\infty} n_k(1 - F(a_k x)) = -\phi(x)$, $\lim_{k\to\infty} n_k F(-a_k x) = \phi(-x)$, $x > 0$, where $\phi(x)$ is finite for $|x| \neq 0$ and increases somewhere in both ranges $x > 0$ and $x < 0$, then the S_n generated by X_j with distribution function $F(x)$ oscillate. $\sum_1^\infty \Pr(S_j > 0, S_{j+1} \leqq 0)$, $\sum_1^\infty \Pr(S_j \leqq 0, S_{j+1} > 0) = \infty$ is shown to be a necessary and sufficient condition for oscillation. It is used to obtain a class of random variables each of which generate sums S_n such that $\Pr\ (\lim_{n\to\infty} |S_n| = \infty) = \Pr\ (\limsup_{n\to\infty} S_n = \infty) = \Pr\ (\liminf_{n\to\infty} S_n = -\infty) = 1$, $\lim_{n\to\infty} \Pr\ (S_n > 0) = 0$. (Received March 14, 1951.)

Topology

358. R. H. Bing: *A homeomorphism between the 3-sphere and the sum of two solid horned spheres.*

A horned sphere plus its bad complementary domain in S^3 is called a solid horned sphere. If two solid horned spheres are "sewn" together in a certain fashion, the question has been raised by R. L. Wilder as to whether or not the resulting set is homeomorphic with S^3. Suppose that the compact continuum M is the sum of three mutually exclusive sets U_1, H, and U_2 such that there is a homeomorphism of $H + U_1$ into the solid horned sphere that carries H into the horned sphere and there is a homeomorphism of $H + U_1$ into $H + U_2$ that leaves each point of H fixed. It is shown that M is topologically equivalent to S^3. Besides answering the question raised by Wilder, this also answers certain questions that have been raised regarding periodic transformations of S^3 into itself. In the homeomorphism of M into S^3, the "bad" points of H go into a closed 0-dimensional set W in S^3 with the following properties: (1) the complement of W is not simply connected; (2) each pair of mutually exclusive closed subsets of W can be separated by a simple surface in S^3; (3) there is a simple closed curve J in $S^3 - W$ such that each simple surface in $S^3 - W$ that separates W intersects J. (Received March 12, 1951.)

359t. R. H. Bing: *Concerning snake-like continua.*

A compact continuum M is snake-like if, for each positive number ϵ, M can be covered by an ϵ-chain. While not each compact atriodic plane continuum which does not separate the plane is snake-like, each one that contains no nondegenerate indecomposable continuum is snake-like. A nondegenerate continuum is hereditarily decomposable (hereditarily unicoherent) if each nondegenerate subcontinuum of it is decomposable (unicoherent). A necessary and sufficient condition that a compact hereditarily decomposable continuum be snake-like is that it be atriodic and hereditarily unicoherent. If M is a hereditarily decomposable snake-like continuum, M contains two points p and q such that for each positive number ϵ, an ϵ-chain from p to q covers M. (Received March 12, 1951.)

360t. M. L. Curtis and G. S. Young: *A theorem on dimension.*

This note is devoted to proving the theorem: A compact metric space X has dimension less than or equal to n if and only if there exists a light mapping of X into the n-dimensional cube. The sufficiency follows from a theorem due to Hurewicz (*Dimension theory*, Theorem VI 7) and the necessity is proved by a technique similar to that used to prove the classical imbedding theorem (Theorem 3 in *Dimension theory*). (Received March 14, 1951.)

361. Samuel Eilenberg and Saunders MacLane: *Cohomology theory of abelian groups and homotopy theory*. III.

Let X be a space with a given homotopy group $\pi=\pi_n(X)$ and all other homotopy groups trivial. It is shown that the singular homology theory of X is equivalent to the abelian homology theory of the group π, on level $n-1$. Specifically, the homology theory of X is that of a certain cell complex $K(\pi, n)$ defined algebraically in terms of the abelian group π and the integer n. For $n=1$, this complex $K(\pi, 1)=A(\pi, 1)$ has a product $*$ of excess zero (Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 657–663). The bar construction B (loc. cit.) applies to any complex with such a product, and yields a new complex with a new product of excess 1. B^+ is written to indicate that all dimensions are raised by 1; the new product then has excess zero. The abelian homology theory of π in level $n-1$ is that of the complex $A(\pi, n)=(B^+)^{n-1}(K(\pi, 1))$ obtained by $n-1$ applications of the bar construction. We obtain an explicit chain equivalence $A(\pi, n)\rightarrow K_a(\pi, n)$, where the subscript a denotes augmentation. The essential devices are the construction of a product in $K(\pi, n)$ which corresponds to the subdivision of the cartesian product of two simplices $\Delta_p\times\Delta_q$ into $(p+q)$-dimensional simplices, the representation of $K(\pi, n+1)$ in terms of $K(\pi, n)$ by a suitable construction W on the latter complex, and the appropriate use of the notion of generic acyclicity (loc. cit). (Received March 14, 1951.)

362. P. R. Halmos: *Normal dilations and extensions of operators*.

Let H be a subspace of a Hilbert space K, let A and B be operators on H and K respectively, and let P be the projection with domain K and range H. The operator B is called a *dilation* of A if $AP=PBP$; the operator B is called an *extension* of A if $AP=BP$. The main purposes of this paper are (1) to prove that if $|A|\leqq 1$, then A has a dilation which is unitary, (2) to prove that if $0\leqq A\leqq 1$, then A has a dilation which is a projection, and (3) to derive necessary and sufficient conditions on A in order that A have an extension which is normal. The first two results enable one to prove that if H is infinite-dimensional, then the weak closure of the set of all unitary operators is the set of all operators A such that $|A|\leqq 1$ and the weak closure of the set of all projections is the set of all operators A such that $0\leqq A\leqq 1$. The necessary and sufficient conditions for the normal extendibility of A are expressed in terms of certain matrices (generalizations of Gramians) associated with the operator A. (Received January 16, 1951.)

363*t*. Tibor Radó: *Cohomology theory for general spaces*.

Let A be a subspace of the topological space X. Cohomology groups $\bar{H}^p(X, A)$ are assigned to the pair (X, A) as follows. A p-function c^p is an integral-valued function $c^p(x_0, \cdots, x_p)$ of $p+1$ points of X. The coboundary operator δ is defined as usual. $C^p(X, A)$ denotes the group of those p-functions which vanish locally on X and vanish on A. For $p<0$, one sets $C^p(X, A)=0$. The group $Z^p(X, A)$ is the group of those elements of $C^p(X, A)$ for which $\delta c^p=0$. The group $B^p(X, A)$ is the group of those elements of $C^p(X, A)$ which are of the form $c^p=\delta c^{p-1}$, where $c^{p-1}\in C^{p-1}(X, A)$.

Finally, $\tilde{H}^p(X, A) = Z^p(X, A)/B^p(X, A)$. Let $H^p(X, A)$ be the cohomology group employed by Spanier (Ann. of Math. vol. 49 (1948) pp. 407–427). The following results are established. If A is a nonempty closed subset of X, then $\tilde{H}^{p+1}(X, A) \approx H^p(X, A)$ for every p. For the case $A=0$, it is shown that $\tilde{H}^{p+1}(X) \approx H^p(X)$ for $p \neq 0$, while $\tilde{H}^1(X)$ is isomorphic to the reduced 0-dimensional cohomology group. (Received March 12, 1951.)

364. E. H. Rothe: *Leray-Schauder index and Morse type numbers in Hilbert space.*

Let E be a Hilbert space and $c(x)$ a real-valued function defined for x in some neighborhood of the zero point o of E. We suppose that o is an isolated critical point of $c(x)$, that is, that $g(x) = \operatorname{grad} c(x)$ equals o for $x=o$ and is not equal to o for all other $x \subset U$. Then the Morse type numbers m^r of dimension r ($r=0, 1, 2, \cdots$) are well defined. If on the other hand $g(x) = x + G(x)$ where $G(x)$ is completely continuous, then also the "index" j, that is, the Leray-Schauder degree at o of $g(x)$ as map $U \rightarrow E$ is well defined. The object of the paper is to establish the relation $j = \sum_r (-1)^r m^r$ (under some additional assumptions about $g(x)$). In the special case that E is a finite-dimensional Euclidean space this relation has been proved in an earlier paper (Mathematische Nachrichten, Erhard Schmidt Festschrift, 1951). (Received March 15, 1951.)

365. M. E. Shanks: *Rings of functions on locally compact spaces.* Preliminary report.

Consider the ring $R_0(X)$ of real continuous functions with compact supports on the locally compact Tychonoff space X. Then the locally compact Tychonoff spaces X and Y are homeomorphic if and only if $R_0(X)$ and $R_0(Y)$ are isomorphic. Actually the full rings are not needed, it is necessary only that they form "Tychonoff classes" of functions. In addition X is compact if and only if $R_0(X)$ has a unit; and if a unit is adjoined, in the noncompact case, the resulting ring determines the one-point compactification of X. If X is a manifold of class C^k, let $R_k(X)$ be the ring of functions of class C^k with compact supports. Then two manifolds X and Y of class C^k are differentiably homeomorphic of class C^k if and only if $R_k(X)$ is isomorphic to $R_k(Y)$. (Received March 15, 1951.)

366. G. H. M. Thomas: *Simultaneous partitionings of two sets.*

A set M is partitionable if, for each positive number ϵ, there exists a finite collection $G = \{U_1, \cdots, U_k\}$ of disjoint, connected, open subsets of M, each of diameter less than ϵ, whose sum is dense in M. G is called an ϵ-partitioning of M. If $M_2 \subset M_1$ and G is a partitioning of M_1 such that $G' = \{U_1M_2, \cdots, U_kM_2\}$ is a partitioning of M_2, then define G to be a simultaneous partitioning of M_1 and M_2. In this paper results are obtained regarding simultaneous partitionings of two sets, which are analogous to those obtained for partitionings by R. H. Bing (*Partitioning a set*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1101–1110). The main theorem is: If M_1 is a compact partitionable set and M_2 is a connected partitionable closed subset of M_1, then there exists a sequence $\{G_i\}$ of simultaneous partitionings of M_1 and M_2, each a refinement of the preceding, such that each element U^i_j of G_i has property S and is of diameter less than $1/i$, and each $U^i_jM_2$ also has property S. (Received March 12, 1951.)

367*t*. R. L. Wilder: *The complex-like character of certain spaces.*

It is well known that the Betti numbers of dimension less than or equal to n of a compact lc^n space S are finite. Using new methods of proof, it is shown that the lc^n requirement may be replaced by the weaker assumption that S have property $(P, Q, \sim)^{n-1}$ and be n-lc. (Definitions may be found in author's Colloquium volume, *Topology of manifolds*, referred to herein as T.M.) Indeed, if a locally compact space S has property $(P, Q, \sim)^{n-1}$ and is n-lc, then it has property $(P, Q)^n$. A number of related results are obtained: For compact spaces, and $k<n$, properties lc$_k^n$ and S$_k^n$ are equivalent. If U is an open subset of a locally compact space, and U has property $(P, Q)^k$ while $q^{k+1}(U; x) \leqq \omega$ for all $x \in \overline{U}$, then U has property $(P, Q)^{k+1}$. (Compare Theorem XI 6.8, T.M.) A nasc that an n-dimensional, locally compact space S should have property $(P, Q)_k$ and be lc$_k^n$, $k \leqq n$, is that $p_r(x) \leqq \omega$ for all $x \in S$ and $r=k, k+1, \cdots, n$ (see Theorem VII 2.25, T.M.). Various other results are obtained, such as generalizations of Theorems VI 5.1, VII 2.26, Theorems 1.3, 2.1, 2.3, 3.6, 3.10, 3.11 of Chapter XI of T.M., and on the decompositions of spaces into "prime parts". (see Amer. J. Math. vol. 63 (1941) pp. 691–697). (Received March 13, 1951.)

368. Oswald Wyler: *Order and topology in a projective plane.*

In a projective plane satisfying the axioms of incidence and of order, a topology of any line is derived from its order. Projectivities are then homeomorphisms. If l is a line, a point-set of the plane is said to be *l-convex* if it contains no point of l, and if its intersection with any line is either void or a point or a segment. Taking sets l-convex with respect to some line l and containing three points not on a line as a defining system of neighborhoods, the plane of points is seen to be a regular topological space. A line is a closed set, and its relative topology is the order-topology. Dually, a topology of the plane of lines is defined. It is shown that the line joining two distinct points is a continuous function of these points, and that the point of intersection of two lines is a continuous function of the lines. No configuration theorem is needed. If the lines are homeomorphic to the real projective line, the plane will be homeomorphic to the real projective plane. An example of such a plane is given, in which the Theorem of Desargues does not hold. (Received February 23, 1951.)

J. W. T. YOUNGS,
Associate Secretary

THE APRIL MEETING IN STANFORD

The four hundred seventieth meeting of the American Mathematical Society was held at Stanford University, Palo Alto, California, on Saturday, April 28, 1951. Approximately 127 persons attended, including the following 91 members of the Society:

H. L. Alder, T. M. Apostol, Richard Arens, H. A. Arnold, H. M. Bacon, W. G. Bade, G. A. Baker, Clifford Bell, R. E. Bellman, Benjamin Bernholz, Z. W. Birnbaum, David Blackwell, W. W. Bledsoe, F. H. Brownell, H. D. Brunk, C. L. Clark, J. G. van der Corput, Richard Courant, A. C. Davis, E. A. Davis, R. Y. Dean, S. P. Diliberto, R. P. Dilworth, Brother V. Dominic, Melvin Dresher, Roy Dubisch, L. K. Durst, Arthur Erdélyi, D. J. Ewy, Harley Flanders, G. E. Forsythe, R. A. Fuchs, C. M. Fulton, P. R. Garabedian, M. A. Girshick, G. E. Gourrich, G. J. Haltiner, J. G. Hardy, C. A. Hayes, G. B. Hedrick, J. G. Herriot, M. R. Hestenes, Edwin Hewitt, J. L. Hodges, C. G. Jaeger, Fritz John, R. M. Lakness, Cornelius Lanczos, G. E. Latta, D. H. Lehmer, R. B. Leipnik, Hans Lewy, J. C. C. McKinsey, A. V. Martin, A. B. Mewborn, W. E. Milne, C. B. Morrey, F. R. Morris, T. S. Motzkin, R. G. Needels, Ivan Niven, I. L. Novak, H. A. Osborn, Edmund Pinney, George Pólya, R. M. Redheffer, M. M. Resnikoff, J. B. Robinson, R. M. Robinson, E. B. Roessler, P. G. Rooney, Herman Rubin, Seymour Sherman, M. L. Stein, Robert Steinberg, R. G. Stoneham, E. G. Straus, A. C. Sugar, M. V. Sunseri, Irving Sussman, Gabor Szegö, R. B. Talmadge, Alfred Tarski, M. A. Weber, J. W. Weihe, F. J. Weyl, P. A. White, A. L. Whiteman, A. R. Williams, G. M. Wing, František Wolf.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Arthur Erdélyi of the California Institute of Technology delivered at 11:00 A.M. an address entitled *The analytic theory of systems of partial differential equations.* Professor Erdélyi was introduced by Professor George Pólya. There were sessions for contributed papers in the morning and afternoon, presided over by Professors Gabor Szegö, R. E. Bellman, and W. E. Milne. Following the meetings, those attending were guests at tea in the Stanford Union.

Following are abstracts of papers presented at the meeting. Papers whose abstracts numbers are followed by "*t*" were presented by title. Paper number 380 was presented by Professor Hyers, number 389 by Professor Hewitt, number 393 by Professor Birnbaum, and number 396 by Professor Straus.

ALGEBRA AND THEORY OF NUMBERS

369*t*. H. W. Becker: *Euler numbers in combinatory analysis.*

The Euler-Andre numbers (Netto, *Combinatorik*, 1927, pp. 105–112) have an interpretation; $2A_n$ is the number of alternating permutations of n letters. Their generating function is $e^{tA} = \sec t + \tan t$, from which $A_{n+1} = (A + jE)^n$, $j = (-1)^{1/2}$, $E_{2m-1} = 0$, $E_{2m} = (-)^m A_{2m}$. A new interpretation is: $A_{n+1} = W_n$ is the number of n letter words such that the mth letter of the alphabet may not occur prior to the mth position in the word, and no letter occurs more than twice. Their generating function is $e^{tW}=(\sec t)e^{tA}$, from which $W_{n+1}=(W+A)^n=(W+S)^{n+1}$, $S_m=\sin \pi m/2$. The dictionary of these words may be classified the usual ways, according to terminal letter, maximum letter, number of different letters, number of singletons, position of last "a," and so on, thus yielding new recurrences and breakdowns of the Euler numbers. If these words are restricted to being rhyme schemes $H_n \subset W_n$, then H_n (also the number of self-conjugate permutations of n letters) is an Hermite number, with generating function $e^{tH}=\exp(t+t^2/2)$. If further restricted to consecutive rhyme, $F_n \subset H_n$, then F_n (isomorphically the number of compositions of n into 1's and 2's) is a Fibonacci number, with generating function $1/(1-tF)=1/(1-t-t^2)$. But it is an unsolved problem to find a generating function for $n! \supset W_n$, the first two terms of which are constituted by e^{tW}. (Received March 12, 1951.)

370*t*. H. W. Becker: *The closed diagonal binomial coefficient expansion.*

The title refers to sums $\|U+V\|^n=\sum U_{n-m}\cdot V_m\cdot\|n, m\|$, the notation emphasizing that for given n the successive $\|n, m\|=(n-m, m)$ are found on the diagonal of the table of binomial coefficients, ascending from the left. Four different combinatory interpretations are given, of the Fibonacci series $F_n=\|1+1\|^n$, and of the Hermite series $H_n=\|1+(n)/2\|^n$, (n) being here the umbra of a Jordan factorial. Application is made to various configurations of Lucas, *Theorie des nombres*, 1891, pp. 218–219. If $N_n=(2n)!/n!(n+1)!$, which incidently has more than forty different combinatory interpretations, then the umbral product $N^{k*}\|N-1\|^n=0$, 1, or $(n+1)(n+2k)_{k-1}/k!$ according as $n\geqq -2k$, $k=0$, or $k>0$. There are such expansions of zero and fractions as $\|1-1\|^{n-1}=(2/3^{1/2})\sin(\pi n/3)$, $\|1-1/2\|^{4n-1}=\|1-1/3\|^{6n-1}=0$, $\|1-1/4\|^n=(n+1)/2^n$, and so on, suggested by a problem of Coxeter, Amer. Math. Monthly vol. 52 (1945) pp. 100–101. (Received March 12, 1951.)

371*t*. P. W. Carruth: *Sums and products of ordered systems.*

A non-empty set R of elements in which a reflexive binary relation $r \geqq r'$ is defined is termed an *ordered system*. Let R be an ordered system, and for each element r in R, let S_r be an ordered system. Then an ordered or lexicographic product and an ordered sum over R of the systems S_r have been defined (see, for example, Day, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 1–43). Necessary and sufficient conditions for an ordered product to be a lattice, complete lattice, chain, cardinal, and so on, respectively are obtained. Analogous results concerning ordered sums also are given. In the proofs use is made of results of Day, loc. cit. (Received February 28, 1951.)

372. L. K. Durst: *Apparition of primes in elliptic sequences (equianharmonic case).*

Let E be the ring of quadratic integers $a+b\rho$, $\rho^3=1$. If $\omega_2/\omega_1=\rho$, $g_2=0$, then $\psi_\mu(u)=\sigma(\mu u)/\sigma(u)^{N\mu}$ generates a mapping of E into the ring $E[\wp(u), g_3]$ which preserves division. An explicit arithmetical restriction is given for the apparition of prime ideals in this divisibility mapping, solving for this case a problem proposed by Morgan Ward (Amer. J. Math. vol. 70 (1948) p. 74). (Received March 15, 1951.)

373. Ivan Niven: *Functions which represent prime numbers.*

Given any real number $c>8/3$, it is proved that there exists a real number A

such that $[A^{c^n}]$ is a prime for every positive integer n; also for any given $A>1$ there exists a real number c yielding the same conclusion. The first part of this result was proved in the case $c=3$ by W. H. Mills [*A prime representing function*, Bull. Amer. Math. Soc. vol. 63 (1947) p. 604] and in the case of positive integers $c>3$ by L. Kuipers [*Prime-representing functions*, Neder. Akad. Wetensch. vol. 53 (1950) pp. 309–310]. The proof of the first part is a slight rearrangement of Mills' proof, and the proof of the second part employs the same basic idea in a different setting. (Received March 9, 1951.)

374*t*. Ernst Snapper: *Completely primary rings*. IV.

In the three papers *Completely primary rings*, I, II, III (Ann. of Math. (2) vol. 52 (1950), vol. 53 (1951)), no chain conditions were imposed on the rings. In the present paper a completely primary ring S whose ideals satisfy the two chain conditions is investigated. (Either chain condition implies the other.) It is shown that S always contains a *coefficient ring* R, that is, a completely primary ring which is a canonical extension of the prime ring of S and whose residue class field is equal to that of S. The first part of the paper is concerned with the properties of R. It is then observed that, since R is always a principal ideal ring and S is an R-module with a finite number of generators, one can introduce the elementary divisors of S, that is, the elementary divisors of the R-module S. The invariants of S which arise from this elementary divisor theory are discussed in the second part of the paper. In the last part of the paper these results are applied to the case when S is a principal ideal ring. (Received March 14, 1951.)

375*t*. Morgan Ward: *The diophantine equation $xy=zw$ in semi-groups*. Preliminary report.

Let I be a commutative semi-group with a unit partially ordered by the usual division relation. Assume that the diophantine equation $xy=zw$ always has solutions in I of the form $x=kl$, $y=mn$, $z=km$, $w=ln$; k, l, m, n in I. If in addition every element has only a finite number of nonequivalent divisors, I may be a group G, a distributive lattice L, or isomorphic to a subset M of the positive integers closed under multiplication. These are essentially the only possibilities; I is always isomorphic to a direct product $G\times L\times M$. (Received March 16, 1951.)

376. A. L. Whiteman: *Numbers of solutions of equations in finite fields*.

Let p be an odd prime and $GF(p^n)$ the Galois field of order p^n. Let g be the generator of the cyclic group formed by the nonzero elements of $GF(p^n)$ under multiplication. Let $m_1, m_2, \cdots, m_s$ be s divisors of p^n-1 and put $p^n-1=m_i m_i'$, $i=1, 2, \cdots, s$. In a series of three papers (Proc. Nat. Acad. Sci. U. S. A. vol. 33 (1947) pp. 236–242; vol. 34 (1948) pp. 62–66; vol. 35 (1949) pp. 681–685) Vandiver considered the problem of finding the number of solutions $(j_1, j_2, \cdots, j_s)$ in $\gamma_1, \gamma_2, \cdots, \gamma_s$ of $1+g^{j_1+m_1\gamma_1}+g^{j_2+m_2\gamma_2}+\cdots+g^{j_s+m_s\gamma_s}=0$, for a fixed set of numbers, $j_1, j_2, \cdots, j_s$ with the j_i in the range $0, 1, \cdots, m_i-1$, and the γ_i in the range $0, 1, \cdots, m_i'-1$. Explicit expressions involving p and n were obtained in the case $s=2$ for the sums $\sum_{i_1,i_2,\ldots,i_s}(j_1, j_2, \cdots, j_s)(j_1+k_1, j_2+k_2, \cdots, j_s+k_s)$, where the j_i range independently over $0, 1, \cdots, m_i-1$, and $k_1, k_2, \cdots, k_s$ is a fixed set of non-negative integers. In the present paper Vandiver's method is extended to derive similar results in the case where $s=3$ and in the general case where the m_i are prime each to each. (Received January 23, 1951.)

ANALYSIS

377. F. H. Brownell: *Translation invariant measure over separable Hilbert space.*

This paper investigates the existence of a translation invariant, nontrivial measure ϕ over the prototype real separable Hilbert space l_2, or more generally over the subset $X=\{x\in l_2 \mid |x_n|\leqq h(n)$ for all $n\}$, where $h(n)$ is a fixed function over the positive integers with values positive real or $+\infty$. Obviously $X=l_2$ if $h(n)\equiv+\infty$. Precisely the conditions on ϕ are that it should be defined non-negative and completely additive over the Borel class of subsets of X, that it be invariant over l_2 vector translations of subsets of X remaining inside X, and to be nontrivial that $\phi(X)>0$ and $\phi(A)<+\infty$ for some nonempty A open in X under some topology. The results found are that if, as when $X=l_2$, $\lim\inf_{n\to\infty} h(n)>0$, then no such ϕ exists for the norm metric topology, and hence none for the Cartesian product topology either. Also if $h(n)=+\infty$ for infinitely many n, no ϕ exists for the product topology. However if $\sum_{n=N}^{\infty} [h(n)]^2<+\infty$ for some finite N, the two topologies coincide, X is locally compact under either, and ϕ exists unique up to a constant factor. Extensions to other subsets of l_2 than X and to the case where $\lim\inf_{n\to\infty} h(n)=0$ but $\sum^{\infty} [h(n)]^2=+\infty$ are also considered. (Received March 16, 1951.)

378. H. D. Brunk: *Three-dimensional analogues of the cardinal series.* Preliminary report.

The cardinal series gives an entire function assuming given values at equally-spaced points on a line. Two three-dimensional analogues of the cardinal series have been studied, and a convergence property has been proved, generalizing to a strip region that of de la Vallee Poussin for the line (Académie Royale de Belgique. Bulletin de la Classe des Sciences vol. 1 (1908) pp. 319–410). (Received March 15, 1951.)

379. J. G. Herriot: *Partial-sum couplings for double Fourier series.*

Let $s_{mn}(x, y)$ be the rectangular partial sums of the double Fourier series of an integrable periodic function $f(x, y)$. It is well known that convergence of these partial sums is not a local property, even if the manner in which $m, n\to\infty$ is restricted by keeping m/n and n/m bounded. By forming suitable couplings of the partial sums similar to those studied by Rogosinski for single Fourier series (Math. Ann. vol. 95 (1926) pp. 110–134), there is obtained a method of summation for double Fourier series which possesses the localization property. Let $\kappa_{mn}(x, y, p, q, h, k) =(1/4)[s_{mn}(x+h, y+k)+(-1)^{p-1}s_{mn}(x+h+p\pi m^{-1}, y+k)+(-1)^{q-1}s_{mn}(x+h, y+k+q\pi n^{-1})+(-1)^{p+q}s_{mn}(x+h+p\pi m^{-1}, y+k+q\pi n^{-1})]$, where p and q are integers. If h_m and k_n are null sequences, then, at points of continuity of $f(x, y)$, $\kappa_{mn}(x, y, p, q, h_m, k_n)$ converges restrictedly to $(1/4)[1+(-1)^{p-1}][1+(-1)^{q-1}]f(x, y)$. For odd values of p and q this sequence exhibits a Gibbs' phenomenon at ordinary discontinuities of $f(x, y)$. If one forms a coupling of $\kappa_{mn}(x, y, p, q, h, k)$ by changing the signs of p, q and h, k, a method of summation is obtained which is effective at points where $f(x, y)$ satisfies more general conditions than continuity. The proofs are obtained by studying the kernels of the transformations. (Received March 12, 1951.)

380. D. H. Hyers and S. M. Ulam: *Approximately convex functions.*

A real-valued function $f(x)$ defined on a convex subset S of Euclidean space E_n will be called *ε-convex* if $f(hx+(1-h)y)\leqq hf(x)+(1-h)f(y)+\epsilon$ for all x and y in S and for $0\leqq h\leqq 1$. Here ϵ is a fixed positive number. The principal theorem is this. If

S is an open convex set (bounded or unbounded) in E_n and if $f(x)$ is ϵ-convex and continuous on S, then there exists a convex function $g(x)$ such that $|f(x)-g(x)| \leqq k\epsilon$, for all x in S, where $2k=1+(n-1)(n+2)/2(n+1)$. The proof is carried out first for polyhedral functions (whose graphs are polyhedral surfaces) which are ϵ-convex and the result is extended to continuous functions by a limiting process. (Received March 5, 1951.)

381. Fritz John: *Solution of parabolic equations by finite differences.*

A differential equation of the form (1) $u_t=F(x, t, u, u_x, u_{xx})$ $(-\infty<x<+\infty)$ with initial values $u(x, 0)=f(x)$ is replaced by the difference equation (2) $\nabla u=F(x, t, u, \Delta u, \Delta^2 u)$, where $\nabla u=[u(x, t+k)-u(x, t)]/k$, $\Delta u=[u(x+h, t)-u(x, t)]/h$, $\Delta^2 u=[u(x-h, t)-2u(x, t)+u(x+h, t)]/h^2$. The functions $f(x)$, $F(x, t, u, p, q)$ shall have continuous derivatives up to the fourth order, which are bounded for bounded t, u, p, q and all x. Moreover F_q shall be positive. It is proved that for given f, F there exists a $T>0$, such that for $0\leqq t\leqq T$ and for all sufficiently small h, k/h^2, the solution of (2) will converge towards a solution of (1) for $h\rightarrow 0$. If (1) is of the special form $u_t=a(x, t)u_{xx}+b(x, t)u_x+c(x, t, u)$ $(a>0)$, the solution of (2) converges for $h\rightarrow 0$ under the less restrictive assumptions that $f(x)$, $c(x, t, u)$ and its first derivatives, $a(x, t)$, $b(x, t)$ and their first and second derivatives are continuous and bounded for bounded t, u. (Received March 12, 1951.)

382. Cornelius Lanczos: *Integration by weighted sums.*

The customary definition of an indefinite integral by means of a sum of equidistant ordinates up to x has generally slow convergence. It is based on the difference equation $\Delta u/\Delta x=f(x)$, letting Δx converge to zero. In this paper the customary procedure is modified by associating with $u(x)$ a new $v(x)$ on the basis of the operational relation $du/dx=\Delta v/\Delta x$. After obtaining $v(x)$ by the customary summation process, one performs the transformation from $v(x)$ to $u(x)$. This transformation amounts to a weighting of the partial sums by pre-assigned weight factors. Numerical integration is thus changed into a simple summation process, augmented by an additional weighting which greatly increases the accuracy obtained. The process can be generalized to linear partial differential operators with constant coefficients. It leads to an increase of the mesh size used in changing the differential equation into a difference equation. Analytically interesting is the fact that sums which do not converge due to too great nearness of a point of infinity of $f(x)$ become convergent after the proper weighting and yield the indefinite integral at every point except the point of infinity. (Received March 9, 1951.)

383. R. B. Leipnik: *Solution of nonhomogeneous linear differential equations with infinitely differentiable coefficients.* Preliminary report.

A formal solution is found for a nonhomogeneous linear differential equation with infinitely differentiable coefficients as a sum of products, one factor of which depends only on the coefficients and their derivatives, the other on the right-hand side only. Convergence is studied. (Received March 14, 1951.)

384. Hans Lewy: *A note on harmonic functions and a hydrodynamical application.*

Let $U(x, y)$ and $V(x, y)$ be conjugate harmonics near the origin for $y<0$ and assume U, V, $\partial U/\partial x$ continuous for $y\leqq 0$. If, for $y=0$, $\partial U/\partial y=A(x, U, V, \partial U/\partial x)$ with A analytic at the argument values occurring on $y=0$, then U and V are analytically extensible across $y=0$. This theorem is proved and applied to the irrotational motion of a homogeneous incompressible liquid subject to gravity. It is shown that in a steady motion relative to a Galilei system the free surface, where the pressure is constant, is necessarily analytic, provided the particle occupying the free surface is not at rest. (Received March 12, 1951.)

385. T. S. Motzkin: *Periodicity regions of sequences with a piecewise linear recursion formula.* Preliminary report.

The iteration of piecewise linear transformations of d-space (or of piecewise analytic transformations of a d-dimensional variety), important i.a. in the theory of the approximative solution of linear (or general) equations and inequalities, may lead to a periodicity behavior essentially different from the usual one (Poincaré, Weyl, F. John) where periodicity either holds for every starting-point or at most for a set of points with measure 0. Already $a_{n+1}=a_n+|a_n|$ (or $(a_n+|a_n|)\exp(-a_n^{-2})$ with piecewise analyticity and differentiability of all orders), $d=1$, has a d-dimensional region of periodicity and of aperiodicity. A more intricate situation holds for $a_{n+3}+a_n=\max(a_{n+1}, a_{n+2})$, $d=3$. To decide on periodicity and to find the length p of the period, proceed until $a_{k-1}\leqq a_k\geqq a_{k+1}$, $a_k\geqq 0$, which will occur for some $k=1, \cdots, 9$, and let $r=(2a_k-a_{k-1}-a_{k+1})/(a_{k-1}+a_{k+1}-a_k)$. Then if r is irrational and >0, there is no period; if $r=0/0$, $p=1$; if $r=-2$, $p=2$; if $r<-2$, $r=\infty$ or $r=0$, $p=8$; if $r=1$, $p=9$; if r or $1/r=1+s/t$ (s and t coprime and >0), $p=8s+18t$. Every even number, with 36 exceptions (4, 6, 10, $\cdots$, 136, 180, 228), can be a period. The sequence is always bounded and attains its upper bound infinitely many times. Some of these results can be generalized to the sequence $a_{n+d}+a_n=\max(a_{n+1}, \cdots, a_{n+d-1})$. (Received April 24, 1951.)

386. M. L. Stein: *On methods for obtaining solutions of fixed end point problems in the calculus of variations.*

Two methods are proposed for constructing solutions to the problem of minimizing an integral in a certain class of arcs joining a pair of fixed points. For both methods an initial estimate of the solution is made. From this estimate a new one is obtained in a definite way, and so forth. One of the proposed procedures is a generalization of Newton's method. It is shown to converge to a solution of the Euler equations associated with the integral to be minimized and (under the proper conditions) to yield a strong relative minimum. The second of the proposed procedures is a "gradient" method. It is shown to be particularly applicable to an integral whose integrand is quadratic. Conditions for convergence to a strong relative minimum are given. (Received March 4, 1951.)

387. F. G. Tricomi: *A new entire function related to a well known noncontinuable power series.*

In the study of the statistical distribution of certain bacteria mutants there arises (see Bull. Amer. Math. Soc. Abstract 57-4-394) a new entire function $G(t)=\sum_{n=1}^{\infty} t^n/[(a^n-1)n!]$ in the case $a=2$. This function is related to the well known noncontinuable power series $\sum_{n=0}^{\infty} z^{a^n}$. Precisely, if one puts $z=e^t$, $F_m(t)=\sum_{n=0}^{m-1}\exp(a^n t)$, then $F_m(t)=m+G(a^m t)-G(t)$. Among other things, the function G is connected to the quite strange doubly periodic function $P(u)=\sum_{m=-\infty}^{\infty} f(u+m)$ where $f(t)=\exp(t/A-e^{t/A})$, $A=1/\log a$, a function which exists in only half of each period parallelogram. Furthermore this function is *almost constant* on the real axis as long as log a is not too large. For instance in the case $a=2$, one has $|P(x)-A| < 0.0000143$. This inequality can be deduced from the elegant Fourier expansion of the function $P(u)=A\sum_{n=-\infty}^{\infty}\Gamma(1-2nA\pi i)\exp(2nu\pi i)$. Moreover the functions $G(u)$ and $P(u)$ are related also to the *principal solution* of the difference equation $\Delta\phi(u)=f(u)$, where f has the previous expression. Actually the function $G(t)$ is an entire function of order *one* which attains any finite value in an infinity of points of the complex t-plane. On the real axis $G(t)$ is *totally monotone* in the sense that all its derivatives are positive there. But, while $G(t)$ grows like $e^{t/a}$ for $t>0$, $-G(t)$ grows logarithmically as $t\rightarrow-\infty$. This fact is connected with the extreme "skewness" of the frequency curve of the bacteria mutants. (Received March 12, 1951.)

388. František Wolf: *Analytic perturbations of operators.*

If (i) $A(\lambda)=\sum_0^\infty A_k\lambda^k$, for $|\lambda|<\delta$, a bounded operator in Hilbert space is normal for $\lambda=\lambda_n$, lim $\lambda_n=0$, (ii) μ_0 is an isolated point of $\sigma(A_0)$, the spectrum of A_0, (iii) $A_1-\mu_1I$ is completely continuous, then there is a sequence of analytic functions $\mu_k(\lambda)$ and a real function $\epsilon(\lambda)$ such that lim $\epsilon(\lambda)=0$ and $\mu\in\sigma(A(\lambda))$ in a certain neighborhood of μ_0 implies either $|\mu-\mu_0-\mu_1\lambda|<\lambda\in(\lambda)$ or μ equal to one of the numbers $\mu_k(\lambda)$. The domains of analyticity D_k of $\mu_k(\lambda)$ might be such that the intersection of the D_k's may be the single point $\lambda=0$. If A_1 does not have μ_1 as an eigenvalue, then the conclusion is "then there exists a sequence of analytic functions $\mu_k(\lambda)$ such that $\mu\in\sigma(A(\lambda))$ near μ_0 implies $\mu=\mu_k(\lambda)$ for some k." This theorem is a generalization of a theorem of F. Rellich (Math. Ann. vol. 113 (1936) pp. 600–619) who restricts the multiplicity of μ_0 as an eigenvalue of A_0 to be finite, but omits condition (iii). He shows that without any additional condition the finiteness of the multiplicity is essential. The author was able to deduce the theorem from a result which covers the asymptotic behavior of $\sigma(A(\lambda))$ near μ_0 for sufficiently small λ in the case of bounded operators in Banach space (cf. abstract at the International Congress of Mathematicians, Cambridge, Mass., 1950). (Received March 25, 1951.)

389. Kosaku Yosida and Edwin Hewitt: *Finitely additive measures.*

Let $\mathcal{M}$ be an algebra of subsets of a set T. A real-valued finitely additive measure ϕ defined on $\mathcal{M}$ is said to be purely finitely additive if for every countably additive measure γ such that $0\leqq\gamma\leqq\phi^+$ or $0\leqq\gamma\leqq\phi^-$, one has $\gamma=0$. It is proved that every finitely additive measure ϕ defined on $\mathcal{M}$ admits a unique resolution into a purely finitely additive part and a countably additive part. If $\mathcal{M}$ is a σ-algebra and a certain σ-ideal $\mathcal{N}\subset\mathcal{M}$ is designated, then the set of all finitely additive real-valued measures which vanish on $\mathcal{N}$ is a representation of the conjugate space to the Banach space $\mathfrak{L}_\infty$ defined for $T,\mathcal{M}$, and $\mathcal{N}$. A number of peculiar finitely additive measures are constructed, using this fact, for the case $T=R$, $\mathcal{M}=$Lebesgue measurable sets, $\mathcal{N}=$set of Lebesgue measure 0. It is also shown that for general T, $\mathcal{M}$, and $\mathcal{N}$, all finitely additive measures have counterparts which are countably additive Borel measures on a suitable compact Hausdorff space. The properties of pure finite additivity and countable additivity are identified in terms of the behavior of these countably additive Borel measures. (Received March 12, 1951.)

APPLIED MATHEMATICS

390. E. A. Davis: *A dynamic economic model.* Preliminary report.

This paper is concerned with the construction of a mathematical model for an economic system. It uses techniques developed by Griffith C. Evans (Econometrica vol. 2 (1934) pp. 37–50), and F. W. Dresch (Bull. Amer. Math. Soc. vol. 44 (1938) pp. 134–141). The model deals with operation of industries over an extended time interval. Each industry is considered to produce either a capital good, a consumer good, or a direct factor of production. It is presumed that capital and consumer good industries fix production rates so as to maximize integrals that define profits under the assumption that output is sold at the rate at which it is produced. In the maximization, input rates for productive factors and period of production are supposed subject to enterpriser control. The existence of non-constant inventory stocks is considered. A banking system is admitted and some investigation of monetary flows is made. An expression for the velocity of circulation of money results. Finally, index quantities (including indices for "period of production") are defined and relations existing among the index quantities are found to be generalizations of fundamental relations holding in a three industry model of the type considered. (Received March 15, 1951.)

391. G. E. Forsythe: *Generation and testing of* 1,217,370 *"random" binary digits on the SWAC.* Preliminary report.

Let $f_i(k)=0$ or 1 ($k=1, \cdots, n_i$; $i=1, \cdots, r$), where $n_1, \cdots, n_r$ are relatively prime. Let each $f_i(k)$ be extended periodically with period n_i over the domain $k=1, 2, \cdots$. The sequence $F_k=\sum_{i=1}^{r} f_i(k)$ (modulo 2) ($k=1, 2, \cdots$) has period at most $N=n_1n_2 \cdots n_r$. J. B. Rosser (unpublished) proposed $F_1, F_2, \cdots, F_{\theta N}$ ($\theta \ll 1$) as possible "random" digits for sampling problems. Let $n=n_1+\cdots+n_r$. Let $\mathfrak{f}$ [$\mathfrak{F}$] be the space of all n-uples f [all N-uples F] of independent binary digits. With each f one may associate functions $\{f_i(k)\}$, and hence may determine a unique $F(f)$ by Rosser's process. Let $Z(F)$ be the number of 1's in F. For $m=0, 1, 2, 3$ the present author shows that the mth moment of $Z(F)$ in $\mathfrak{F}$ equals the mth moment of $Z[F(f)]$ in $\mathfrak{f}$. For $r=4$, $n_1=35$, $n_2=34$, $n_3=33$, $n_4=31$, $N=1,217,370$, and for one choice of f, the corresponding N-uple $F(f)$ was generated on the National Bureau of Standards Western Automatic Computer (SWAC). Also on the SWAC, all sums s_j of 100 consecutive digits F_k were formed, and distributions were made of the s_j in twelve disjoint groups of 1000 consecutive sums s_j. The χ^2 test showed 11 of the 12 distributions to fit reasonably well the theoretical (binomial) distribution of 1000 s_j. The twelfth was a bad fit: $\chi^2=47.24$ for 23 degrees of freedom ($P \ll .01$). (Received March 12, 1951.)

392t. G. E. Forsythe and T. S. Motzkin: *Acceleration of the optimum gradient method.* Preliminary report.

In Cauchy's optimum gradient method for determining the solution x^* of the system $Ax=b$ ($|A| \neq 0$) of n linear equations in n unknowns, one lets $F(x)=|Ax-b|^2$ and successively determines the $x_{k+1}=x_k-\alpha \operatorname{grad} F(x_k)$ which minimizes $F[x_{k+1}(\alpha)]$. For $n=3$ the present authors have proved [Bull. Amer. Math. Soc. Abstract 57-3-231] and for $n \geqq 3$ now conjecture that x_k-x^* is asymptotically a linear combination of the eigenvectors belonging to the largest and least eigenvalues of A^TA ($A^T=$ transpose of A). (If there are eigenvectors orthogonal to x_0, disregard the corresponding eigenvalues.) When this asymptotic relationship holds for a given x_0, the sequence $\{(x_k-x^*)/|x_k-x^*|\}$ is asymptotically of period 2; then the usually slow convergence of x_k to x^* can be accelerated by occasionally determining the $x_{k+1}=\beta x_k+(1-\beta)x_{k-2}$ which minimizes $F[x_{k+1}(\beta)]$. Numerical experiments (on desk machines and I.B.M. card-programmed calculator) with a diagonal matrix ($n=36$) and a general matrix ($n=6$) confirm the conjecture and show the efficiency of the acceleration procedure. Compared to some other accelerations of iterative processes, this method has the advantages for automatic digital computation of being easily programmed and seeming to converge reasonably well. The accelerating shortcut can also be applied to the minimization of any well-behaved real-valued function F of the vector x; cf. the above reference. (Received March 12, 1951.)

STATISTICS AND PROBABILITY

393. Z. W. Birnbaum and F. H. Tingey: *One-sided confidence contours for probability distribution functions.*

Let X be a random variable with a continuous probability distribution function $F(x) = \text{Prob.}\ \{X < x\}$. An ordered sample $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ of X defines the "empirical distribution function": $F_n(x) = 0$ for $x < X_1$, $F_n(x) = k/n$ for $X_k \leqq x < X_{k+1}$, $k = 1, 2, \cdots, n-1$, $F_n(x) = 1$ for $X_n \leqq x$; the function $F_{n,\epsilon}(x) = \min \{F_n(x) + \epsilon, 1\}$ will be called "upper confidence contour." It is known that the probability $P_n(\epsilon) = \text{Prob.}\ \{F(x) \leqq F_{n,\epsilon}(x) \text{ for all } x\}$ is independent of $F(x)$. An expression for $P_n(\epsilon)$ was given by A. Wald and J. Wolfowitz in determinant form (Ann. Math. Statist. vol. 10 (1939) pp. 105–118), and an asymptotic expression for $n \to \infty$ is due to W. Feller (Ann. Math. Statist. vol. 19 (1948) pp. 177–189). In the present paper it is shown that $P_n(\epsilon) = 1 - \epsilon \sum_{j=0}^{[n(1-\epsilon)]} \cdot C_{n,j}(1-\epsilon-j/n)^{n-j}(\epsilon+j/n)^{j-1}$ for $0 \leqq \epsilon \leqq 1$. Furthermore, numerical solutions of the equation $P_n(\epsilon) = 1 - \alpha$ are tabulated for $n = 5, 8, 10, 20, 40, 50$ and $\alpha = .10, .05, .01, .001$, and it is observed that for $n = 50$ these solutions agree very closely with those obtained from Feller's asymptotic expression for $P_n(\epsilon)$. (Received March 15, 1951.)

394. F. G. Tricomi: *On the statistical distribution of bacteria mutants.*

The study of the statistical distribution of bacteria which, thanks to a suitable *mutation*, become resistant to some antibiotica or other killing agents, leads to a new problem of calculus of probability, which can be considered as a far-reaching generalization of that leading to the well known Poisson distribution. In fact, under a certain hypothesis, the obtained distribution differs formally only slightly from that of Poisson. This hypothesis, however, is *not* satisfied in the bacteriological case which requires a very accurate treatment. In this case the author found the following (approximate) parametric representation of the frequency curve: $x = G(t)$, $y = \alpha^{1/2}[2\pi G'(t)]^{-1/2} \exp \{\alpha[G(t) - tG^*(t)]\}$. Here $\alpha > 0$ is a parameter of the problem, $G(t)$ a new entire function defined by the expansion $G(t) = \sum_{n=1}^{\infty} t^n/[(2^n - 1)n!]$, $G'(t)$ its derivative and $G^*(t)$ its integral over $(0, t)$. If α is small with respect to 1 (which is generally the case) the frequency curve has only a maximum at about $A \log (2A\alpha)$ where $A = \log_2 e = 1.44270$ and descends exceedingly quickly *on the left* of it. Furthermore the first moments of the distribution can be (rigorously) evaluated in any case, as well as a recurrence formula for the relevant probabilities. The new function $G(t)$ and some related ones are studied in a further paper of the author. (See Bull. Amer. Math. Soc. Abstract 57-4-387.) (Received March 12, 1951.)

TOPOLOGY

395*t*. B. H. Arnold: *Topologies defined by bounded sets.*

Let S be a real vector space and $\mathfrak{B}$ a subset of 2^S. The elements of $\mathfrak{B}$ are called bounded subsets of S and are subjected to six intuitively satisfactory axioms. Example: The union of two bounded sets is bounded. These axioms completely determine the finite-dimensional bounded subsets of S as being the usual ones. Definition: A subset G of S is open if for each g in G, and B in $\mathfrak{B}$, there is a $\lambda>0$ such that $g+\lambda B\subset G$. This definition makes S a T_1 space; it may fail to be a Hausdorff space, and addition may fail to be continuous, but the cases where S is a normed linear space are easily characterized. If $\mathfrak{T}$ is the collection of all subsets of S which are topologically bounded, then $\mathfrak{T}\supset\mathfrak{B}$; the set $\mathfrak{T}$ is characterized in terms of $\mathfrak{B}$. The set $\mathfrak{T}$ satisfies the six basic axioms on bounded sets and the topologies in S generated by $\mathfrak{T}$ and $\mathfrak{B}$ are identical. Convergence in S is discussed and several examples are given, for example, a convergent directed system such that no cofinal subsystem is a bounded set. (Received March 12, 1951.)

396. E. G. Straus and F. A. Valentine: *A characterization of convex sets.*

Let S be a continuum in an n-dimensional Euclidean space R_n. *If S has the property that each point in S belongs to a unique maximal convex subset of S of dimension $\geqq n-1$, then S is convex.* This result, together with an obvious converse, yields a characterization of convex bodies in R_n. As an interesting corollary in R_2, we have: If S is a continuum in R_2 each point of which belongs to a unique maximal line segment of S, then S is a straight line segment (a segment contains at least two points). (Received March 15, 1951.)

397. R. B. Talmadge: *The representation of Baire functions.*

Let S be any topological space, $B(S)$ the set of real-valued Baire functions of class one on S. It is shown that *if S is normal, then $B(S)$ is isomorphic to the lattice of continuous functions on the Boolean space associated with the Boolean algebra of ambiguous sets of Borel class one.* Similar results are obtained for functions of class α in case S is a metric space. (Received March 9, 1951.)

398. P. A. White: *Extensions of the Jordan-Brouwer separation theorem and its converse.*

It is proved that if K is an $(n-1)$-generalized closed manifold ($n-1$-gcm) (see R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 32, 1949) (not necessarily connected) contained in the connected orientable n-gcm S, such that $S-K=A\cup B$ separate, then $\bar{A}$ and $\bar{B}$ are generalized manifolds with boundaries each consisting of some of the components of K. It is proved that if $A\subset S$ is open, uniformly-r-locally connected, $0\leqq r\leqq n-1$, and has an $(n-1)$-dimensional boundary, where S is a connected orientable n-gm, then $\bar{A}$ is an n-generalized manifold with boundary. This generalizes a similar result of Wilder where the additional hypothesis that the $(n-1)$-dimensional betti number of $S=0$ was included. To eliminate this hypothesis the following generalization of the Alexander duality theorem was proved: If K is a closed subset of the connected orientable n-gcm S such that $S-K$ has m components (m may be infinite), then $m-1\leqq p^{n-1}(K)\leqq(m-1)+p^{n-1}(S)$. (Received March 5, 1951.)

J. W. GREEN,
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JOSEPH FELS RITT

All Columbia College mathematics students came under the influence and scrutiny of J. F. Ritt in his introductory course on the solution of differential equations. To many teachers this subject presents no particular fascination; we appear to have here a not too well connected collection of facts and methods whose presentation would not at first glance seem very enlivening. But Ritt gave this material a characteristic glow. In all branches of mathematics he was an avid student of the past; in the field of differential equations he was a particularly distinguished authority. In his classes his own mind recreated the greatness of a golden period of our subject with special emphasis on the contributions of the French school, ever dear to him. It was in this course that the writer first met Professor Ritt. In the ensuing twenty-four years, my relations with him were constant as a younger colleague and friend. His recent death has terminated a career of outstanding distinction. His scientific achievements in the contemporary mathematical scene are truly impressive. A necessarily brief summary of his life work is given in the later pages of this article. It should be recorded here that on numerous occasions Ritt expressed the belief that a man's work should speak for him and that vitae or obituaries were artificial and superfluous commentaries on the ebb and flow of history.[1]

Ritt was born on August 23, 1893 in New York City. His early education in the city included two years at the College of the City of New York. He obtained the degree of Bachelor of Arts from George Washington University in 1913. The same institution honored him with the bestowal of the degree of Doctor of Science in 1932. After some years as a graduate student at Columbia University, where he came under the influence of Professors Fite and Kasner, he obtained in 1917 the Ph.D. for a work which is fresh even now. Ritt referred frequently and with affectionate pride to his dissertation. Except for two years spent as a Master Computer in the Ordnance Department, (1918–1919), he spent the remainder of his life at Columbia. In 1928 he married Miss Estelle Fine who for over twenty years was her husband's devoted companion. Together they labored at the preparation of his many manuscripts, they relaxed in travel to various countries and around the world, and they shared every success which came to him. The end came suddenly and without warning to his friends, on January 5, 1951, at a time when he was engaged in some of the most important work of his life.

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[1] In the preparation of the manuscript valuable help was received from many friends. Mrs. Ritt very kindly checked the bibliography. The members of the department of mathematics at Columbia as well as E. L. Post and B. P. Gill made valuable suggestions. Two paragraphs on some deeper aspects of the work in differential equations (low-power theorem, difference equations) are due to Ellis Kolchin.

Ritt had an enviable reputation as a teacher. His lectures were etched on the minds of his hearers who carried away a feeling of masterfulness. His wit was out of the ordinary. Both in a completely impromptu sally and in studied and carefully polished excursions, the unexpectedness and sure impact of his phrases were unmatched. His colleagues referred to them as ritticisms. In the writing of his papers he was capable of devoting weeks of worry to the proper balance of a single thought occurring in one paragraph. This will reveal the dogged tenacity of his mind—at once a strong quality in his own work, and yet not without elements of weakness. His memory was astonishing. Being much taken with the tempo of Mr. Churchill's prose, he memorized in a single reading whole sections of speeches made by the latter during the dark days of the war, and would regale his hearers months and years later with their recital. Although thus endowed, he was definitely not encyclopedic in mathematics. His wise comment on this was: "A man will know as much as he needs." He was an amateur of languages although he did not derive social enjoyment from prolonged conversation in other idioms. For example, he learned to read mathematical Russian before that became fashionable.

He laid the greatest emphasis on the question of style in mathematical papers. His ideas had been developed from intensive study of the writings of the masters of the eighteenth and nineteenth centuries. Here he found a distinction which one does not readily discover in the desiccated writing of our own day,—carried on in the shadow of a hawk-like editor who parsimoniously counts the pages to his petitioner. All of Ritt's books and papers were written with the greatest stylistic care. He did not consider his contributions as being local in time but rather definitive; for that reason he wished them to be in final form. At times the striving for smoothness of pace and ease of ascent may impress the reader with technical craftsmanship rather than well-balanced and intuitive artistry. The fact is that both are present in all of his writing.

The excellence of his work was recognized early, by his University, by the professional society of which he was a member, and by the world at large. At the age of 39 he was Colloquium Lecturer of the Society. Shortly after, he was elected to membership in the National Academy of Sciences. Two of his books appear in the Colloquium Publications, an unusual honor. He was vice-president of the

Society from 1938 to 1940. Other elected positions he held in the Society were: Trustee, 1923–1924; member of the Council, 1926–1928; representative on the Board of Editors of the American Journal of Mathematics, 1936–1940; and member of the Colloquium Editorial Committee, 1943–1948. His works received acclaim in Europe. For example, Bieberbach in his invited address to the 1932 International Congress in Zürich devoted one-half of his lecture to a discussion of his writings. At home he had a score of young students of distinction. During the last decade he occupied the Davies Chair of Mathematics. At the 1950 International Congress in Cambridge, he gave an invited address on his latest work in which he developed a Lie theory of differential groups. Only one mark of esteem escaped him. He was never the recipient of one of our great prizes. There can be no doubt of the fact that this was a source of more than disappointment to him. As one views his writings both as a young man and later as a mature scientist, one is compelled to admit with generosity his claim to fame. That he failed on successive occasions to receive this particular symbol of greatness suggests a weakness in our social organization. It is true that our field of endeavor is a blessed paradise compared to certain others in this matter. But perhaps our situation is not ideal. Is it not possible that we have too few prizes? or perhaps too many?

From acquaintance with the grown man, one would surmise that the boy was shy, probably excessively so. As his life was governed by a strict and occasionally inflexible code, his sensitive spirit was never reconciled to the chaotic organism which is society. During the early years, the growing boy became conscious of his own genius and with this developed a tremendously strong feeling of independence. The need for collaboration and association with a scientific school of his day was not present since all alone he could evaluate, plan, and execute. In order to satiate the unquenchable universal need for companionship, he turned to the great of all time. Here one could find the noblest inspiration as well as freedom from strife and uncertainty. The fascination of the grandeur of bygone days had an immeasurable influence on him. His heroes were Lagrange, Laplace, Abel, Jacobi, Liouville, Cauchy, Poincaré, and Hilbert. From them he derived standards of excellence; of them he hoped to be worthy. His study of mathematics was through the man. This is the normal approach in the arts and letters but is much less usual in the sciences. Most of us appreciate the man through a preliminary mastery of the subject.

His own inclinations and abilities, and the moulding forces in his personality which have been adumbrated above, channelled his creative energies into the field of classicism. His media are complex function theory of the nineteenth century and differential equations. Much of his work could have been written a half century earlier. Instinctively he sought for problems to solve. His papers and numerous conversations underline this constant search for problems of importance in a completely established domain beyond the shadow of controversy. The "modern" exploratory spirit in algebra, topology, or linear spaces was not for him. He did not understand it nor did he wish to. The Lebesgue integral he begrudgingly recognized. But fields of characteristic p were special objects of scorn. That all of classic number theory was bound up with them he preferred not to contemplate. Yet his own instincts were so sure that only once did this disregard for new developments conduce to an imperfection in his written work. In his chosen specialties he exhibited unmatched virtuosity. The problems were so natural that one marvelled that they had not received attention before. The solution was carried through like the traditionally perfect surgical operation. Yet with this distrust of the modern, and with this emphasis on classical analysis, it is nevertheless a fact which is borne out by his lifelong work that Ritt was fascinated by the elemental idea from which springs all of contemporary algebra—the notion of structure. Some of his very best papers lead to a unique factorization theorem, the very essence of rich algebra. Details will make this clear shortly. A favorite operation which comes under his scrutiny is that of composition of functions (forming $f(g(z))$ from $f(z)$ and $g(z)$) which gives an associative "multiplication." Finally, in algebraic differential equations the theory of the manifold of solutions is described in terms of intersection of prime ideals.

The dissertation [5][2] was presented to the Society in April 1916. Here are considered the properties of the linear homogeneous differential operator of infinite order with constant coefficients: $A = \prod_1^\infty (1-D/a_i)$ where $\sum_1^\infty |a_i|^{-1} < \infty$ and D represents d/dz. Application of A to an analytic function $f(z)$ yields again such a function. Any solution of $A\Phi(z)=0$ which is analytic on the circumference of a circle is analytic throughout the interior of the circle, hence no solution may have an isolated singularity. These two results will give an idea of the material developed.

After a brief period of orientation come two brilliant papers *Prime and composite polynomials* [14] and *On algebraic functions which can be expressed in terms of radicals* [15]. In the first, a polynomial $F(z)$ is called composite if there exist polynomials $\phi_1(z)$ and $\phi_2(z)$ of degree higher than one such that $F(z)=\phi_1(\phi_2(z))$. Thus we are considering the problem of factoring polynomials into primes—$F=\phi_1\phi_2 \cdots \phi_s$. It is proved that any two decompositions contain the same number of polynomials and that the degrees are the same in both decompositions. Essentially distinct decompositions are completely analyzed. In the second, Ritt determines all algebraic functions w defined by a polynomial $F(w, z)=0$ which can be expressed in terms of z by means of radicals. Here is certainly a beautiful problem in the most classic tradition! As the author put it: "Why the results obtained here should not have been found before this time is a question which has puzzled us as much as it will puzzle the reader." For the case of genus zero, the rational functions whose inverse can be expressed in terms of radicals can be obtained from the functions z^n; $f_n(z)$ where $\cos nu=f_n(\cos u)$; the rational functions which occur in the formulas for the transformations of the periods of the function $\wp(u)$; and certain other functions associated with elliptic functions in the lemniscatic ($g_3=0$) and the equianharmonic ($g_2=0$) cases. In both of these papers we have masterful treatment of monodromy groups of Riemann surfaces, in particular of those associated with trigonometric polynomials and elliptic functions. The same techniques are used in one of Ritt's most important papers *Permutable rational functions* [18]. The problem is to find conditions under which rational functions $\phi(z)$ and $\psi(z)$ satisfy $\phi(\psi(z))=\psi(\phi(z))$. This paper finds Ritt competing with some of his contemporaries in the solution of a specific problem for probably the only time in his scientific life. Julia and Fatou in France had obtained preliminary results in the direction of a solution, but Ritt's results, arrived at by completely different methods, are much more definitive. This paper marks the high point of the first period in his career. The particular problem with which it deals was probably more in the public eye than those which follow it. After this publication it was clearly evident that a master had stepped forth.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

We turn now to his contributions in the theory of elementary functions. The field was essentially in the state in which Liouville had left it eighty-five years before. Ritt's great tour de force here was the determination of the structure of the elementary functions $y=f(x)$ whose inverses $x=g(y)$ are also elementary. He showed in a tough paper [22] that they are precisely the functions $f(x)$ for which a "factorization" exists: $f=\phi_1\phi_2 \cdots \phi_s$, where the functions $\phi_i(x)$ are either $e(x)$, $\log x$, or an algebraic function of x. Any such function $f(x)$ evidently has an elementary inverse; note for example that the function $x+\log x$ does not have an elementary inverse. He showed as well that the elliptic integrals $w=g(z)$ do not satisfy an elementary equation in two variables, $F(w, z)=0$, thus improving on a result of Liouville who had demonstrated that the integrals themselves were not elementary, and proving that the elliptic functions are not elementary. We note here once more the fascination with the notion of factorization. The methods of attack based on a classification of the elementary functions are due to Liouville. One can trace the influence of this classification several years later in the totally unconnected field of algebraic differential equations. In the latter, the coming to grips with the problem is based on a very ingenious classification of rank of differential polynomial. Ritt offered a course of lectures on elementary functions at frequent intervals. In 1947 he wrote a short monograph on the subject [C] making available for the first time in book form the salient points of this theory. It is strange to think of writing a book on a branch of mathematics which almost certainly will be the definitive work for the next fifty to seventy-five years. Yet this seems to be the case here. It is probable, nevertheless, that Ritt overemphasized the importance of this subject. It is true, as he contended, that younger students deserve to have the answer to various questions which arise in the calculus; but the elaboration of this answer contributes little to the opening of vistas into the domain of vibrantly living mathematics.

In the next few years most of the papers deal with exponential polynomials. The problem of factoring (once more!) the expression $\sum_1^n a_i e(\alpha_i z)$ into irreducible exponential polynomials is completely solved [31]. In another paper it is shown that if every zero of one exponential polynomial is a zero of a second polynomial, the quotient of the second by the first is an exponential polynomial [35]. In a third paper of this series it is established that if w is a uniform function satisfying $\sum_0^n P_i w^i = 0$ where the P_i are exponential polynomials, then w is the quotient of two exponential polynomials [34].

We come now to the outstanding work in Ritt's career, to which he devoted almost twenty years of his life and for which above all else he will be remembered decades hence—his creation of a theory of ordinary and partial algebraic differential equations. The basic theory appears full-fledged in his paper *Manifolds of functions defined by systems of algebraic differential equations* [37]. In the score of years that follow, there is a flow of some twenty papers on the subject, from his own pen as well as a large number from his students. Every aspect of the theory is studied in detail. It is impossible for several reasons to give here an extensive review of the material covered. The most important portions of it will be found in his two Colloquium Publications, volumes 14 and 33 [A, D]. For a preliminary orientation, one might recommend chapter I of volume 33 as well as an address delivered at the semi-centennial celebration of the Society, *Algebraic aspects of the theory of differential equations* [49]. We shall make a few remarks concerning the situation under scrutiny. Ritt considers differential polynomials or forms in the unknown functions $y_1, \cdots, y_n$ and their derivatives with coefficients which are functions meromorphic in some domain. Given a system Σ of such forms, he shows that there exists a finite subsystem of Σ whose manifold of solutions is identical with that of Σ. Furthermore, if a form G vanishes for every solution of the system of forms $H_1, \cdots, H_r$, then some power of G is a linear combination of the H_i and their derivatives. Define a basis of Σ to be a finite subset Φ of Σ such that every G in Σ has the property that for some positive integer p, G^p is a linear combination of the forms in Φ and their derivatives. It then follows from the above that every infinite system of forms has a basis.

Next, Ritt considers the question of reducibility. A system Σ is said to be reducible if there exist two forms G and H such that every solution of Σ is a solution of GH while some solutions of Σ are not solutions of G, and others are not solutions of H. There follows then the fundamental theorem that every system Σ is equivalent to a finite set of irreducible systems. Let $\{\Sigma\}$ represent the perfect differential ideal generated by Σ. Let $\mathfrak{M}$ represent the manifold of solutions of Σ and hence of $\{\Sigma\}$. Suppose the essential irreducible systems to which Σ is equivalent are $\Sigma_1, \cdots, \Sigma_p$; suppose $\mathfrak{M}_i$ is the manifold of Σ_i. Then clearly $\{\Sigma_i\}$ is a prime ideal. Also, we have $\mathfrak{M}=\mathfrak{M}_1\cup \cdots \cup\mathfrak{M}_p$ by the above fundamental theorem. But this statement is precisely equivalent to the statement $\{\Sigma\} = \{\Sigma_1\}\cap \cdots \cap\{\Sigma_p\}$. This is the theorem on the unique factorization into primes of a given perfect ideal. The analogy to systems of polynomials in the theory of ordinary polynomials will now be clearer. One should keep in mind however that the introduction of differentiation introduces entirely new phenomena and thus the present theory is not a "mere" generalization of polynomial ideal theory.

Ritt defines next the notions of a general solution of a single equation $A=0$. One may advance the thesis that all his work on differential equations owes its origin to the very unsatisfactory state of the nature of general and singular solutions in the literature at large and more specifically in the writings of Laplace, Lagrange, and Poisson which were most carefully examined by Ritt. Let A be algebraically irreducible and suppose A involves some particular y_j.

Let S_j be the partial derivative of A with respect to the highest derivative of y_j occurring in A. Let $\{A\} = \{\Sigma_1\} \cap \cdots \cap \{\Sigma_p\}$ be the factorization of $\{A\}$ into prime ideals. Then there is precisely one $\{\Sigma_i\}$ whose solutions do not cause all S_j to vanish. Furthermore this $\{\Sigma_i\}$ is independent of j. The manifold of $\{\Sigma_i\}$ is called the general solution of $A=0$. The remaining manifolds determined by the $\{\Sigma_k\}$, $k \neq i$, constitute the singular manifolds.

Thus all the pivotal results were obtained by him in the early stages of his program. However, the definitive ideal-theoretic formulation of the theory (intersection theorem) is due to Raudenbush, one of his students, who discovered it shortly after completing his dissertation under him. It is a tribute to Ritt's perfectionism that he considered that he had failed in not perceiving himself the primary role played by ideal theory in his problem. Yet failure it most certainly was not. Rather is was a poignant experience that can arise only in a genuinely rich scientific career.

A second complex of results in his theory was secured by him in papers dating from 1936 to 1945. He proved that every irreducible component of the manifold of a differential polynomial A is the general solution of some irreducible differential polynomial B. Furthermore he gave a simple algorithm (the "low power theorem") by which one can decide, given two differential polynomials A and B, when the general solution of B is a component of the manifold of A. Also, he attempted to discover an algorithm for deciding when a given solution of A belongs to the general solution of A; this is evidently a very difficult question. For the case in which A is an ordinary differential polynomial in one unknown and is of order $\leqq 2$, he gave a complete answer, albeit not a simple one. These and certain other results were all established by the construction of certain formal power series solutions (the "polygon process"), and seem to lie deeper than the earlier part of the theory.

Shortly after he launched his theory of algebraic differential equations he perceived the possibility of an analogous theory for algebraic difference equations. With the occasional collaboration of some of his students he succeeded in establishing such a theory. Actually this theory deals with equations more general than difference equations. For example it handles substitutional equations $P(x, y(x), y(\phi(x)), \cdots, y(\phi(\cdots \phi(x) \cdots))) = 0$, where P is a polynomial in $y(x), y(\phi(x)), \cdots, y(\phi(\cdots \phi(x) \cdots))$, and $\phi(x)$ is a given function, or more generally equations $P(y, \sigma y, \cdots, \sigma^n y) = 0$, where P is a polynomial with coefficients in a field K and σ is an endomorphism of K.

The last phase of his writing began three years ago. Ritt came upon problems which generalize the Lie theory. Consider an infinite series $P_1+P_2+\cdots$ where the P_i are homogeneous differential polynomials of degree i in the indeterminates $y_1, \cdots, y_p$ and their derivatives with coefficients in a differential field. (Derivatives are written y_{ij}, or in case $p=1$ as y_j.) One may make a substitution by replacing each y_i by an expression of the above type in the new indeterminates $y_1', \cdots, y_p'$. Specifically, consider two sets of indeterminates u_i, v_i, $i=1, \cdots, n$, and n series of this type $A_i(u_i, \cdots, u_n; v_i, \cdots, v_n)$, $i=1, \cdots, n$. Write the above for short as $A(u, v)$. The equation $A[A(u, v), w]=A[u, A(v, w)]$ states that A is an associative operation. Consider as an example $A(u, v)=u+v$. A differential group is a set of power series $z_i=A_i(u, v)$ such that each A_i starts with the polynomial u_i+v_i; $z_i=u_i$ if $v=0$ and $z_i=v_i$ if $u=0$; the $A_i(u, v)$ have the associative property described above.

In his first paper [63], Ritt determines all groups of order 1, that is, those for which $n=1$. They are the additive group $z=u+v$ and the group of substitutions. The latter may be made more explicit as follows: If $z(x)=u(x)+v[x+u(x)]$, then the usual Taylor series gives $z=u+v+\sum_{n=1}^{\infty} v_n u^n/n!$; if one sets $z'(x)=x+z(x)$, $u'(x)=x+u(x)$, $v'(x)=x+v(x)$, then $z'(x)=v'[u'(x)]$. Thus at this point we already see a difference between these results and those of the classical Lie theory. The case of groups of order 2 is even more revealing. In his invited address to the 1950 International Congress in Cambridge, Ritt described thirteen distinct types of groups of this order. In another paper [64], he gives identities satisfied by the structure constants of differential groups which resemble those given in Lie's third fundamental theorem. Other results not known to the writer are still on the printer's desk, awaiting publication at this very moment. The theory had just entered upon maturity when its creator laid down his pen. This and the older work is our inheritance.

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BOOK REVIEWS

Operational calculus. By B. van der Pol and H. Bremmer. Cambridge University Press, 1950. 14+415, pp. $10.00.

Any new book on the Heaviside calculus or on the Laplace transformation must prove its worth in competition with a large and established literature containing such favorites as Titchmarsh, Doetsch, and Widder for the mathematician, or McLachlan and Churchill for the engineer and scientist. Even in this heavy competition the book under review is likely to succeed, for it offers some points of novelty of considerable interest, its choice of material and style of exposition is "best possible" for a certain class of readers, and it is very well written.

Some novel features of the book will be noted later, but it must be mentioned already here that the style of the exposition is likely to become the greatest asset of this book, and that it is somewhat of a novelty—at any rate as far as literature in English is concerned. Every mathematical book written primarily for engineers encounters this peculiar difficulty that a purely formal presentation is admittedly inadequate even from the engineer's point of view, and yet a mathematically sound and rigorous presentation under sufficiently general conditions is far beyond the scope of such a book; and insofar as a "sound" presentation remains unintelligible to many of its readers, it defeats its purpose. The reaction of the best authors to this situation varies. Some (for instance McLachlan in his last book on operational calculus) develop the mathematical matter that is necessary for the understanding of the finer points of the exposition. Others assume a certain standard of mathematical education, and give a sound presentation under suitably simplified conditions. For instance Churchill, in his *Modern operational mathematics in engineering,* develops the theory of the Laplace transforms of sectionally continuous functions, although he points out that more general types, for instance functions with infinite discontinuities, appear in very many applications. Van der Pol and Bremmer adopt a different attitude. They make a point of stating the results in a form sufficiently general for all applications which they envisage. Instead of sectionally continuous functions, they talk of functions of bounded variation, admit integrable infinite discontinuities, mention Stieltjes integrals, and in the inversion formula consider Cauchy principal values and integrals which are Cesàro summable rather than convergent. Of course they cannot prove their results in this general form. Instead of attempting a proof under more restrictive conditions, they altogether dispense with proofs of the basic theorems: they give a clear statement accompanied by a careful expanation of the terms, and reference to standard mathematical works (chiefly the books by Doetsch and Widder). The space saved by the omission of the proofs, and far more than that space, is devoted to a lavish discussion of the theorems. Illustrative examples are used to show the meaning of the various conditions, the role they play, and the practical use of the theorems. To the reviewer's mind this is an excellent plan—when it is carried out with as much understanding of the mathematical background and with as wide a knowledge of the applications as in the book under review. The technologist is put in a position to use the theorem under rather general conditions with comparative safety, and gets a much more intelligent grasp of the situation than an abstract proof would give him. The mathematical reader, on the other hand, is spared long routine proofs, and the references put him in the position to read up on the theorems with which he is not familiar. It must be admitted, though, that somehow this exposition lacks the last touch of precision. At no stage is it made quite clear just on what class of functions the authors defined the Laplace transform, and while the trained mathematician will have no difficulty in recognizing the conditions under which each result is valid, the engineer may be uncertain at times.

The authors deviate from the usual practice by basing the operational calculus, in chapter I, on the two-sided Laplace transform

$$f(p) = p\int_{-\infty}^{\infty} e^{-pt}h(t)dt \tag{1}$$

rather than on the one-sided transform (with 0 and ∞ as the limits of integration). They hold that the new operational calculus embraces a larger class of functions, simplifies the so-called rules, and leads to a more rigorous treatment. Actually, every complete discussion should (and the present book like many others does) include both the one-sided and the two-sided transformation, and there is little evidence to show that there is much to choose between the two as a starting point. Personal taste seems to be decisive.

In chapter II the Fourier integral in the complex domain is discussed, and tentative results on the domain of convergence of a Laplace integral are obtained. The notation used for the relation (1) is $f(p) \doteq h(t)$ or $h(t) \doteq f(p)$. The function $h(t)$ is called the *original*, and $f(p)$ the *image*. One-sided Laplace transforms may be written as $f(p) \doteq U(t)h(t)$ where Heaviside's unit function $U(t)$ is defined to be 1 for positive t, 1/2 for $t=0$, and 0 for negative t. A number of useful images are computed in chapter III, and the rules of the operational calculus are explained in chapter IV. At this stage the derivation is more or less formal, and although explanatory remarks are added, the conditions of validity of the various rules are not defined very precisely. Take for instance the differentiation rule. From $h(t) \doteqdot f(p)$ it follows that $h'(t) \doteqdot pf(p)$. It is pointed out that this formula is based on an interchange of two limiting processes, and must be handled with caution. Well-chosen examples illustrate the various possibilities, but no precise conditions of validity are given.

Chapter V is an excellent chapter on the unit function and its derivatives. Beside the definition given above, the authors give several families of continuous functions which approximate to the unit function. The impulse function $\delta(t)$ is defined as the derivative of the unit function, and approximations for it are obtained. The history of this function is traced through the writings of Cauchy, Poisson, Hermite, Kirchhoff, and more recent authors. The principal property of this function can be expressed by the "sifting integral"

$$\int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau = h(t), \tag{2}$$

and the authors are careful in pointing out that this integral, and also the image of the delta function, can be given a mathematical meaning as a Stieltjes integral. They discuss several functions approximating to the delta function, and point out that not every function which has the sifting property (that is, not every singular kernel) can be regarded as an approximation for the delta function.

The main part of the mathematical theory of the Laplace transformation is contained in chapters VI and VII. In chapter VI, the region of convergence is investigated. The formulas for the abscissae of convergence, absolute convergence, and uniform convergence are stated without proof, but with reference to books where proofs can be found, and a number of examples serve to familiarize the reader with the use of the formulas. After a digression on summable series and integrals, the behavior of the Laplace integral on the boundary of the region of convergence, and the (complex) inversion formula are discussed. It is pointed out that the same image in two different strips may arise from two different originals, and operational relations having a line of convergence (rather than a strip of convergence) are introduced. One of the advantages of the bilateral transformation is that it exhibits a symmetry between the definition integral and the inversion integral.

The operational treatment of infinite series, and of asymptotic expansions, is taken up in chapter VII. The basic Abelian and Tauberian theorems are explained and applied to derive operational equalities. Asymptotic series are introduced and the usual asymptotic expansions are proved. This part of the book is not only more thorough, but also more detailed than that of any other book written for engineers—a very valuable feature since asymptotic expansions occur so frequently in applications of the operational calculus. The operational interpretation of (ascending or descending) power series, and Heaviside's expansion theorem are also discussed in this chapter. The various possibilities are outlined, but no very precise and satisfactory conditions of validity are given. This chapter concludes with Widder's inversion formula.

A little more than half of the book is devoted to the applications of operational calculus. There is no attempt at a general theory in these chapters, but the technique is explained and illustrated by a number of well-chosen (worked) examples. There are examples taken from electrical engineering problems, there are others illustrating the application of operational calculus to the investigation of special functions of mathematical physics, and yet other applications to the functions of number theory. At least the first two of these groups occur in other books on the subject, but the reviewer knows no book which gives such an extensive and well-balanced training in all three. Partial differential equations and integral equations are not illustrated to the same extent, but in view of Carslaw and Jaeger's book, and of the chapter on integral equations in Titchmarsh's book, this was not really necessary.

Chapter VIII is on linear differential equations with constant coefficients, including differential equations with one-point boundary conditions. Most examples here are taken from electrical network theory. The same is true of the examples of chapter IX on systems of linear differential equations with constant coefficients. Linear differential equations with variable coefficients are discussed in chapter X. Laguerre and Hermite polynomials, Bessel, Legendre, and hypergeometric functions are discussed in great detail, and as a matter of fact this chapter in conjunction with many of the examples in other chapters forms an excellent introduction to the special functions of mathematical physics. Of general theory there is none, and it appears that so far no book on the Laplace transformation ventured on a presentation of the Birkhoff-Horn theory of integration of linear differential equations by Laplace integrals.

Samples of the problems discussed in chapter XI are: the relation between the originals belonging to $f(p)$ and to $p^{k}f(1/p)$, the connection between the images of $h(t)$ and $h(g(t))$ where $g(t)$ is a standard function such as t^{-1}, t^2, e^t, sinh t, and so on, interesting applications to hypergeometric series, the Parseval relation, again with interesting applications to Bessel function identities. Chapter XII introduces step- and discontinuous functions. This is the chapter in which the functions of number theory play such a conspicuous and welcome part. While the work is an excellent practice in operational technique, the results are also relevant for the theory of numbers. The discussion of difference equations in chapter XIII is another opportunity for deriving many relations between special functions.

Chapter XIV explains the technique of solving integral equations by means of the operational calculus. Chapter XV, on partial differential equations, is somewhat disappointing. It makes stimulating reading, the technique is explained carefully, and is illustrated by excellent examples; yet the work is more or less formal. The peculiar difficulties involved in verifying the solution are not even mentioned.

In chapter XVI the operational calculus is extended to several variables, and it is shown how this simultaneous operational calculus can be utilized for the solution of partial differential equations, and other problems.

Chapter XVII contains ten pages of general formulas and rules of the operational calculus, and is aptly labelled *Grammar*. Chapter XVIII, *Dictionary*, contains 27 pages of well-classified operational transform pairs.

The book developed from lectures given by van der Pol in 1938 and 1940, the original Dutch manuscript was prepared during the late war, and the English translation was edited by Dr. C. J. Bouwkamp. The translation deserves special praise for preserving the freshness and flavor of the original. It goes without saying that it is always clear what the authors mean, even where they do not express themselves idiomatically. Passages which may lead to misunderstandings, as for instance on p. 135 where the authors say "the exponents of x may increase" when they presumably mean "the exponents of x are assumed to increase" are very rare. A very special praise is also due to the printing of this book, one of the most beautifully printed of the recent mathematical books.

A. ERDÉLYI

Existence theorems in partial differential equations. By Dorothy L. Bernstein. (Annals of Mathematics Studies, no. 23.) Princeton University Press, 1950. 10+228 pp. $2.50.

An enormous number of papers have been written that deal with various aspects of the theory of partial differential equations. On the other hand, there are only very few books that give an adequate and up-to-date account of the main facts that have been established in this field, as anybody can testify who has ever had to teach a course on the subject. Dr. Bernstein's book, prepared under a contract with the Office of Naval Research, has as its purpose to collect existence theorems that may be of use in formulating problems for solution by computing machines. There is no doubt that existence theorems constitute an important guide for the arrangement of the computation. If the proof of the theorem is constructive, it may be possible to translate it immediately into a scheme for numerical computation. The theorem may also shed light on the nature of the dependence of the solution on the data, which often is of importance in judging the soundness of an approximation scheme. Thus a scheme that does not make use of all the data theoretically needed for determining the solution can be discarded as useless from the beginning. Unfortunately, existence theorems have been established only for relatively few of the types of problems encountered in applications for which numerical solutions are desired.

The following subjects are discussed by the author in greater detail and with outlines of proofs: First order equations (solution of Cauchy problem by the method of characteristics); Systems of first order equations (solution of Cauchy problem for analytic systems); Hyperbolic second order equations (characteristic Goursat problem and Cauchy problem for two independent variables; Cauchy problem for linear equations in more independent variables); Parabolic equations of second order (essentially the linear and quasi-linear cases); Second order elliptic equations (mostly the linear case); Equations of higher order (isolated results for special equations in addition to the Cauchy-Kowalesky theory).

This book is mostly concerned with problems in the small (except in the case of first order equations) and concentrates on existence theorems obtained by general methods rather than those based on explicit solutions. The author took great pains to state the precise assumptions needed in the proofs of the theorems given.

This reviewer feels that an even greater emphasis on methods and ideas in preference to isolated results would have served better the purposes for which this book is intended. Systems of first order equations might have been grouped with equations of higher order rather than with those of first order. There are a fairly large number of misprints and minor inaccuracies in the text that might have been eliminated by more careful proofreading. Many of the papers referred to in the text cannot be found in the bibliography.

Aside from these minor criticisms, this book should be of great value to any pure or applied mathematician working with partial differential equations, and not only to those interested in numerical solutions. A person engaged in research in the field of partial differential equations will find it necessary to go beyond the theorems stated to the original literature. However, the material given in the book should be ample enough to prevent his overlooking major contributions to a problem he is interested in, and to facilitate the task of informing himself on progress made by other workers in the field.

FRITZ JOHN

The Farey series of order 1025, *displaying solutions of the diophantine equation* $bx-ay=1$. Designed and compiled by E. H. Neville. (Royal Society Mathematical Tables, vol. 1.) Cambridge University Press, 1950. 29+405 pp.+1 plate. $18.50.

The Farey series of order n is the table obtained if one arranges in order of magnitude the rational numbers having denominators not greater than n and lying between 0 and 1 inclusive; each rational number is understood to be written in its "lowest terms," the numbers 0 and 1 being given in the forms 0/1 and 1/1 respectively. The table under review is by far the largest of its kind ever published in full and probably will remain so for some time to come, since the Farey series of order 1025 consists of 319,765 fractions.

Even though the fractions a/b and $(b-a)/b$ are given together, the Farey series of order 1025 itself occupies 400 pages. Except for the last page, each page contains 400 pairs of fractions, arranged in lines of 20 each. Besides the main table there is an appendix containing the following smaller tables: (1) the Farey series of order 50 with decimal equivalents, (2) the Farey series of order 64, (3) the denominators of the Farey series of order 100. In addition there is a brief introduction in which the author discusses the background, construction, and use of the tables. (See also the author's article *The structure of Farey series,* Proc. London Math. Soc. (2) vol. 51 (1949) pp. 132–144 and the third chapter of Hardy and Wright's *Introduction to the theory of numbers.*)

The principal interest of such a table, aside from its obvious usefulness in finding rational approximations to irrational numbers, lies in the fact that if a/b and x/y are consecutive fractions in the Farey series of order n, then $bx-ay=1$. Thus the present table can be thought of as a table of solutions of the diophantine equation

$bx-ay=1$ for all pairs of integers a, b such that $1\leqq a\leqq b\leqq 1025$ and $(a, b)=1$, where the entries are arranged in order of magnitude of the ratio a/b. Of course it is possible to extend the usefulness of the table well beyond its apparent range.

Since the main table of the work under review does not give the decimal equivalents of the fractions listed, there is considerable difficulty in locating a given fraction in the series or in fixing a given irrational number (or rational number with denominator greater than 1025) between two fractions of the series. While it would clearly be out of the question to give the decimal equivalent of every fraction listed, it seems to the reviewer that it would have been quite feasible to give at the end of each line of the table the decimal equivalents of the last pair of fractions in the line. This would have enhanced the value of the table considerably, for it would have made the location problem relatively easy. As it is, the user of the table is expected to locate a given fraction in the series (or to determine in what interval a number not in the series would fall) by appealing to the uniform distribution of the Farey fractions. (Cf. problem 189 of section II of Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin, Springer, 1925.) To be sure, the author gives a table of the locations of the fractions equivalent to $k/1000$ ($k=0, 1, 2, \cdots, 500$), but between two consecutive such fractions there are on the average about 320 other fractions.

This work is the first volume in a series of mathematical tables which is to be published by the Royal Society and which is intended as a continuation of the well known series of tables published by the British Association. The format and printing of this first volume are very satisfactory. Although this work will be a mere curiosity to most mathematicians and will certainly not have widespread use, number-theoretic experimenters will find it of considerable interest.

P. T. BATEMAN

A theory of cross-spaces. By Robert Schatten. (Annals of Mathematics Studies, no. 26.) Princeton University Press, 1950. 7+153 pp. $2.50.

The subject of this book is the generalization to Banach spaces of the construction of the algebraic direct product of two finite-dimensional vector spaces. The main problems arise from the variety of possible norms which can be used in the algebraic product space and the resulting profusion of Banach spaces which have honest claim to the title of direct product space. The results, mostly due to the author and to J. von Neumann, are outlined in a long introductory chapter; in the next paragraphs we state briefly the argument of the text. The author has provided a clear exposition of the carefully limited topic he set himself.

Given two Banach spaces, B_1 and B_2, with conjugate spaces B_1^* and B_2^*, the algebraic direct product $B_1 \bigcirc B_2$ consists of suitable equivalence classes of the set of all formal sums of formal products, $\sum_{i \leqq n} f_i \otimes g_i$, $f_i \in B_1$, $g_i \in B_2$; the definition of equivalence is so chosen that the correspondence of $\sum_{i \leqq n} f_i \otimes g_i$ with the operator A from B_1^* to B_2 defined by $A(F) = \sum_{i \leqq n} F(f_i)g_i$ is an isomorphism between $B_1 \bigcirc B_2$ and a linear subspace of the space $L(B_1^*, B_2)$ of linear bounded operators from B_1^* to B_2. Let $\lambda(\sum_{i \leqq n} f_i \otimes g_i)$ equal the norm of the corresponding A.

A norm α in the linear space $B_1 \bigcirc B_2$ is called a crossnorm if $\alpha(f \otimes g) = |f| \, |g|$. It is shown that λ is a crossnorm and that there is in $B_1 \bigcirc B_2$ a greatest crossnorm γ.

Each element of $B_1^* \bigcirc B_2^*$ determines a linear functional on $B_1 \bigcirc B_2$ by $\phi \leftrightarrow \sum_{j \leqq m} F_j \otimes G_j$ if $\phi(\sum_{i \leqq n} f_i \otimes g_i) = \sum_{j \leqq m} \sum_{i \leqq n} F_j(f_i)G_j(g_i)$. Hence every crossnorm α on $B_1 \bigcirc B_2$ determines an associate norm α' on $B_1^* \bigcirc B_2^*$. λ is the least crossnorm whose associate is also a crossnorm. On $B_1^* \bigcirc B_2^*$, γ' and λ are equal.

Completion of $B_1 \bigcirc B_2$ under crossnorm α gives the cross-space $B_1 \otimes_\alpha B_2$. If $\alpha \geqq \lambda$, completion of $B_1^* \bigcirc B_2^*$ under the associate crossnorm α' yields the associate space $B_1^* \otimes_{\alpha'} B_2^*$; the natural embedding of this associate space into the conjugate space $(B_1 \otimes_\alpha B_2)^*$ is an isometry but need not fill up all of $(B_1 \otimes_\alpha B_2)^*$. A representation of $(B_1 \otimes_\alpha B_2)^*$ as a renormed linear subspace of $L(B_1, B_2^*)$ is obtained; in particular, $(B_1 \otimes_\gamma B_2)^*$ is all of $L(B_1, B_2^*)$ with the usual norm.

Specialization of B_1 and B_2 to two copies of Hilbert space, H and H^*, gives results on the ideal of completely continuous operators on H. $H \bigcirc H^*$ can be interpreted as the set of operators from H to H of finite rank. If E is the set of all operators $X \in L(H, H)$ such that $\sigma(X) = (\sum_i |X\phi_i|^2)^{1/2} < \infty$ for a fixed complete orthonormal set $\{\phi_i\} \in H$, then σ is a crossnorm on $H \bigcirc H^*$ and E with the norm σ is equivalent to $H \otimes_\sigma H^*$. Similarly, the space $H \otimes_\lambda H^*$ can be interpreted as I, the ideal of all completely continuous operators belonging to $L(H, H)$. The conjugate and associate spaces of $H \otimes_\lambda H^*$ coincide with $H \otimes_\gamma H^*$, which, in turn, can be represented as the closure of $H \bigcirc H^*$ under the norm $m(A) = t(|A|)$, where $|A| = (A^*A)^{1/2}$ and $t(X)$, the trace of X, $= \sum_i (X\phi_i, \phi_i)$, $\{\phi_i\}$ a complete orthonormal system. Finally, I^{**} is equivalent to $(H \otimes_\gamma H^*)^*$ which is equivalent to $L(H, H)$.

Appendices discuss properties of a crossnorm α which are connected with equality of conjugate and associate spaces.

MAHLON M. DAY

Coefficient regions for schlicht functions. By A. C. Schaeffer and D. C. Spencer. With a chapter on *The region of values of the derivative of a schlicht function* by Arthur Grad. (Amer. Math. Soc. Colloquium Publications, vol. 35.) New York, 1950. 14+311 pp., 2 plates. $6.00.

In the theory of schlicht functions there is considerable difference in depth between the elementary results exemplified by Koebe's distortion theorem and the advanced theory exemplified by Löwner's proof of the inequality $|a_3| \leqq 3$. Through Löwner's paper variational techniques were introduced in the coefficient problem, and since that time they have provided a more and more efficient tool. Löwner's result was at first quite isolated. To obtain similar results of a more general nature Schiffer had to overcome specific difficulties which seemed rather formidable. Later on the techniques were perfected by Schiffer himself, by the authors of the book under discussion, and by many others. The time was ripe for a systematic presentation of such methods and results, and the team of Schaeffer and Spencer has obliged us with an impressive monograph on the subject.

The historical introduction is brief and leads almost directly to the formulation of the main problem. One considers the class $\mathfrak{S}$ of normalized functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ which are regular and schlicht in $|z| < 1$. The problem is to characterize the sequences $\{a_n\}$ which define such functions; this problem will be solved if one can determine the region V_n in $(2n-2)$-dimensional space to which the point $(a_2, a_3, \cdots, a_n)$ is confined. The most likely way to success is by determination of the boundary of V_n through the extremal properties of the corresponding functions.

This problem is not the most general and not even necessarily the most natural; it is difficult to agree that the series development has special merits in comparison with other problems of interpolation. However, the coefficient problem is certainly typical in the sense that more general problems can very probably be treated by analogous methods, and the amount of interest it derives from the fact that Bieberbach's conjecture $|a_n| \leqq n$ has neither been proved nor disproved is not negligible.

Chapter II introduces the authors' specific variational method on which most of the book is based. A rough description of the method follows. Let C be the unit circle $|z| < 1$, S the Riemann sphere over the w-plane, and $w = f(z)$ a schlicht mapping. Draw a simple analytic arc γ in C, and introduce an analytic correspondence, different from the identity, between the two edges of a slit along γ. Through identification of corresponding points a new abstractly defined Riemann surface C^* is obtained which is conformally equivalent with C. A similar identification of the corresponding points on the image $f(\gamma)$ produces a Riemann surface S^* conformally equivalent with S. Map C conformally onto C^*, let this be followed by the mapping f of C^* into S^*, and finally by a conformal mapping of S^* onto S. This series of mappings produces a mapping of C into S which, properly normalized, can serve as a variation of $f(z)$.

It turns out that the mappings can be expressed explicitly by means of line integrals, in a first approximation, and the authors derive the existence of a normalized schlicht function given by

$$\begin{aligned} f^*(z) &= f(z) \\ (1) \qquad &+ \frac{\epsilon}{2\pi i}\int_\gamma \frac{p(u)}{2u^2}\left[\left(\frac{uf'(u)}{f(u)}\right)^2 \frac{2f(z)^2}{f(u)-f(z)} - zf'(z)\frac{u+z}{u-z} \right. \\ &\left. + f(z)\right] du \quad + \frac{\bar{\epsilon}}{2\pi i}\int_\gamma \frac{\overline{p(u)}}{2\bar{u}^2}\left[zf'(z)\frac{1+\bar{u}z}{1-\bar{u}z} - f(z)\right] d\bar{u} + o(\epsilon), \end{aligned}$$

$p(u)$ being a fairly arbitrary weight function.

In the reviewer's opinion the author's choice is a somewhat artificial compromise. It is conceptually simpler to define C^* by the introduction of a new metric, defined for instance by $ds = |dz + h(z)d\bar{z}|$. The computations are fairly trivial and lead to (1) with a double integral in the place of the line integral. On the other hand, the results are not stronger, and the authors' method has the advantage that it need not be based on the general uniformization theorem.

By means of (1) it is proved that every function which maximizes a real-valued differentiable function of the coefficients must satisfy a differential equation of the form

$$(2) \qquad \left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z)$$

with

$$P(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{w^\nu}, \qquad Q(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu}.$$

In addition, $Q(z) \geqq 0$ on $|z| = 1$ with at least one zero there.

Chapters III–VI are devoted to a study of the differential equation (2) and especially those solutions which belong to the class $\mathfrak{S}$. Although the variables can be separated, the multiple-valuedness of the integrals make this study far from trivial. The aim is of course to characterize those solutions which can correspond to boundary points of the coefficient region V_n. Such investigations tend to be rather laborious, the main chore being to refute certain undesirable possibilities. In places the presentation is not very lucid, although the painstaking reader will ultimately become convinced that the argument is correct. On the whole the authors seem to prefer blunt frontal attacks to simpler but more devious reasonings. A typical case is the proof of Lemma XI which requires almost three pages and a concentrated effort on the part of the reader. In the reviewer's opinion it would take much less effort to relate the angles, the Euler characteristic, and the signatures of the singularities by a general formula.

In the next chapter the authors use Teichmüller's method to prove the converse, namely that the solutions of (2) belong to the boundary of V_n. There is thus one-to-one correspondence between boundary points and solutions, but it may happen that a solution belongs to several differential equations of the type (2). This complication is due to the fact that V_n will be bounded by several hypersurfaces which intersect in manifolds of lower dimension.

By this time the solution of the coefficient problem begins to take rather definite shape. The extremal functions map the unit circle onto the complement of a "tree" whose branches are characterized by the condition $\operatorname{Im} P(w)^{1/2}dw/w=0$. It is ultimately possible to characterize these trees by their topological nature and by the requisite number of numerical parameters. In this sense the solution is complete, but it leads to practically no explicit statements concerning the coefficients.

An interesting chapter deals exhaustively with the lowest nontrivial case, the inequalities between a_2 and a_3. Since a_2 may be supposed real the problem is to determine a three-dimensional region. In this case all functions can be expressed in terms of elementary functions; numerical tables are given, and two plates give excellent visual impressions. The authors fail, however, to give the geometric interpretation which in the eyes of the reviewer is the most revealing. Remove the points $\eta \leqq 0$, $|\xi| \leqq 1$ from the $\zeta=(\xi+i\eta)$-plane, and identify boundary points which are symmetric with respect to the imaginary axis; the closed surface obtained in this manner may be considered as a model of the whole complex plane. In this model we introduce a rectilinear slit with one end point at the origin, and consider the slit surface as a model of the unit circle. If we let $\zeta=\infty$ correspond to $z=0$, $w=0$ and $\zeta=0$ to $w=\infty$ the identical mapping can be considered as a conformal mapping $w=f(z)$ of $|z|<1$ into the w-plane; it is the coefficients a_2, a_3 of these mappings which form the boundary of V_3. Two cases are possible: the slit may consist of a single segment of length ρ forming an angle $\phi \neq 0, \pi$ with the real axis, or the slit may lie on the real axis with the pieces of absolute value ≤ 1 identified with each other, in which case the slit appears as a fork with end points α, β. Accordingly, the boundary of V_3 consists of one surface corresponding to the parameters ρ, ϕ, and another surface with the parameters α, β.

The last chapter, written by Arthur Grad, gives an explicit determination of the region of values taken by $f'(z)$ at a fixed point z. It is an excellent illustration of the method in a case different from but of the same degree of difficulty as the determination of V_3. The reader who is anxious to learn the technique from the point of view of actual application will find this chapter most rewarding.

The authors can be congratulated on the accurate work they have accomplished. Great professional skill and painstaking detailed analysis are dominating features throughout the book. Books of this sort are never easy to read, and they offer little to the impatient skimmer. This book is definitely written by conscientious authors for conscientious readers.

LARS V. AHLFORS

BRIEF MENTION

Funzioni quasi-periodiche. By S. Cinquini. (Scuola Normale Superiore, Pisa, Quaderni Matematici, no. 4.) Pisa, Tacchi, 1950. 132+7 pp.

The class of Bohr almost periodic functions includes in particular the subclass of exponential polynomials $\sum a_n e^{i\lambda_n x}$, and Bohr's uniqueness theorem states that the whole class arises from the subclass on closing the latter by the norm

$$\|f\| = \sup_{-\infty<x<\infty} |f(x)|. \tag{1}$$

In the periodic case this corresponds to the C-functions, and the so-called generalizations of almost periodic functions are various modes for introducing closure norms that would give the analogues to all L_p-functions as well. The narrowest such generalization known is due to Stepanoff, and it uses the norm

$$\|f\| = \sup_{-\infty<x<\infty} \left(\int_0^l |f(x+\xi)|^p d\xi \right)^{1/p} \tag{2}$$

for some (and hence any) finite length l; the purpose of the present tract is to give an account of the theory that would feature the Stepanoff functions in the main.

The tract presupposes but little and it is written in a pleasant comfortable style, so that not too much material need be expected from it; and, in fact, the author more or less carries it only to the point where he presents the following theorem which he had published previously:

If $\psi(w)$ is non-negative convex in $0 \leqq w < \infty$, and $\psi(0)=0$, if the Stepanoff function $f(x)$ is such that for $\psi(2|f(x)|)$ a certain integrability condition is fulfilled, and if a sequence $\{\sigma_n(x)\}$ of Bochner-Fejér polynomials of $f(x)$ is formally convergent towards it, then

$$\int_0^l \psi|f(x+\xi) - \sigma_n(x+\xi)| \, d\xi \to 0$$

uniformly in $-\infty < x < \infty$.

The latter theorem generalizes a known theorem for pure periodic functions, and it confirms a general statement made by the reviewer repeatedly with illustrations [see for instance, *Properties of Fourier series of almost periodic functions*, Proc. London Math. Soc. (2) vol. 26 (1926) pp. 433–452], that any property of pure periodic functions whatsoever, no matter what its scope or origin, may be adapted to Stepanoff functions as well, either to the entire class as in the present case, or to some suitable sub-class containing all pure periodic elements occurring, as in many other cases.

We also recall that Stepanoff functions may also be viewed as particular cases of abstractly-valued functions [see for instance, Bochner, *Abstrakte fastperiodische funktionen*, Acta Math. vol. 61 (1933) pp. 149–184, last section] and we venture to say that no comprehensive account of the theory of Stepanoff functions would be complete without introducing the viewpoints thus arising—but these remarks of ours are not intended to be a criticism of the work the author has accomplished according to his own plans.

S. Bochner

The problem of moments. By J. A. Shohat and J. D. Tamarkin. (Mathematical Surveys, no. 1.) American Mathematical Society, New York, 1943. Reprinted 1950. 14+144 pp. $3.35.

In this printing the theorem on p. xiii on extending a non-negative linear functional has been corrected and a supplementary bibliography of papers up to 1949 has been added. Otherwise, except for minor corrections, this printing is identical with the printing of 1943.

NOTES

As part of the Semicentennial Celebration of the National Bureau of Standards, a Symposium on the Solution of Simultaneous Equations and the Determination of Eigenvalues will be held at the Institute for Numerical Analysis (a section of the NBS Applied Mathematics Laboratory) at Los Angeles, California, on August 23–25, 1951.

Dr. Christine Williams Ayoub of Cornell University has been awarded a post doctoral fellowship by Sigma Delta Epsilon.

Professor Einar Hille of Yale University has been elected a foreign member, first class, of the Swedish Academy of Sciences.

Professor G. T. Whyburn of the University of Virginia and Professor H. P. Robertson of the California Institute of Technology have been elected to membership in the National Academy of Sciences.

Professor Jakob Nielsen of the Technical University of Denmark has been appointed to a professorship at the University of Copenhagen.

Professor C. B. Allendoerfer of Haverford College has been appointed to a professorship at the University of Washington.

Mr. A. F. Bartholomay of Rutgers University has accepted a position as research mathematician at the Laboratory for Electronics, Inc., Boston, Massachusetts.

Professor Z. W. Birnbaum of the University of Washington is on leave of absence and has been appointed to a visiting professorship at Stanford University.

Dr. Nat Edmonson of Fairchild Engine and Airplane Corporation has accepted a position as mathematician at the Oak Ridge National Laboratories.

Associate Professor R. E. Gaskell of Iowa State College of Agriculture and Mechanic Arts is on leave of absence and has accepted a position at Boeing Aircraft Corporation, Seattle, Washington.

Dr. Saul Gorn of the Wright-Patterson Air Force Base has accepted a position as principal research mathematician at the Computing Laboratory and as a member of the Scientific Staff of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Dr. Emil Grosswald of the University of Saskatchewan is on leave of absence and has been appointed a member of the Institute for Advanced Study for the year 1951–1952.

Assistant Professor R. C. James of the University of California has been appointed to an associate professorship at Haverford College.

Assistant Professor Jacob Korevaar of Purdue University has been appointed to a professorship at the Institute of Technology, Delft, Netherlands.

Mr. Rubin Kuritsky of Michigan State College has accepted a position as research engineer at North American Aviation Co., Los Angeles, California.

Assistant Professor W. W. Leutert of the University of Maryland has accepted a position as mathematician at the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Dr. George Lorentz has been appointed to an assistant professorship at the University of Toronto.

Professor Charles Loewner of Syracuse University has been appointed to a professorship at Stanford University.

Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College is on leave of absence and has been appointed to a visiting professorship at Stanford University.

Dr. W. L. Murdock of Cornell University has accepted a position as mathematician at General Electric Company, Syracuse, New York.

Assistant Professor Saul Rosen of Drexel Institute of Technology has accepted a position at Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Dr. Louis Sacks of the Carnegie Institute of Technology has been appointed to an assistant professorship at the University of Pittsburgh.

Assistant Professor W. C. Sangren of the University of Miami has accepted a position as senior mathematician at the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Assistant Professor R. H. Somers of the University of New Hampshire has retired.

Dr. George Springer of the Massachusetts Institute of Technology has been appointed to an assistant professorship at Northwestern University.

Dr. Jacob Wolfowitz of Columbia University has been appointed to a professorship at Cornell University.

The following promotions have been announced:

R. P. Dilworth, California Institute of Technology, to a professorship.

H. Margaret Elliott, Washington University, to an assistant professorship.

Zeev Nehari, Washington University, to a professorship.

Rohn Truell, Brown University, to a professorship.

The following appointment to an instructorship is announced:

University of Oregon: Mr. R. D. Stalley.

Professor Emeritus G. A. Bliss of the University of Chicago died on May 8, 1951 at the age of seventy-four years. He had been a member of the Society for fifty-one years.

Dr. R. B. Robbins of Charlottesville, Virginia died on February 11, 1951 at the age of sixty-five years. He had been a member of the Society for thirty-seven years.

Dean H. L. Slobin of the University of New Hampshire died on February 23, 1951 at the age of sixty-eight years. He had been a member of the Society for forty-two years.

Professor Emeritus W. D. A. Westfall of the University of Missouri died on April 28, 1951 at the age of seventy-two years. He had been a member of the Society for forty-nine years.

Mr. A. H. Wheeler of Worcester, Massachusetts, died on December 19, 1950. He had been a member of the Society for twenty-seven years.

NEW PUBLICATIONS

Appell, P. *Analyse mathématique à l'usage des candidats au certificat de mathématiques générales et aux grandes écoles.* Vol. 1. *Analyse des courbes, surfaces et fonctions usuelles, intégrales simples.* 6th ed. Paris, Gauthier-Villars, 1951. 9+408 pp. 2,000 fr.

Appell, P. *Analyse mathématique à l'usage des candidats au certificat de mathématiques générales et aux grandes écoles.* Vol. 2. *Équations différentielles, développements en séries, nombres complexes, intégrales multiples, probabilités, déterminants, exercises.* 6th ed. Paris, Gauthier-Villars, 1951. 442 pp. 2,200 fr.

Appert, A., and Fan, K. *Exposés d'analyse générale.* XV. *Espaces topologiques intermédiaires. Problème de la distanciation.* (Actualités Scientifiques et Industrielles, no. 1121.) Paris, Hermann, 1951. 160 pp.

Arrow, K. J. *Social choice and individual values.* (Cowles Commission Monograph, no. 12.) New York, Wiley, 1951. 11+99 pp. $2.50.

Baumgardt, C. *Johannes Kepler. Life and letters.* With an introduction by A. Einstein. New York, Philosophical Library, 1951. 209 pp. 2 plates. $3.75.

Béghin, H., and Julia, G. *Exercises de mécanique.* Vol. 1, no. 2. 2d ed. Paris, Gauthier-Villars, 1951. 1+244 pp.

Bieberbach, L. *Einführung in die analytische Geometrie.* 4th ed. Bielefeld, Verlag für Wissenschaft und Fachbuch, 1950. 168 pp. 8.50 DM.

Borel, E. *Probabilité et certitude.* Paris, Presses Universitaires de France, 1950. 136 pp.

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THE ANALYTIC THEORY OF SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS[1]

A. ERDÉLYI

1. **Introduction.** The theory of ordinary linear differential equations in the complex domain has always attracted particular attention, and topics such as singularities of the regular type and equations of the Fuchsian class belong to those well-rounded gems for which the classical theory of functions of a complex variable is so justly famous.

Almost simultaneously with the fundamental researches of Fuchs and others on ordinary differential equations, efforts were made to develop a similar theory for systems of partial differential equations. Although mathematicians of the rank of Appell, Goursat, and Picard were among those interested in the matter, at the time these efforts were doomed to failure because of the insufficient understanding of functions of several complex variables, and also of some relevant topics in multiple series, algebraic geometry, topology, and groups. Today the problem could be attacked with greater hope of success, and the present survey seeks to prepare the ground for such an attack.

2. **Ordinary differential equations.** It will be useful to review briefly the theory of ordinary linear differential equations

$$\frac{d^n z}{dx^n} + g_1(x)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + g_n(x)z = 0 \tag{1}$$

in the complex domain. The $g_i(x)$ are one-valued analytic functions of the complex variable x, and we shall assume that each $g_i(x)$ has at most a finite number of singularities. It is well known that such an equation has n linearly independent solutions, and that any $n+1$ solutions are linearly dependent. If all the $g_i(x)$ are regular in a neighborhood of $x=a$, then a is an *ordinary* point of the equation, and in a neighborhood of any such point the equation possesses a fundamental system of n analytic solutions represented by convergent power series. A singularity of any of the $g_i(x)$ is a *singular point* of the differential equation.

An address delivered before the Stanford meeting of the Society on April 28, 1951 by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors April 30, 1951.

[1] Prepared under Contract N6onr-244 Task Order XIV of the Office of Naval Research with the California Institute of Technology.

Let $z(x)$ be a vector of n linearly independent analytic solutions $z_1(x), \cdots, z_n(x)$. If x describes a closed curve Γ every point of which is an ordinary point, then $z(x)$ can be continued analytically along Γ, and when we return to the point of departure, $z(x)$ will have changed into $\bar{z}(x)$. Since every component of $\bar{z}(x)$ is a solution of the differential equation, and hence a linear combination of $z_1(x), \cdots, z_n(x)$, we have a relation $\bar{z}(x)=Az(x)$ where A is an $n\times n$ matrix of constant elements. For a given differential equation, A depends on Γ; for a fixed Γ, different fundamental systems give rise to *equivalent* matrices, and thus Γ determines the *canonical form* of A. The linear transformations A for all possible closed curves form a group, the *monodromic group* of the equation, and this group is generated by the transformations corresponding to simple closed curves encircling one of the singularities.

Let $x=a$ be a singular point. If the canonical form of the corresponding matrix A is diagonal,

$$\begin{bmatrix} e^{2i\pi\rho_1} & & \\ & \ddots & \\ & & e^{2\pi i\rho_n} \end{bmatrix},$$

then there is a fundamental system of the form $z_j=(x-a)^{\rho_j}y_j(x)$ where the y_j are holomorphic in a deleted neighborhood of $x=a$, and hence can be represented by Laurent series. If the canonical form of A is not diagonal, logarithmic solutions occur in the fundamental system but otherwise the situation is unchanged. In case the Laurent series have only a finite number of terms with negative exponents, the ρ_j can be adjusted so that the y_j are all analytic at $x=a$. A point at which this is possible is a *regular* singular point, and equations possessing a finite number of regular singularities, and no other singularities, are said to be of the *Fuchsian class.*

A linear differential equation determines its monodromic group; and Riemann's investigations of hypergeometric functions show that at any rate Fuchsian equations are essentially determined by their monodromic group. More precisely, for an equation of the nth order with m regular singularities, the singularities can be prescribed arbitrarily. If $mn\leqq 2(n+1)$, the monodromic group is subject to $(n^2-1)-n(n-1)m/2$ conditions; and any group satisfying these conditions can be prescribed and then determines the equation completely. If $mn>2(n+1)$ any (consistent) monodromic group can be prescribed and determines the differential equation up to $n(n-1)m/2-(n^2-1)$ arbitrary constants (the *accessory parameters*).

3. **Systems of partial differential equations.** The extension of all these ideas to several complex variables was envisaged as early as 1880 when Appell [1][2] introduced certain series in two variables x and y which are analogous to the hypergeometric series $F(\alpha, \beta; \gamma; x)$. Appell's series satisfy a system of partial differential equations of the second order of the form

$$\begin{aligned} r &= as + bp + cq + ez, \\ t &= fs + gp + hq + kz \end{aligned} \tag{2}$$

where z is the unknown function, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, subscripts denote partial differentiation, and $a, b, c, \cdots, k$ are given (in this case rational, more generally analytic) functions of the independent variables x and y. Investigations by Bouquet, Méray, and Mayer give information about the number of linearly independent solutions of (2), and a certain amount of information about the singularities; but it is significant that in no case is complete information available. The best known system is that belonging to Appell's function F_1, that is, the system

$$\begin{aligned} &x(1-x)r + y(1-x)s + \{\gamma - (\alpha + \beta + 1)x\}p \\ &\qquad\qquad\qquad\qquad\qquad - \beta yq - \alpha\beta z = 0, \\ &y(1-y)t + x(1-y)s + \{\gamma - (\alpha + \beta' + 1)y\}q \\ &\qquad\qquad\qquad\qquad\qquad - \beta' xp - \alpha\beta' z = 0 \end{aligned} \tag{3}$$

(α, β, β', γ constants), and although 10 solutions of (3) were known as early as 1893 (LeVavasseur) [17], it turns out that these 10 solutions are insufficient to provide fundamental systems for the neighborhood of all singular points. For this another 15 solutions are necessary and these were obtained only recently (Erdélyi [4]) by rather ad hoc methods, not from the general theory.

Much of this survey will be restricted to a discussion of systems of the form (2). The most significant exception from this rule is the next section which contains a brief account of the modern investigations into the existence theory of analytic solutions of systems of partial differential equations. Here we can handle a much more general situation.

4. **Existence theorems.** Given a system of partial differential equations in any number of variables and unknown functions, we may ask whether the system is consistent, whether it has analytic solutions, and also we may ask for the number of arbitrary constants

---

[2] Numbers in brackets refer to the references at the end of the paper.

(or arbitrary analytic functions) contained in the general (analytic) solution of the system.

Such questions were first investigated by Cauchy and Sonja Kovalewska [16] and it was found that a formal power-series solution is no guarantee for the existence of an analytic solution. A more detailed investigation was commenced by Méray [20] and Riquier [23; 24], and continued by Janet [13; 14; 15]. With a view of ordering derivatives integers, called "cotes," were attached to derivatives, and majorization was used to prove the existence of analytic solutions. Riquier actually succeeded in proving the existence of analytic solutions for so-called *orthonomic systems*, and even for more general systems in case the defining functions and initial determinations satisfy certain inequalities.

A more modern version of the theory is due to J. M. Thomas [27; 28; 29; 30; 31; 32]. Eventually Thomas was able to dispense with "cotes," and develop an algebraic procedure which resulted in existence theorems wider than Riquier's.

Let $x_1, \cdots, x_n$ be the independent variables, $u_1, \cdots, u_r$ the unknown functions.

$$D = \frac{\partial^{i_1+\cdots+i_n} u_j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

is a typical derivative. We shall associate it with the monomial

$$m = x_1^{i_1} \cdots x_n^{i_n} u_j,$$

and write $D \sim m$. We consider a system S consisting of a finite number of partial differential equations of finite order, each solved explicitly for one of the derivatives. Thus S consists of equations of the form

$$D_i' = f_i(x, D''), \qquad i = 1, \cdots, k, \tag{4}$$

each of which involves only a finite number of derivatives D''. A set of constants for the x and the D which substituted in (4) satisfy each equation is called a *numerical determination*, and the values of the x in a numerical determination are the *initial values*. It is assumed that each $f(x, D'')$ is an analytic function of the x and the D, holomorphic in some neighborhood of the numerical determination.

Let D' be a derivative occurring on the left-hand side of an equation (4), and D'' any of the derivatives occurring on the right-hand side of the same equation: $D' \sim m'$, $D'' \sim m''$. If there are positive constants x, u satisfying all inequalities $m' > m''$, $x_i > 1$, Thomas says that the first inequality system is consistent, and calls the system S *admissible.* The derivatives can be ordered by the adopted solution of the first inequality system. An admissible system can be *prolonged* by differentiating each equation with respect to the x in every possible way, taking account of the dependence of the D on the x_j; the infinite system so obtained is the *prolongation* of S. An initial determination of the prolongation gives initial values of all D, and thus a formal power series solution, known as the *tentative* solution; the number of arbitrary D in the initial determination gives information about the number of arbitrary constants in the general solution of S.

Next a second inequality system is obtained whose consistency means, roughly, that a certain principal part S^* of the prolongation can be ordered. An admissible system whose second inequality system is consistent is an *orthonomic system.* In general, S^* can be decomposed, and if every component of S^* has a consistent second inequality system, then S is said to be *orderly.* The numerical determination is part of the orderly system. The expansions of a tentative solution of an orderly system converge in a neighborhood of the initial values; whether or not the tentative solution actually satisfies the system S depends on the particular initial determination chosen. However, if a series of further conditions is satisfied, the system is *passive,* and it can be proved that a passive orderly system has a unique analytic solution corresponding to each arbitrarily given (relevant) initial determination.

Clearly the theory thus sketched is restricted to *ordinary* points of the system, that is, to points in the neighborhood of which the general (analytic) solution is regular.

5. **Total differential equations.** We now take a system of linear partial differential equations in two independent variables x and y and one unknown function z, and use the notation introduced in §3 for partial derivatives. First we shall show that such a system can be converted into a system of *total differential equations* in two independent variables (see Appell and Kampé de Fériet [2], Chapter 3, and the literature quoted there).

We take a system of the form (2) and assume at first that $1-af$ does not vanish identically. Differentiate the first equation with respect to y, and the second with respect to x. Since $r_y=s_x$ and $t_x=s_y$, we obtain two equations for s_x and s_y, and since $1-af$ does not vanish identically, these two equations can be solved in the form

$$s_x = \alpha s + \beta p + \gamma q + \delta z, \qquad s_y = \epsilon s + \eta p + \theta q + \zeta z.$$

In this case we obtain the following system in the four unknown functions z, p, q, s

$$
\begin{aligned}
dz &= p\,dx + q\,dy,\\
dp &= (as + bp + cq + ez)dx + s\,dy,\\
dq &= s\,dx + (fs + gp + hq + kz)dy,\\
ds &= (\alpha s + \beta p + \gamma q + \delta z)dx + (\epsilon s + \eta p + \theta q + \zeta z)dy.
\end{aligned} \tag{5}
$$

On the other hand, if $1-af$ vanishes identically, then all third-order derivatives can be eliminated from the differentiated equations (2), and the result is an equation of the form

$$\alpha r + \beta s + \gamma t = \delta p + \epsilon q + \eta z, \tag{6}$$

so that in this case (2) is equivalent to a system of *three* partial differential equations of the second order. If

$$\begin{vmatrix} 1 & -a & 0 \\ 0 & -f & 1 \\ \alpha & \beta & \gamma \end{vmatrix} = \begin{vmatrix} 1 & -a & 0 \\ 0 & -f & 1 \\ ag & a(h+f_x)+f(b+a_y) & cf \end{vmatrix} \tag{7}$$

does not vanish identically, the system (2), (6) can be replaced by

$$r = Az + Bp + Cq, \quad s = Dz + Ep + Fq, \quad t = Gz + Hp + Kq, \tag{8}$$

and we have now a system of total differential equations in the three unknown functions z, p, q of x and y,

$$
\begin{aligned}
dz &= p\,dx + q\,dy,\\
dp &= (Az + Bp + Cq)dx + (Dz + Ep + Fq)dy,\\
dq &= (Dz + Ep + Fq)dx + (Gz + Hp + Kq)dy.
\end{aligned} \tag{9}
$$

If the determinant (7) vanishes identically, a further reduction takes place, and one obtains a system in less than three unknown functions.

In any event, a system of linear partial differential equations can be converted into a system S of *total linear differential equations* in two independent variables,

$$dz_k = \sum_j \left[f_{kj}(x, y)dx + g_{kj}(x, y)dy\right]z_j, \qquad k, j = 1, \cdots, n. \tag{10}$$

Such a system must satisfy the *compatibility conditions*

$$\frac{\partial}{\partial y}\left(\sum_j f_{ij}z_j\right) = \frac{\partial}{\partial x}\left(\sum_j g_{ij}z_j\right)$$

or

$$\sum_i (f_{ij}g_{jk} - g_{ij}f_{jk}) = \frac{\partial g_{ik}}{\partial x} - \frac{\partial f_{ik}}{\partial y}, \qquad i, j, k = 1, \cdots, n. \tag{11}$$

If these conditions are satisfied, it is known that (10) has a fundamental system of n solutions z_k^m, $k, m = 1, \cdots, n$, which are analytic at every point at which all the f_{kj} and g_{kj} are analytic (*ordinary point*).

6. **Singularities.** For the further discussion let us assume that the f and g in (10) are *rational functions* of x and y which satisfy (11). From the foregoing it is clear that singularities of the differential equations are caused by the denominators of the f or g. Let $\phi(x, y)$ be an irreducible factor of one of the denominators. The algebraic curve $\phi(x, y) = 0$ is a *singular curve* of the system (7).

Every closed curve Γ in the four-dimensional x, y space which avoids all singular curves of (10) corresponds to a *transformation* of our fundamental system, and the totality of transformations corresponding to all such Γ form the *monodromic group* of the equation. It appears that not much progress has been made in the investigation of the monodromic group.

Let the irreducible algebraic curve $\phi(x, y) = 0$, or C, be a singular curve of S. A point (a, b) of C is called a nonsingular point of C if either $\phi_x(a, b) \neq 0$ or $\phi_y(a, b) \neq 0$; and it is called a general point of C if it is nonsingular and if C is the only singular curve of the system which passes through that point. If there is a fundamental system of solutions which can be represented in the canonical form

$$z_k^m = \phi^{\rho_m} p_k^m(z - a, y - b)$$

(or in special cases in a similar form containing terms in $\log \phi$), where the $p(x-a, y-b)$ are power series convergent in some neighborhood of (a, b); and if such a representation holds in the neighborhood of each general point (a, b) of C, with the ρ_m independent of k, a, b, x, y, then $\phi = 0$ may be called a singular curve of the *regular* type.

A point (a, b) will be called a *general intersection* of the two irreducible singular curves $\phi(x, y) = 0$, $\psi(x, y) = 0$, if it is a nonsingular point both of $\phi = 0$ and $\psi = 0$, if no other singular curve passes through it, and if the Jacobian $\partial(\phi,\psi)/\partial(x, y) \neq 0$ at (a, b). In the neighborhood of a general intersection of two singular curves $\phi = 0$, $\psi = 0$ of the regular type, a canonical fundamental system is of the form

$$\overset{m}{z}_k = \phi^{\rho_m} \psi^{\sigma_m} \overset{m}{p}_k (x - a, y - b)$$

(or logarithmic solutions). It appears that at present the general theory cannot handle a point which is an intersection of three or more singular curves, or at which two curves are in contact ($\partial(\phi, \psi)/\partial(x, y) = 0$), or which is a singular point of a singular curve ($\phi_x = \phi_y = 0$). As J. F. Ritt remarks in the introduction to his *Differential equations from the algebraic standpoint* [25], canonical forms, which are usually assumed in dealing with systems of differential equations, are inadequate for the representation of general systems since in the process of reduction to canonical forms one encounters so many exceptions, possibilities of degeneration, and possibilities of the occurrence of extraneous factors as to make the general theory somewhat less fruitful than the investigation of special systems. The study of particular examples suggests an entirely novel occurrence in the analytic theory of differential equations: there does not seem to be any fundamental system which is canonical in the entire neighborhood of such a point. In the case of a general intersection of three singular curves the examples indicate the existence of three distinct canonical systems.

Returning to the system of partial differential equations (2), we have to distinguish two cases. If $1-af$ does not vanish identically, then the system can be transformed into a system of total differential equations of the form (5); and if the compatibility conditions are satisfied for this system, (2) has four linearly independent solutions. The singular curves of (2) are, in this case, the singular curves of $a, \cdots, k$ and the curve or curves determined by $1-af=0$.

If $1-af$ vanishes identically, but the determinant (7) does not vanish identically, then the system (2) can be transformed into a system of total differential equations of the form (9), and if the compatibility conditions for this system are satisfied, (2) has three linearly independent solutions. The singular curves of (2) are, in this case, the singular curves of $a, \cdots, k$ and the zero curve or curves of the determinant (7).

There is no need to give a discussion of further special cases. It is clear that in any given case an equivalent system of total differential equations can be derived, the compatibility conditions can be checked, and the prospective singular curves of the system can be obtained. The curves so obtained may be real or apparent singular curves of the system (2), and if they are in reality singular curves of the system they may be of the regular or irregular type. It appears that no general criterion is known for finding out which of these possibilities is present in any given case. Furthermore, even if it is known that certain singular curves are regular singular curves of the system, the general theory furnishes the form of the canonical fundamental system only for a general point of regular singular curve, or for a general intersection of two singular curves. All points of a more complicated nature require a special investigation, and except for a very few examples there is little hint as to the nature of this investigation.

7. **Examples.** At this stage it seems worth while to consider a few examples and see how the general theory applies, and wherein it fails to give the required answers. We shall choose hypergeometric systems of partial differential equations, because about these a considerable amount of information is available.

The system (3) is satisfied by Appell's series

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum \sum \frac{(\alpha)_{\lambda+\mu}(\beta)_\lambda(\beta')_\mu}{(\gamma)_{\lambda+\mu}\lambda!\mu!} x^\lambda y^\mu$$

and is thus compatible. It can be written in the form (2), $1-af$ of that form vanishes identically, and apart from an irrelevant factor the determinant (7) is $x-y$, so that the system has three linearly independent solutions. Clearly the singular curves of the system are

$$x = 0, \quad x = 1, \quad x = \infty, \quad y = 0, \quad y = 1, \quad y = \infty, \quad x - y = 0,$$

and it is easy to see that they all are of the regular type. Each of the six singular points $(0, 1)$, $(0, \infty)$, $(1, 0)$, $(1, \infty)$, $(\infty, 0)$, $(\infty, 1)$ is a general intersection of two regular singular curves. In the neighborhood of each of these points (a, b) there is a fundamental system of three solutions of the canonical form

$$(x - a)^\rho (y - b)^\sigma p(x - a, y - b),$$

and these solutions have been known for a long time.

The remaining three singular points, $(0, 0)$, $(1, 1)$, and (∞, ∞), are of a very different nature. At each of these singular points (a, a) three singular curves intersect. There is indeed one solution of the form

$$(x - a)^\rho (y - a)^\sigma p(x - a, y - a)$$

for each of these singular points, but no fundamental system of three linearly independent solutions of the canonical form is available. Several investigations were carried out in order to determine an appropriate form of the fundamental system. No system can be found which is canonical for every (sufficiently small) closed Γ in the neighborhood of (a, a) and for this reason Horn [10; 11] has proposed to investigate the behavior of solutions as (x, y) *approaches* (a, a) along a straight line rather than encircles (a, a). Actually it is not really necessary to exclude closed curves Γ encircling (a, a). It turns out (Erdélyi [4]) that the entire neighborhood of the singular point (a, a) must be broken up into three parts, according as to which of the three quantities $|x-a|$, $|y-a|$, $|x-y|$ is the smallest. For a closed curve Γ which remains entirely within one of the three "conical" neighborhoods, there is a canonical fundamental system: when proceeding from one conical neighborhood to another, one must change from one canonical system to another. To begin with, one obtains six forms of canonical systems which are indicated below together with the supplementary conditions defining that part of the neighborhood of (a, a) in which they are valid.

$$(x-a)^{\rho}(y-a)^{\sigma}p\left(\frac{x-a}{y-a},\ y-a\right), \qquad \left|\frac{x-a}{y-a}\right| < 1, \tag{12}$$

$$(x-a)^{\rho}(x-y)^{\sigma}p\left(\frac{x-a}{x-y},\ x-y\right), \qquad \left|\frac{x-a}{x-y}\right| < 1, \tag{13}$$

$$(x-a)^{\rho}(y-a)^{\sigma}p\left(x-a,\ \frac{y-a}{x-a}\right), \qquad \left|\frac{y-a}{x-a}\right| < 1, \tag{14}$$

$$(x-y)^{\rho}(y-a)^{\sigma}p\left(x-y,\ \frac{y-a}{x-y}\right), \qquad \left|\frac{y-a}{x-y}\right| < 1, \tag{15}$$

$$(x-a)^{\rho}(x-y)^{\sigma}p\left(x-a,\ \frac{x-y}{x-a}\right), \qquad \left|\frac{x-y}{x-a}\right| < 1, \tag{16}$$

$$(y-a)^{\rho}(x-y)^{\sigma}p\left(y-a,\ \frac{x-y}{y-a}\right), \qquad \left|\frac{x-y}{y-a}\right| < 1. \tag{17}$$

Clearly, (12) and (13) are equivalent in that conical part of the neighborhood of (a, a) in which $|x-a| < \min(|y-a|, |x-y|)$. Similarly, (14) and (15), and also (16) and (17), coincide in one or the other of the conical neighborhoods. It should be noted, however, that there is a two-dimensional surface along which none of the six canonical systems is valid.

This example shows clearly how to construct canonical fundamental systems for the various conical parts of the neighborhood of a general intersection of three singular curves of the regular type, but in general it is rather difficult to determine the linear transformations which transform any one of these canonical systems into any other one.

As another example we choose the system of partial differential equations

$$
\begin{aligned}
&x(x+1)r - ys - y^2t + \{(\alpha+\beta'+1)x + 1 - \beta\}p \\
&\qquad\qquad + (\beta' - \alpha - 1)yq + \alpha\beta' z = 0, \\
&- x^2r - xs + y(y+1)t + (\beta - \alpha - 1)xp \\
&\qquad\qquad + \{(\alpha+\beta+1)y + 1 - \beta'\}q + \alpha\beta z = 0
\end{aligned} \tag{18}
$$

which is satisfied by Horn's series

$$G_1(\alpha, \beta, \beta'; x, y) = \sum\sum (\alpha)_{\lambda+\mu}(\beta)_{\mu-\lambda}(\beta')_{\lambda-\mu} \frac{x^\lambda}{\lambda!}\,\frac{y^\mu}{\mu!}$$

and hence is consistent. This system is not of the form (2) but it can easily be transformed into that form. An investigation shows that the system has three linearly independent solutions and that its singular curves are $x=0$, $x=\infty$, $y=0$, $y=\infty$, $x+y+1=0$, $4xy-1=0$. Each of the three points $(0, 0)$, $(0, -1)$, $(-1, 0)$ is a general intersection of two regular singularities, and a canonical fundamental system for each of these points can be obtained in the form predicted by the general theory. On the other hand, at the point $(-1/2, -1/2)$ the two singular curves $x+y+1=0$ and $4xy-1=0$ are in contact, and the general theory does not give any information there. The integration of the system by contour integrals suggests the transformation

$$
\begin{aligned}
\xi &= \frac{1}{2x}\left[1 + 2x + (1 - 4xy)^{1/2}\right], \\
\eta &= \frac{1}{2x}\left[1 + 2x - (1 - 4xy)^{1/2}\right], \\
\zeta &= (1 + x + y)^\alpha z
\end{aligned}
$$

and this transformation carries the two singular curves which have a common tangent at $x=y=-1/2$ into a *general* intersection of the *three* singular lines $\xi=0$, $\eta=0$, $\xi-\eta=0$ of the system of partial differential equations satisfied by ζ as a function of ξ and η; and we already know how to deal with such an intersection.

There are other examples showing how to "reduce," by a transformation, a singular point, for instance a cusp, of a singular curve. Generally speaking, given any type of singular point of a system of partial differential equations, it is usually easy to reduce this singular point locally, that is, to find a transformation which will map this singular point into a finite number of singular points each of which is a general intersection of two singular curves. There is no guarantee, however, that this can be performed *simultaneously* for *all* singular points. Given a finite number of algebraic curves, Professor H. Busemann informs me that in general there is no birational transformation of the x, y plane into itself which will reduce (in the fashion described above) all singular points of these curves, and all their intersections; and although it is known from algebraic geometry that the singular points of any algebraic curve can be reduced to ordinary double points by a birational transformation defined on the curve only (Riemann transformation), such transformations do not appear to be very useful in connection with differential equations.

8. Riemann's problem. We shall say that an analytic function of x is of canonical form, and has exponent ρ, at $x=a$, if it is of the form $(x-a)^{\rho}p(x-a)$ where p is a power series convergent in some neighborhood of a. When $a=\infty$, the appropriate form is $x^{-\rho}p(1/x)$. Similarly, an analytic function of x and y is of canonical form, and has exponent ρ, at the singular curve $\phi(x, y)=0$ if in the neighborhood of any general point (a, b) of that curve it is of the form $\phi^{\rho}p(x-a, y-b)$.

Riemann has shown that the differential equation

$$x(1-x)\frac{d^2z}{dx^2}+[\gamma-(\alpha+\beta+1)x]\frac{dz}{dx}-\alpha\beta z=0$$

can be identified as the differential equation satisfied by the family P of analytic functions of x described as follows.

(i) α, β, γ are given constants; $\alpha-\beta$, γ, $\gamma-\alpha-\beta$ are not integers.

(ii) Each function of P is holomorphic everywhere except possibly for $x=0, 1, \infty$.

(iii) Any three functions of P are linearly dependent.

(iv) At $x=0$ there are two functions of P which are in canonical form with exponents 0 and $1-\gamma$, similarly at $x=1$ with exponents 0 and $\gamma-\alpha-\beta$; and at $x=\infty$ with exponents α and β.

Picard [21] attempted to characterize in a similar manner the system (3) of partial differential equations satisfied by Appell's function F_1 as the system satisfied by all members of a family P of analytic functions of x and y with the following properties.

(i) $\alpha, \beta, \beta', \gamma$ are given constants, $\alpha-\beta$, $\alpha-\beta'$, $\beta+\beta'$, $\gamma-\beta$, $\gamma-\beta'$, $\gamma-\alpha-\beta$, $\gamma-\alpha-\beta'$ are not integers.

(ii) Each function of P is holomorphic, except when $x=0, 1, \infty$, or $y=0, 1, \infty$, or $x=y$.

(iii) Any four functions are linearly dependent.

(iv) At each of the singular curves there are three linearly independent functions of P with the exponents given in the table.

Singular curve	Exponents
$x = 0$	$0, 0, 1 + \beta' - \gamma$
$x = 1$	$0, 0, \gamma - \alpha - \beta$
$x = \infty$	α, β, β
$y = 0$	$0, 0, 1 + \beta - \gamma$
$y = 1$	$0, 0, \gamma - \alpha - \beta'$
$y = \infty$	α, β', β'
$x = y$	$0, 0, 1 - \beta - \beta'$

If z_1, z_2, z_3 are any three linearly independent functions of P, then any function of P satisfies the three partial differential equations

$$\begin{vmatrix} r & p & q & z \\ r_1 & p_1 & q_1 & z_1 \\ r_2 & p_2 & q_2 & z_2 \\ r_3 & p_3 & q_3 & z_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} s & p & q & z \\ s_1 & p_1 & q_1 & z_1 \\ s_2 & p_2 & q_2 & z_2 \\ s_3 & p_3 & q_3 & z_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} t & p & q & z \\ t_1 & p_1 & q_1 & z_1 \\ t_2 & p_2 & q_2 & z_2 \\ t_3 & p_3 & q_3 & z_3 \end{vmatrix} = 0.$$

Arguing on the exponents, Picard could show that these equations are of the same *form* as those satisfied by F_1, but he could not identify all constants, except by deriving the ordinary differential equation (of the third order) satisfied by any function of P when y is fixed and x varies (or vice versa), and using the theory of Fuchsian equations, in this particular case investigations by Pochhammer [22].

A similar problem relating to Appell's series F_3, whose system of partial differential equations has *four* linearly independent solutions, was discussed by Goursat [5]. The list of singular curves and exponents follows:

Singular curve	Exponents
$x = 0$	$0, 0, 1 + \alpha' - \gamma, 1 + \beta' - \gamma$
$x = 1$	$0, 0, 0, \gamma - \alpha - \beta$
$x = \infty$	$\alpha, \alpha, \beta, \beta$
$y = 0$	$0, 0, 1 + \alpha - \gamma, 1 + \beta - \gamma$
$y = 1$	$0, 0, 0, \gamma - \alpha' - \beta'$
$y = \infty$	$\alpha', \alpha', \beta', \beta'$
$x + y - xy = 0$	$0, 0, 0, 1 + \gamma - \alpha - \alpha' - \beta - \beta'$

Riemann's investigation of the hypergeometric differential equation leads to the Fuchsian theory of linear differential equations with regular singularities. A corresponding theory of systems of partial differential equations is not feasible at present because of the lack of convenient criteria for singular curves of the regular type.

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BOUNDED ANALYTIC FUNCTIONS

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If D is a domain in the complex z-plane, then the family $B=B(D)$ of bounded analytic functions in D is defined as consisting of those analytic functions $f(z)$ which are regular and single-valued in D and which satisfy the inequality $|f(z)|<1$ at all points of D. The classical investigations of the family $B(D)$ were restricted to the case in which D is a simply-connected domain. In fact, D was generally taken to be the interior of the unit circle, a restriction which is apparent rather than real since most properties of bounded functions are either invariant with respect to a conformal mapping of D, or else are transformed in a simple manner. The use of the simple properties of the unit circle led to a large number of results which are distinguished both by their elegance and their preciseness. However, since the proofs leading to these results lean heavily on the special features of the unit circle, they gave little or no indication as to their possible generalization to the case of bounded functions in multiply-connected domains.

In the classical treatment of bounded functions, the family B was occasionally replaced by the more general class of analytic functions $w=f(z)$ whose values—for $z\in D$—are contained in a specified simply-connected domain D' in the w-plane. The family B corresponds to the case in which D' is the unit circle $|w|<1$. Other special cases are the family of functions with a positive real part—to be denoted by $P=P(D)$—obtained if D' is the right half-plane $\operatorname{Re}\{w\}>0$, and the family of functions with a bounded real part—denoted by $B_R=B_R(D)$—for which D' is the infinite strip $-1<\operatorname{Re}\{w\}<1$. These families are obtained from B by means of the conformal transformations which carry $|w|<1$ into the various domains D'. For instance, we have

$$g(z)=\frac{1+f(z)}{1-f(z)},\qquad f(z)\in B,\ g(z)\in P, \tag{1}$$

$$\phi(z)=\frac{4}{\pi}\arctan f(z),\qquad f(z)\in B,\ \phi(z)\in B_R. \tag{2}$$

Apart from their intrinsic interest, these classes are often useful in the investigation of the functions of B since the special features of

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the classes P and B_R make it possible to use certain simple yet powerful techniques whose adaptation to the class B would be a matter of considerable difficulty.

The novel features encountered in the attempt to generalize the classical results on bounded functions to the case of multiply-connected domains are due to the failure of the monodromy theorem. We are therefore faced by two different types of problems, according as we consider the family of all functions which are regular and bounded in D, or confine ourselves to those functions which are, in addition, single-valued in D. The first case introduces no new features since, by considering the function on the universal covering surface of D, we may reduce it to the simply-connected case. In the case of single-valued functions, however, this device is obviously unable to yield precise results, and different methods have to be used.

The first to consider problems of this kind seem to have been Carlson [5][1] and Teichmüller [20] who obtained a sharpened version of Hadamard's three circle theorem for functions which are single-valued in a circular ring, and Heins [11] who considered a number of extremal problems for functions regular and single-valued in a doubly-connected domain. The case of bounded functions in domains of arbitrary finite connectivity was first treated by Grunsky [9, 10], who considered generalizations of the lemmas of Schwarz, Julia, and Loewner. The extension of Schwarz' lemma aimed at by Grunsky consists in the following problem: Given a domain D of connectivity n and two distinct points—say ζ and z_0—of D, to find a function $f_0(z)$ of $B(D)$ such that $f_0(\zeta)=0$ and $|f(z_0)| \leqq |f_0(z_0)|$, where $f(z)$ is any other function of $B(D)$ which vanishes at ζ. The existence of such a function $f_0(z)$ follows, of course, from the compactness of the family $B(D)$. Grunsky showed [10] that the extremal function $w=f_0(z)$ yields a conformal mapping of D onto the n-times covered unit circle $|w|<1$.

The existence of such mappings had been conjectured by Riemann and it was proved later [4, 8] that there exist a large variety of essentially different conformal mappings of a domain of connectivity n onto the m-times covered unit circle, where $m \geqq n$. In [10] it is shown that any function of B can be approximated by functions mapping D onto the multiply-covered unit circle and that it is therefore sufficient to consider functions of this latter type. The solution of the problem $|f(z_0)| = \max$, $f(\zeta)=0$ among these functions is then shown to have not more than $n-1$ zeros in D other than ζ, and this is equivalent to the characterization of the function $f_0(z)$ given above.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

The fact that the extremal function yields a $(1, n)$ mapping of D onto the unit circle is not sufficient to characterize the extremal function completely, since there exist an infinity of essentially different mappings of this type. The problem thus remains of characterizing the extremal mapping within this class. This problem was solved by Ahlfors [1], who considered a generalization of the Schwarz lemma slightly different from the one mentioned above, namely, the problem: $f(z) \in B(D)$, $\zeta \in D$, $|f'(\zeta)| = \max$. Ahlfors shows that the extremal function $F(z)$ yields a $(1, n)$ mapping of D onto the unit circle and that the zeros $\zeta, z_1, z_2, \cdots, z_{n-1}$ of $F(z)$ in D are distinguished by the fact that there exists a differential $p'(z)dz$ such that

$$\frac{1}{i} p'(z)dz > 0, \qquad z \in C \tag{3}$$

(C being the boundary of D), where $p'(z)$ is regular in D except for a simple pole of residue 1 at ζ and $z_1, \cdots, z_{n-1}$ are the $n-1$ zeros of $p'(z)$ in D (the fact that a function $p'(z)$ with one pole in D and with the boundary behavior (3) has precisely $n-1$ zeros in D is an immediate consequence of the argument principle). We remark that in the case in which the boundary C of D consists of n closed analytic curves—and it is sufficient to consider this case—it follows from (3) and the Schwarz reflection principle that $p'(z)$ is regular on C.

The proof of this property of the extremal is based on the observation that the function $\log |F(z)|$ is—apart from a number of logarithmic poles—a harmonic function in D whose harmonic conjugate has periods about the boundary components of D which are integral multiples of 2π. By slight variations of the locations of the logarithmic poles—which amounts to adding and subtracting Green's functions of D with suitable points of reference—and by variations of the boundary values of $\log |F(z)|$, a function $\log |F^*(z)|$ is obtained. In order to insure that $F^*(z)$ is also single-valued in D and is thus a competing function in the original problem, it is necessary to carry the condition that the periods of the harmonic conjugate of $\log |F(z)|$ remain unchanged in these variations. This is achieved by a Lagrange multiplier technique, which finally leads to the above result. In a more recent paper [2], Ahlfors shows that a similar procedure can be applied to bounded functions on an open Riemann surface of genus higher than zero, and that it leads to similar results. The extremal map is again a covering surface of the unit circle; if p is the genus of the surface and n is the number of its boundary components, then the number m of the sheets of the extremal map satisfies $n \leqq m \leqq n + 2p$.

The investigation of the extremal problem $f(z) \in B(D)$, $|f'(\zeta)| = \max$ was carried further by Garabedian [6] who showed that there exists an interesting connection between this problem and the problem

$$\int_C |h(z)| ds = \min, \tag{4}$$

where $h(z)$ is regular in D except for a double pole at ζ with the principal part $(z-\zeta)^{-2}$, and where the boundary behavior of $h(z)$ is such that the integral in (4) exists. To see this, consider the function $q(z)$ defined by

$$q(z) = \frac{p'(z)F'(\zeta)}{F(z)}, \tag{5}$$

where $F(z)$ is Ahlfors' extremal function and $p'(z)$ is the function appearing in (3). Since the $n-1$ zeros of $p'(z)$ coincide with the zeros of $F(z)$ other than $z=\zeta$, $q(z)$ has no zeros and is regular in D except for a double pole at $z=\zeta$. It follows from (3) and (5) that the boundary relation

$$-iF(z)q(z)dz > 0, \qquad z \in C, \tag{6}$$

holds. The principal part of $q(z)$ at $z=\zeta$ is $(z-\zeta)^{-2}$. Indeed, if there were a term of the form $\alpha(z-\zeta)^{-1}$, it would follow from (6) and the fact that $|F(z)|=1$ on C that[2]

$$\begin{aligned} 2\pi\alpha &= \frac{1}{i}\int_C q(z)dz = \frac{1}{i}\int_C [F(z)]^* F(z)q(z)dz \\ &= \frac{1}{i}\left[\int_C F^2(z)q(z)dz\right]^*, \end{aligned}$$

and this vanishes since, in view of $F(\zeta)=0$, the integrand is regular in D.

The fact that $F(z)$ is the extremal function of the generalized Schwarz lemma and that $q(z)$ solves the problem (4) was deduced by Garabedian from (6) in the following elegant manner. If $f(z)$ is a function of $B(D)$, it follows from the residue theorem and the inequality (6) that

---

[2] Complex conjugates are denoted by asterisks.

$$
\begin{aligned}
|f'(\zeta)| &= \left|\frac{1}{2\pi i}\int_C f(z)q(z)dz\right| \leqq \frac{1}{2\pi}\int_C |q(z)dz| \\
&= \frac{1}{2\pi}\int_C \left|\frac{1}{i}F(z)q(z)dz\right| = \frac{1}{2\pi i}\int_C F(z)q(z)dz = F'(\zeta),
\end{aligned} \tag{7}
$$

where the fact that $|F(z)| = 1$, $z \in C$, has been used. This shows the extremal property of $F(z)$. As to the problem (4), it follows from the properties of the functions $h(z)$ that

$$\int_C |h(z)|\,ds \geqq \left|\int_C h(z)F(z)dz\right| = 2\pi F'(\zeta).$$

By (7), we have

$$\int_C |q(z)|\,ds = 2\pi F'(\zeta), \tag{8}$$

and therefore

$$\int_C |q(z)|\,ds \leqq \int_C |h(z)|\,ds.$$

Thus, $q(z)$ solves the extremal problem (4), and the extremal values of (4) and of the quantity associated with the generalization of Schwarz' lemma are connected by the simple relation (8).

The fact that our maximum problem in the theory of bounded functions is associated with a "dual" minimum problem of the type (4) is not an isolated occurrence. This duality was recognized by Fr. Riesz [18] in a variety of extremal problems concerned with bounded functions in the unit circle. In all these cases, the two associated extremal functions form a positive differential in the manner of (6), and their extremal properties follow from the positivity of the differential by judicious use of the residue theorem. In the case of the unit circle, the existence of pairs of functions with suitable properties which form positive differentials is trivial and they can easily be written down explicitly in terms of elementary functions. In the case of general multiply-connected domains D this is not true any more and the relevant existence theorems appear to lie somewhat deeper than the existence theorems for the usual harmonic and analytic domain functions of D. A direct, but rather difficult, proof for the existence of two functions $F(z)$ and $q(z)$ which are connected by (6) and have the other properties specified above was given by Garabedian and Schiffer [7]. A short proof which, however, uses the existence of the Green's function of D can be found in [15].

Another important aspect of the problem $f(z)\in B(D)$, $|f'(\zeta)| = \max$ is the connection between the extremal $F(z)$ and the Szegö kernel function [19] established in [6]. It is not difficult to show that the function $q(z)$ in (6) can be written in the form $q(z)=4\pi^2L^2(z, \zeta)$, where $L(z, \zeta)$ is regular and single-valued in $D+C$ except for a simple pole of residue $(2\pi)^{-1}$ at $z=\zeta$. If we denote the regular function $F(z)L(z, \zeta)$ by $K(z, \zeta)$, that is, we write

$$F(z) = \frac{K(z, \zeta)}{L(z, \zeta)}, \tag{9}$$

we can deduce from (6) that $K(z, \zeta)$ and $L(z, \zeta)$ are connected by the relation

$$[K(z, \zeta)]^*ds = -iL(z, \zeta)dz, \qquad z\in C. \tag{10}$$

If $\phi(z)$ is any function which is regular and single-valued in D and such that $|\phi(z)|^2$ can be integrated over C, it follows from (10) and the residue theorem that

$$\int_C [K(z, \zeta)]^*\phi(z)ds = \frac{1}{i}\int_C L(z, \zeta)\phi(z)dz = \phi(\zeta).$$

The function $K(z, \zeta)$ has therefore the characteristic reproducing property

$$\phi(\zeta) = \int_C [K(z, \zeta)]^*\phi(z)ds$$

of the Szegö kernel function [19] of D and is thus identical with it. Since, by (9),

$$F'(\zeta) = 2\pi K(\zeta, \zeta), \tag{11}$$

and since kernel functions can be numerically computed in terms of complete orthonormal sets of functions [3], (11) yields the complete numerical solution of the problem $f(z)\in B(D)$, $|f'(\zeta)| = \max$.

As pointed out above, it is useful to consider also functions of the classes P and B_R, which are related to the functions of B by the equations (1) and (2), respectively. The special properties of these classes make it possible to devise methods for the solution of extremal problems which are considerably simpler than those required for a direct attack on the class B [14, 17]. To illustrate the method appropriate for problems within the class P [14], we consider the extremal problem $g(z)\in P$, $g(\zeta)=1$, $|g'(\zeta)| = \max$ which is identical with the generalization of Schwarz' lemma discussed further above. From a trivial transformation it follows that it is sufficient to consider the problem $g'(\zeta)=\max$ within the subclass of functions of P for which $g'(\zeta)>0$. Let now $-ir'(z)dz$ be a non-negative differential on C, that is,

$$-ir'(z)dz \geqq 0, \qquad z \in C, \tag{12}$$

where $r'(z)$ is regular and single-valued in D (and, by the reflection principle, also on C) except at $z=\zeta$ where

$$r'(z) = -\frac{1}{(z-\zeta)^2}+\frac{\alpha}{z-\zeta}+r_1(z) \tag{13}$$

and $r_1(z)$ is regular. In view of (12), α is necessarily positive. If $g(z)$ is a function of P which is integrable over C, it follows from (12), (13), and the residue theorem that

$$\begin{aligned}\mathrm{Re}\,\{\alpha - g'(\zeta)\} &= \mathrm{Re}\left\{\frac{1}{2\pi i}\int_C g(z)r(z)dz\right\} \\ &= \frac{1}{2\pi i}\int_C \mathrm{Re}\,\{g(z)\}r(z)dz \geqq 0,\end{aligned}$$

where the fact that $\mathrm{Re}\,\{g(z)\}\geqq 0$ has been used. Since both α and $g'(\zeta)$ are positive we find that

$$g'(\zeta) \leqq \alpha. \tag{14}$$

In order to obtain the best estimate of this type, we have to use the function (13) for which $\alpha=\min$ under the condition (12). This minimum problem presents no difficulties since the positive differentials (12) form an n-parameter family which can be expressed in terms of the Green's function and the harmonic measures of D. It is found that the minimizing differential is characterized by the fact that the associated function $r'(z)$ has a double zero—say z_ν—on each boundary component C_ν $(\nu=1,\cdots,n)$ of D. For the α belonging to this differential, the inequality (14) will then be sharp, and the extremal $w=G(z)$ of our original problem will necessarily yield a $(1, n)$ mapping of D onto the right half-plane $\mathrm{Re}\,\{w\}>0$. Indeed, as shown in [4, 8], there exists a completely determined mapping $w=f(z)$ of D onto the n-times covered unit circle such that $f(\zeta)=0$, $f'(\zeta)>0$, and $f(z_\nu)=1$, $\nu=1,\cdots,n$, where $z_\nu\in C_\nu$. Passing to the class P by means of (1), we obtain a function $w=G(z)$ which yields a $(1, n)$ mapping of D onto $\mathrm{Re}\,\{w\}>0$ and whose only singularities on C are simple poles at the points z_ν. Since these are the zeros of the minimizing differential $r'(z)$, we may evaluate the integral

$$\frac{1}{2\pi i}\int_C G(z)r'(z)dz$$

by the residue theorem. In view of (12), (13), and the fact that $\operatorname{Re}\{G(z)\}=0$ for $z\in C$, $z\neq z_\nu$, we obtain $\alpha-G'(\zeta)=0$, which shows that (14) is sharp and that $G(z)$ solves our problem.

By using positive differentials which show a singular behavior different from that indicated in (13), it is possible to solve a large variety of extremal problems within the family P. In all these cases, the extremal functions yield mappings of D onto covering surfaces of the right half-plane [14]. It is interesting to note that while the extremal function of the Schwarz lemma is essentially unique, there will be no uniqueness in the case of those problems whose associated positive differentials have more than two poles in D.

In the case of the class B_R, that is, the class of the functions $\phi(z)$ which are regular and single-valued in D and satisfy $|\operatorname{Re}\{\phi(z)\}|<1$, different methods are indicated [17]. If we write $\phi(z)=u(z)+iv(z)$, where $u(z)$ and $v(z)$ are real, we are dealing with those harmonic functions $u(z)$ which satisfy $|u(z)|<1$ in D and which possess a single-valued harmonic conjugate. The latter condition is equivalent to the $n-1$ conditions

$$\int_C u(z)\frac{\partial\omega_\nu(z)}{\partial n}ds = 0, \qquad \nu = 1,\cdots,n-1, \tag{15}$$

where $\omega_\nu(z)$ is the harmonic measure associated with C_ν, that is, $\omega_\nu(z)$ is harmonic in D and has the boundary values $\delta_{\nu\mu}$ on C_μ. To illustrate the method of [17] by a simple example, we consider the problem $\phi(\zeta)=0$, $|\phi(\eta)|=\max$ (ζ, $\eta\in D$, $\zeta\neq\eta$) which is equivalent to the generalization of Schwarz' lemma treated in [10]. From trivial transformations it follows that it is sufficient to consider the problem $u(\zeta)=0$, $u(\eta)=\max$. Using Green's formula and taking into account the conditions (15), we have

$$\begin{aligned} u(\eta) = -\frac{1}{2\pi}\int_C u(z)\Bigg[&\frac{\partial g(z,\eta)}{\partial n}+\alpha\frac{\partial g(z,\zeta)}{\partial n} \\ &+\sum_{\nu=1}^{n-1}\lambda_\nu\frac{\partial\omega_\nu(z)}{\partial n}\Bigg]ds, \end{aligned} \tag{16}$$

where $g(z,\eta)$ is the Green's function of D and α and the λ_ν are arbitrary real parameters. Since $|u(z)|\leq 1$, it follows from (15) that

$$
\begin{aligned}
u(\eta) &\leqq \frac{1}{2\pi}\int_C \left| \frac{\partial g(z,\eta)}{\partial n} + \alpha \frac{\partial g(z,\zeta)}{\partial n} + \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial \omega_\nu(z)}{\partial n} \right| ds \\
&= \frac{1}{2\pi}\int_C |P|\, ds,
\end{aligned} \tag{17}
$$

where P is an abbreviation for

$$
P = \frac{\partial g(z,\eta)}{\partial n} + \alpha \frac{\partial g(z,\zeta)}{\partial n} + \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial \omega_\nu(z)}{\partial n}, \qquad z \in C. \tag{18}
$$

To obtain the best possible inequality (17), we minimize the right-hand side of (17) with respect to the arbitrary parameters $\alpha, \lambda_1, \cdots, \lambda_{n-1}$. This leads to the conditions

$$
\int_C \operatorname{sgn} P \frac{\partial g(z,\zeta)}{\partial n}\, ds = 0,
$$

$$
\int_C \operatorname{sgn} P \frac{\partial \omega_\nu(z)}{\partial n}\, ds = 0, \qquad \nu = 1, \cdots, n-1,
$$

where sgn P is equal to 1 or -1, according as P is positive or negative, and remains undetermined if $P=0$. If we introduce a harmonic function $V(z)$ by the Dirichlet problem

$$
U(z) = -\operatorname{sgn} P, \quad z \in C, P \neq 0, \qquad |U(z)| \leqq 1, \quad z \in D, \tag{19}
$$

these conditions read

$$
\int_C U(z) \frac{\partial g(z,\zeta)}{\partial n}\, ds = 0, \tag{20}
$$

$$
\int_C U(z) \frac{\partial \omega_\nu(z)}{\partial n}\, ds = 0, \qquad \nu = 1, \cdots, n-1. \tag{21}
$$

(20) shows that $U(\zeta)=0$, while a comparison of (21) and (15) shows that the harmonic conjugate $V(z)$ of $U(z)$ is single-valued in D. The analytic function $\phi(z) = U(z) + iV(z)$ thus belongs to B_R. From (19) and the easily demonstrable fact that an expression of the type (18) cannot have more than $2n$ zeros on C, it further follows that $w=\phi(z)$ yields a $(1, n)$ mapping of D onto the infinite strip $-1 < \operatorname{Re}\{w\} < 1$. The function $\phi(z)$ solves our extremal problem. Indeed, we have $U(\zeta)=0$ and, in view of (17), (19), and (18),

$$
u(\eta) \leqq \frac{1}{2\pi}\int_C |P|\, ds = -\frac{1}{2\pi}\int_C U(z) P\, ds = U(\eta).
$$

This method is easily extended to the case in which the class B_R is restricted by various types of side conditions. As an example, we mention the problem: $\phi(z) \in B_R$, $\phi(a_\mu) = b_\mu$, $\mu = 1, \cdots, m$, $|\phi(\zeta)| = \max$ where we have to require, of course, that there exist at least one function of B_R which satisfies the given interpolation conditions. The extremal is found to yield a $(1, k)$ mapping of D onto the strip $-1 < \operatorname{Re}\{w\} < 1$, where $n \leqq k \leqq n+m-1$. This result, which represents a generalization of the well known Pick-Nevanlinna interpolation theory, was first proved—by different methods—by Garabedian [6]. Another derivation, based on a lemma on positive harmonic functions, has recently been given by Heins [12].

By suitable modifications of the basic method of proof in [17], a large number of extremal problems in the class B_R and in related classes can be treated. For instance, if $\phi(z) \in B_R$, $\phi'(\zeta) = A$ is given and η is such that $\phi'(\eta)$ cannot vanish, then the problem $|\phi'(\eta)| = \min$ is solved by a function mapping D onto the m-times covered strip $-1 < \operatorname{Re}\{w\} < 1$, where $n \leqq m \leqq n+2$. A similar result is obtained if the condition $|\phi'(\eta)| = \min$ is replaced by $|\phi(u) - \phi(v)| = \min$, where the points u and v are such that $\phi(u)$ and $\phi(v)$ cannot coincide if $\phi'(\zeta) = A$. From the last two results it is not difficult to deduce a result concerning the radius of univalence about a given point ζ of those functions $f(z)$ of B for which $|f'(\zeta)| \geqq A$, where A is a given positive number. It is found that, among the functions satisfying these conditions, the smallest radius of univalence is obtained for a function mapping D onto the m-times covered unit circle, where $n \leqq m \leqq n+2$. This generalizes a classical result of Landau [13] for the case $n = 1$, in which $m = 2$.

The class B can be generalized in a number of ways. One generalization is obtained by replacing the condition $|f(z)| < 1$, $z \in D$, by $\limsup_{z \to z_0} |f(z)| \leqq \lambda(z_0)$, where $\lambda(z_0)$ is a positive continuous function on each boundary component of D and z approaches the boundary point z_0 from inside D [16]. The discussion of the problem $|f'(\zeta)| = \max$ in this class of functions leads to two domain functions $K_\lambda(z, \zeta)$ and $L_\lambda(z, \zeta)$ which generalize the functions $K(z, \zeta)$ and $L(z, \zeta)$ associated with the analogous problem in the class B. $K_\lambda(z, \zeta)$ is regular in D, $L_\lambda(z, \zeta)$ is regular in D except for a simple pole of residue $(2\pi)^{-1}$ at $z = \zeta$, and on the boundary we have the identity

$$[K_\lambda(z, \zeta)]^* ds = -i\lambda(z) L(z, \zeta) dz, \qquad z \in C, \tag{22}$$

which generalizes (10). The maximal value of $|f'(\zeta)|$ is given by $2\pi K_\lambda(\zeta, \zeta)$. If $g(z)$ is regular and single-valued in D and $|g(z)|$ is square-integrable on C, it follows from (22) and the residue theorem that

$$\int_C \lambda^{-1}(z) g(z) [K_\lambda(z, \zeta)]^* ds = g(\zeta).$$

This shows that $K_\lambda(z, \zeta)$ is a kernel function which can be computed by the expansion

$$K_\lambda(z, \zeta) = \sum_{\nu=1}^{\infty} g_\nu(z) [g_\nu(\zeta)]^*$$

where the $g_\nu(z)$ are a complete set of functions with the same properties as $g(z)$ which are orthonormalized by the conditions

$$\int_C \lambda^{-1}(z) g_\nu(z) [g_\mu(z)]^* ds = \delta_{\nu\mu}.$$

Another generalization of the class B is obtained by extending B to include, in addition to bounded analytic functions, also bounded complex harmonic functions which are not necessarily analytic. This class, say B', will thus consist of functions of the form $f(z)+g^*(z)$, where $f(z)$ is analytic and $g^*(z)$ is the complex conjugate of an analytic function, such that

$$|f(z) + g^*(z)| < 1, \qquad z \in D. \tag{23}$$

Both $f(z)$ and $g(z)$ are supposed single-valued in D.

The treatment of the class B' can be modelled on that of the class B_R. To illustrate the procedure, we consider the problem $|f'(\zeta) + [g'(\zeta)]^*| = \max$, $\zeta \in D$. By Green's formula, we have

$$\begin{aligned} f'(\zeta) &+ [g'(\zeta)]^* \\ &= -\frac{1}{2\pi} \int_C [f(z) + g^*(z)] \left[\frac{\partial^2 g(z, \zeta)}{\partial n \partial \xi} + \sum_{\nu=1}^{n-1} \frac{\partial \omega_\nu(z)}{\partial n} \right] ds, \qquad (24) \\ \zeta &= \xi + i\eta, \end{aligned}$$

where the arbitrary complex constants λ_ν take account of the fact that $f(z)$ and $g(z)$ are single-valued in D. By (23), it follows that

$$|f'(\zeta) + [g'(\zeta)]^*| \leq \frac{1}{2\pi} \int_C |P| \, ds, \tag{25}$$

where

$$P = \frac{\partial^2 g(z, \zeta)}{\partial n \partial \xi} + \sum_{\nu=1}^{n-1} \lambda_\nu \frac{\partial \omega_\nu(z)}{\partial n}. \tag{26}$$

We now minimize the right-hand side of (25) with respect to the arbitrary complex parameters λ_ν. This leads to the necessary conditions

$$\int_C \exp\{i \arg P\} \frac{\partial \omega_\nu}{\partial n} ds = 0, \qquad \nu = 1, \cdots, n-1.$$

If we define a complex harmonic function $h(z) = u(z) + iu_1(z)$ by the boundary value problem

$$h(z) = \exp\{-i \arg P\}, \qquad z \in C, \tag{27}$$

this condition reads

$$\int_C u \frac{\partial \omega_\nu}{\partial n} ds + i \int_C u_1 \frac{\partial \omega_\nu}{\partial n} ds = 0, \qquad \nu = 1, \cdots, n-1.$$

This shows that the harmonic conjugates $v(z)$ and $v_1(z)$ of $u(z)$ and $u_1(z)$, respectively, are single-valued. Hence, $2u(z) = p(z) + p^*(z)$ and $2iu_1(z) = q(z) - q^*(z)$, where the analytic functions $p(z)$ and $q(z)$ are single-valued in D. The function $h(z)$ defined by (27) is therefore of the form

$$\begin{aligned} h(z) &= \frac{1}{2}[p(z) + q(z)] + \frac{1}{2}[p(z) - q(z)]^* \\ &= F(z) + G^*(z), \end{aligned} \tag{28}$$

where $F(z)$ and $G(z)$ are regular and single-valued in D. Since, by (27), $|h(z)| = 1$ on C and since the maximum principle is also valid for complex harmonic functions, it follows thus that $h(z)$ belongs to B'.

That the function $h(z)$ solves our extremal problem is now easily seen. In view of (27), we have

$$|P| = Ph(z), \qquad z \in C. \tag{29}$$

Inserting this in (25), and observing (24), (26), and (28), we obtain

$$|f'(\zeta) + [g'(\zeta)]^*| \leqq F'(\zeta) + [G'(\zeta)]^*,$$

which expresses the extremal property. As in the case of the corresponding problem within the class B, the extremal function is associated with a positive differential. Since $g(z, \zeta)$ and $\omega_\nu(z)$ are constant on each boundary component of D, the expression P defined in (26) satisfies

$$Pds = -iQ(z)dz, \qquad z \in C,$$

where $Q(z)$ is a single-valued analytic function which is regular in D except for a double pole with the meromorphic part $(z-\zeta)^{-2}$. Combining this with (29) and (28), we find

$$-i[F(z)+G^*(z)]Q(z)dz \geqq 0, \qquad z \in C,$$

a boundary relation analogous to (6).

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WASHINGTON UNIVERSITY

THE JUNE MEETING IN PULLMAN

The four hundred seventy-first meeting of the American Mathematical Society was held at the State College of Washington, Pullman, Washington, on Saturday, June 16, 1951, following the meeting on Friday of the Pacific Northwest Section of the Mathematical Association of America.

Approximately 65 persons attended the meetings including the following 43 members of the Society:

H. A. Antosiewicz, T. M. Apostol, R. W. Ball, C. R. Ballantine, J. P. Ballantine, R. A. Beaumont, J. S. Biggerstaff, Z. W. Birnbaum, J. L. Bottsford, J. L. Brenner, L. G. Butler, Harold Chatland, P. A. Clement, K. L. Cooke, C. M. Cramlet, D. B. Dekker, W. F. Donoghue, A. L. Duquette, Trevor Evans, R. E. Fullerton, K. S. Ghent, J. W. Green, F. L. Griffin, M. E. Haller, A. E. Halteman, I. M. Hostetter, R. D. James, J. M. Kingston, M. S. Knebelman, R. B. Leipnik, J. J. Livers, A. T. Lonseth, R. E. Lowney, L. H. McFarlan, H. B. Mann, A. F. Moursund, D. C. Murdoch, Ivan Niven, L. J. Paige, M. F. Ruchte, J. R. Vatnsdal, R. M. Winger, Fumio Yagi.

The meeting opened Saturday morning with a session for contributed papers, Professor M. S. Knebelman presiding. Following this was the invited address, *The asymptotic density of sequences*, by Professor Ivan Niven, of the University of Oregon. Professor R. D. James presided at the invited address.

On Friday evening there was a joint dinner for the Society and Association, followed by a social hour in the Library. On Saturday afternoon, those attending were entertained for tea at the home of Professor Knebelman.

Following the invited address, a resolution was passed to request the Council of the Society to schedule a meeting in Eugene, Oregon in June, 1952, in conjunction with the June meeting there of the Pacific Northwest Section of the Mathematical Association of America, and in connection with celebration of the seventy-fifth anniversary of the founding of the University of Oregon.

Abstracts of papers presented at the meeting follow. Papers whose titles are followed by *t* were presented by title. Mr. McGregor's paper was presented by Professor R. D. James. Mr. Kiefer was introduced by Professor Jacob Wolfowitz, Mr. Liao by Professor E. H. Spanier.

ALGEBRA AND THEORY OF NUMBERS

399*t*. T. A. Botts: *Representation of algebras by filters.*

A very simple construction by abstract filters, equivalent to a construction of Dunford's [Nelson Dunford and M. H. Stone, *On the representation theorem for Boolean algebras*, Revista de Ciencias (Lima) vol. 43 (1941) pp. 447–453] by well-ordering, is given for a homomorphic point-set representation of any commutative, associative, and idempotent multiplicative system with zero, the construction being made so as to yield classical representation theorems for distributive lattices and for Boolean rings. (Received April 23, 1951.)

400. J. L. Brenner: *The problem of unitary equivalence.*

In this article, a canonical form is defined for an arbitrary (singular or nonsingular) n^2 matrix A_1 over the complex numbers, under unitary transformation. The group of matrices which fix the canonical form is found, and more generally a canonical form is defined for any matrix under transformation by any such group. This gives an effective method for considering the simultaneous equations $XA_iX^*=B_i$, $XX^*=1$ $(i=1, 2, \cdots)$. The steps in the inductive definition of a canonical form involve either rational operations or the finding of characteristic roots of matrices of order n or less. The first step in the analysis is to transform A_1 so that $A_1A_1^*$ is diagonal. This article will be published in Acta Mathematica. (Received April 30, 1951.)

401*t*. D. R. Fulkerson: *Hermite forms of row-finite matrices.*

Let $\mathfrak{M}$ denote the ring of row-finite matrices over a field F. If $A\in\mathfrak{M}$, there exists a nonsingular $P\in\mathfrak{M}$ such that $PA=H$ has the following form: the nonzero rows ϕ_i of H have lengths l_i with $l_i<l_{i+1}$; ϕ_i has 1 in the l_i position, zeros in the positions $l_1, \cdots, l_{i-1}$. The matrix H is said to be an Hermite form of A. Two Hermite forms are left associates if and only if they differ by a permutation of rows, and consequently arbitrary A, $B\in\mathfrak{M}$ are left associates if and only if they have the same row space and nullity. From an Hermite form of A one readily deduces a theorem due to Toeplitz: A is equivalent to T where each row of T has at most one 1, zeros elsewhere, and the nonzero rows have increasing lengths. T is referred to as a Toeplitz form of A. Two Toeplitz forms are equivalent if and only if they differ by a permutation of rows and columns, whence A and B are equivalent if and only if they have the same rank, nullity, and defect. The reduction to Toeplitz form is used to show that $\mathfrak{M}$ is regular and has only one proper two-sided ideal. Finally, row-finite linear systems of equations are treated by reducing the matrix of coefficients to Hermite form. (Received April 26, 1951.)

402*t*. R. A. Macauley: *Polynomial valuations.*

If R is a field on which all (nonarchimedean) valuations are known, then all valuations on $R[x]$, where x is transcendental over R, are also known. Ostrowski described such valuations of $R[x]$ by means of pseudo-convergent sequences in the algebraic completion A of R. MacLane showed that if all valuations of R are discrete, then any valuation V of $R[x]$ can be represented by certain "key" polynomials in $R[x]$. The present paper exhibits the connection between these two treatments. This is achieved by first determining keys for the valuation which a pseudo-convergent sequence defines on $A[x]$, and then relating these keys to those for V. (Received May 1, 1951.)

403. H. B. Mann: *On the number of integers in the sum of two sets of positive integers.*

Let α be the density of the set A, $\bar{\alpha}$ its asymptotic density, α^* its Besicovitch density, and α_1 its modified density as defined by Erdös (Ann. of Math. vol. 43 (1942) pp. 65–68). Let further $A(n)$ denote the number of positive integers in A that do not exceed n. It is proved if $A+B=C$ and if n is not in C then (1) $C(n) \geqq \alpha_1 n + B(n)$. This is an improvement over a previous result of the author (Ann. of Math. vol. 43 (1942) pp. 523–527) according to which $C(n) \geqq \alpha^* n + B(n)$. Equation (1) immediately yields: $\bar{\gamma} \geqq \alpha_1 + \bar{\beta}$. The latter inequality is sometimes better and sometimes not as good as Erdös' $\bar{\gamma} \geqq \bar{\alpha} + \bar{\beta}/2$, which however has been established only if there exists a number $b \subset B$ such that $b+1 \subset B$. (Received May 4, 1951.)

404*t*. M. A. Melvin: *Metric classification of the bounded affinity groups.*

Consider all groups of bounded affine transformations about a fixed point in a real or complex euclidean space of n dimensions (n finite or infinite). Both finite and infinite orders occur. For many geometric and physical purposes (see, for instance, the point of view developed in *Symmetry and affinity of electromagnetic fields charges and poles*, M. Avramy Melvin, Proceedings of the Canadian Mathematical Congress 1949 (in press); there, however, most of the groups involved are unbounded, and will be discussed in a subsequent paper), the usual replacement of any such bounded affinity group (A) by its affinely equivalent unitary group (U) is not desirable. This is because such a transformation effaces the *dilatometric* properties of the affine group, replacing the component strain ("ellipsoid" or Hermitian matrix) associated with each affine transformation by the identity. Thus we propose a further classification of affinity groups in which $T^{-1}(A)T$ is considered equivalent to (A) only when T is unitary. This determines the *metric class* to which (A) belongs, and which is included with other metric classes in the broader *affine class* represented by (U). A general prescription will be given, and applied, for the construction of the metric classes associated with any given affine class. Except for the trivial case $n=1$, there are infinitely many such metric classes. However, for a finite-dimensional group the number of "degrees of freedom" of this infinity, or the *metric multiplicity* of the affine class, is a finite integer. (Received May 2, 1951.)

405. L. J. Paige: *Simultaneous reduction of matrices to triangular form.*

The author proves: A necessary and sufficient condition for two matrices A and B to have an eigenvector in common is that A and B commute on an invariant subspace L of A and B. A simple inductive proof of Frobenius' theorem regarding the eigenvalues of $f(A, B)$ is then given. The same method may be used to obtain a result of McCoy (Bull. Amer. Math. Soc. (1936) p. 593). (Received May 3, 1951.)

406*t*. G. N. Raney: *Completely distributive complete lattices*. Preliminary report.

If L is a complete lattice of which it is assumed only that meets are completely distributive with respect to joins, then the mapping which takes every M-closed subset of L into its join is a complete-homomorphism of the lattice of M-closed subsets of L onto L. This is based on the fact that in a complete lattice the intersection of any family of M-closed subsets is the same as the set of meets of selections over the family. Furthermore, the family of M-closed subsets of any lattice is a complete ring of sets. The lattice of a complete ring of sets is complete and completely distributive, and the image of a completely distributive complete lattice under a complete-homomorphism is a completely distributive complete lattice. Therefore, L is completely distributive and, in particular, joins are completely distributive with respect to meets in L. In this manner it is shown that either condition for complete distributivity implies the dual condition, and that a lattice is complete and completely distributive if and only if it is a complete-homomorphic image of a complete ring of sets. (Received April 19, 1951.)

ANALYSIS

407. T. M. Apostol: *On the Lerch zeta function.*

The function $\phi(x, a, s) = \sum_{n=0}^{\infty} \exp(2n\pi ix)(n+a)^{-s}$ satisfies $\phi(x, a, 1-s) = (2\pi)^{-s}\Gamma(s) \exp(-2\pi iax) \{\exp(\pi si/2)\phi(-a, x, s) + \exp(2a\pi i - \pi si/2)\phi(a, 1-x, s)\}$ which was first proved by Lerch using contour integration as in Riemann's first proof of the functional equation for $\zeta(s) = \phi(0, 1, s)$. Riemann's second method, based on the transformation theory of theta functions, does not yield Lerch's relation as might be expected but leads instead to the formula $\Lambda(x, a, 1-s) = 2(2\pi)^{-s} \cos(\pi s/2)\Gamma(s) \cdot \exp(-2\pi iax)\Lambda(-a, x, s)$ where $\Lambda(x, a, s) = \phi(x, a, s) + \exp(-2\pi ix)\phi(-x, 1-a, s)$. Lerch's relation is derived from this using, in addition, the differential-difference equations for ϕ. The paper also contains two methods for evaluating $\phi(x, a, s)$ for negative integral values of s as finite sums of elementary functions. The results are analogous to the evaluation of the Hurwitz zeta function $\zeta(s, a) = \phi(0, a, s)$ in terms of Bernoulli polynomials for $s = 0, -1, -2, \cdots$. (Received April 24, 1951.)

408t. T. M. Apostol: *Remark on the Hurwitz zeta function.*

Although the Hurwitz zeta function $\zeta(s, a)$ is obtained from the Lerch zeta function $\phi(x, a, s)$ by putting $x = 0$ (see Bull. Amer. Math Soc. Abstract 57-5-407 for definitions), the functional equation for $\zeta(s, a)$ cannot be so obtained from that of ϕ. In this paper the author proves that $k^{1-s}\zeta(s, a/k) - \zeta(s,a) = \sum_{t=1}^{k-1} \phi(t/k, a, s)$ for any positive integer k and uses this to derive Hurwitz's relation $\zeta(1-s, a) = 2\Gamma(s)(2\pi)^{-s}\sum_{n=1}^{\infty} \cos(\pi s/2 - 2\pi an)n^{-s}$ from Lerch's functional equation for ϕ. The proof requires a knowledge of the behavior of $\zeta(s, a)$ for $a \to 0$ in certain regions of the s-plane. (Received April 24, 1951.)

409t. Bent Fuglede and R. V. Kadison: *On a conjecture of Murray and von Neumann.*

A ring of operators M (that is, a weakly closed self-adjoint algebra of operators) is said to be "normal" when each subring of M is the set of all operators in M which commute with the set of operators in M which commute with the given subring (cf. F. J. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. vol. 37 (1936) pp. 116–229). Murray and von Neumann prove that all factors of type I are normal and exhibit examples of factors of type II which are non-normal. They conjecture that all factors of type II are non-normal. This conjecture is proved in the present note, and, in fact, it is shown that one can produce a subfactor of the given factor which violates the normalcy. (Received May 2, 1951.)

410. J. W. Green: *On an expansion method for parabolic partial differential equations.*

Let $L(u) \equiv u_t - au_{xx} - bu_x - cu$, where a, b, and c are functions of x and t. The problem considered is to solve $L(u) = F(x, t)$, in the region $t > 0$, $0 < x < \pi$, subject to the condition that u vanish on the boundary. The method of solution is an adaptation of a method developed by S. E. Faedo (Publicazioni dell' Istituto per le Applicazioni del Calcolo, no. 247, Rome, 1949) for the solution of hyperbolic equations. An approximation is determined in the form $u_n = \sum_{j=1}^{n} c_{n,j}(t) \sin jx$ where the $c_{n,j}(t)$ are determined by the system of ordinary differential equations, $\int_0^\pi L(u_n) \sin jx dx = 0$, $j = 1, 2, \cdots, n$. By employing partial integration in a manner analogous to that in which uniqueness is usually proved, bounds are obtained for the integrals of the square of u and its various derivatives, permitting convergence of the u_n to a solution u to be shown. Estimates for $u - u_n$ and $\int_0^\pi (u - u_n)^2 dx$ are obtained. (Received May 2, 1951.)

411. A. T. Lonseth: *On the error incurred in solving nonhomogeneous linear functional equations by Galerkin's method.*

Suppose that an equation $Lu = f$ is to be solved, where L is linear, and that u must also satisfy certain conditions B on the boundary of its region R of definition. Let ϕ_0 satisfy B, and let $\phi_1, \cdots, \phi_n$ be orthonormal on R and satisfy the homogeneous boundary conditions corresponding to B. Then solution u is approximated by a linear combination u_n of the ϕ's, with coefficients determined so that $Lu_n - f$ is orthogonal to each coordinate function $\phi_1, \cdots, \phi_n$ on R. With quadratic norm, $\|u - u_n\|$ is bounded above if L has a bounded inverse; bounded below if L is bounded. Applications are made to classes of differential and integral equations. (Received May 6, 1951.)

412. J. L. McGregor: *An integral of Perron type.*

In the definition of the Perron integral of a function $f(x)$ over a closed interval $[a, b]$ a major function $M(x)$ and a minor function $m(x)$ are required to satisfy the conditions (i) $M(x)$ and $m(x)$ are continuous on $[a, b]$ and $M(a) = m(a) = 0$; (ii) $-\infty \neq D_* M(x) \geqq f(x) \geqq D^* m(x) \neq +\infty$. It is shown that without restricting the generality of the integral one may impose the additional condition (iii) $M(x)$ and $m(x)$ are differentiable on $[a, b]$. (Received May 1, 1951.)

413t. E. J. McShane: *Order-preserving maps and integration processes.*

Daniell's method of defining the Lebesgue integral is to start with an elementary integral, which is an order-preserving map of a subset of a lattice of functions into the real number system satisfying a certain set of postulates, and by devices based primarily on order properties to extend the domain of definition of the elementary integral so as to obtain a mapping with the well known continuity and closure properties possessed by the Lebesgue integral. By methods resembling those of Daniell, it is shown that an order-preserving map of a subset of a lattice F into a partially ordered set G can, under suitable assumptions, be extended so as to obtain a mapping of a larger subset of F into G having continuity and closure properties like those of the Lebesgue integral. As a preliminary, it is necessary to devise suitable definitions of closure, completeness, and convergence in partially ordered sets. Even when F is a lattice of functions and G the system of real numbers the present postulates are less restrictive than those of Daniell, so that the special cases include non-absolutely-convergent integrals as well as the Lebesgue-Stieltjes integral. By choosing G to be the set of bounded hermitian operators on a Hilbert space, one obtains the spectral resolution of such operators, and of bounded self-adjoint algebras of linear operators. (Received May 1, 1951.)

414. George Marsaglia: *On the compatibility of distribution functions.*

Consider an index set T and elements x_t, $t \in T$. Let the integer k be fixed, and suppose that for every subset $t_1, \cdots, t_n$ of T, with $n \leqq k$, there is assigned a distribution function $F_{x_{t_1},\cdots,x_{t_n}}(a_1, \cdots, a_n) = P[x_{t_1} \leqq a_1, \cdots, x_{t_n} \leqq a_n]$. Can such a system be extended to a stochastic process? Yes. For each $t \in T$, let P_t be the semi-ring of half-open intervals (possibly infinite) of R_1, the reals. Let $\mathfrak{M}$ be the class of cartesian products $\prod_{t \in T} M_t$, where $M_t \in P_t$ and $M_t \neq R_1$ for at most a finite number of t's; let the number of exceptions to $M_t = R_1$ be the index of a set in $\mathfrak{M}$, and let $\mathfrak{M}_k$ be the sets of $\mathfrak{M}$ having index not greater than k. The principal result is to the effect that such a system of distribution functions determines a set function on $\mathfrak{M}_k$ which may be extended to $\mathfrak{M}_{k+1}$ and hence to the semi-ring $\mathfrak{M}$. Well known methods then yield a measure which determines the stochastic process. (Received April 30, 1951.)

415*t*. K. S. Miller: *The one-sided Green's function.*

A function $H(x, \zeta)$ called the one-sided Green's function is defined. It bears the same relation to Volterra type integral equations as the classical Green's function $G(x, \zeta)$ does to Fredholm integral equations. Various properties of $H(x, \zeta)$ are deduced. For example, if $H(x, \zeta)$ is the Green's function for the linear differential operator L, then a fundamental system of solutions can be constructed in terms of H and its derivatives evaluated at the initial point. This function, which is closely related to the impulsive response of linear systems, is of use in various physical problems (particularly network theory) where interest is centered on one boundary point only. (Received May 9, 1951.)

416*t*. L. B. Robinson: *A complete system of tensors.* II.

The author computes a complete system of tensors for the case $(r=2)$ associated with the system (A) $y_1'' + p_{11}y_1' + py_2' + q_{11}y_1 + qy_2$, $y_2'' + py_1' + p_{22}y_2' + qy_1 + q_{22}y_2$. Write $V_{11} \equiv \Theta^{1/2}\{y_1^2 I_{22} - 2y_1y_2I_{12} + y_2^2 I_{11}\} = F_{11}$, $V_{12} \equiv \Theta^{1/4}\{y_1Y_1I_{22} - (y_1Y_2 + y_2Y_1)I_{12} + y_2Y_2I_{11}\} = F_{12}$, $V_{22} \equiv \{Y_1^2 I_{22} - 2Y_1Y_2I_{12} + Y_2^2 I_{11}\} = F_{22}$, $V_{13} \equiv D\{y_1I_{23} - y_2I_{13}\} = F_{13}$, $V_{23} \equiv \Theta^{-1/4}D \cdot \{Y_1I_{23} - Y_2I_{13}\} = F_{23}$. and $V_{33} \equiv \Theta^{-1/2}D^2I_{33} = F_{33}$. The F's are arbitrary functions of the covariants which have been long known. If the V's are solved with respect to the I's, the complete system of tensors results. (Received April 23, 1951.)

STATISTICS AND PROBABILITY

417*t*. J. C. Kiefer: *On Wald's complete class theorems.*

Terminology of A. Wald (*Statistical decision functions*, New York, Wiley, 1950) is used in what follows. Let $\mathcal{D}$ be the class of all decision functions available to the statistician, let $\mathcal{D}_b$ be the class of all decision functions with bounded risk functions, let C_w be the class of all Bayes solutions in the wide sense, and let C_s be the class of all Bayes solutions in the strict sense. Wald showed that under assumptions 3.1 to 3.6 of the above reference, C_w is complete relative to $\mathcal{D}_b$ (Theorem 3.17); and that under assumptions 3.1 to 3.7, C_s is complete relative to $\mathcal{D}_b$ (Theorem 3.20). (The same results were also proved by Wald in previous publications under stronger assumptions.) The present author proves that the above theorems remains true if in their statement $\mathcal{D}_b$ is replaced by $\mathcal{D}$. Under suitable assumptions, the method of proof is applicable to games whose payoff functions are bounded from below but not from above. (Received May 23, 1951.)

TOPOLOGY

418*t*. S. T. Hu: *Cohomology rings of compact connected groups and their homogeneous spaces.* I.

It is a classical result of Elie Cartan that the cohomology ring $H(X)$ with real coefficients of a homogeneous space $X=G/G_*$ of a compact connected Lie group G over a connected subgroup G_* is a local property, namely, it depends only on the Lie algebra L of G and the subalgebra L_* of L corresponding to G_*. In the present paper, this theorem is generalized to the compact connected topological groups as follows. Let a compact connected topological G operate transitively on a Hausdorff space X and let G_* denote the subgroup of G consisting of the elements of G which leave a given point x_* fixed. Then the Alexander cohomology ring $H(X)$ with real coefficients of the homogeneous space X depends only on the points of any given open neighborhood U of x_* in X and the operations on the points of U determined by the elements of G_* and those of any given open neighborhood V of the neutral element e in G. (Received April 30, 1951.)

419*t*. S. T. Hu: *Cohomology rings of compact connected groups and their homogeneous spaces.* II.

Let V be a given local group and V_* a subgroup of V. A cohomology ring $H(V, V_*)$ with real coefficients is defined as follows. For $p \to 0$, a p-function of V mod V_* is a continuous real function φ: $W^p \to R$ defined on $W^p \subset V^p$ for some open neighborhood W of e in V such that $\phi(gv_1h_1, \cdots, gv_ph_p)=\phi(v_1, \cdots, v_p)$ for all $v_i \in W$, $g \in V_*$, $h_i \in V_*$ provided that $gv_ih_i \in W$ $(i=1, \cdots, p)$. Two p-functions ϕ and ψ are said to be equivalent if they are equal for the arguments sufficiently near e. Then the p-functions are thus divided into equivalence classes called the p-cochains, C^p. Put $C^0=R$ and $C=\sum_p C^p$. C forms a graded ring with coboundary operator δ: $C \to C$ as follows. For any p-function ϕ and any q-function ψ, define $(\phi \cup \psi)(v_1, \cdots, v_{p+q}) =\phi(v_1, \cdots, v_p)\psi(\overset{-1}{v_p}v_{p+1}, \cdots, \overset{-1}{v_p}v_{p+q})$, $(\delta\phi)(v_1, \cdots, v_{p+1})=\phi(\overset{-1}{v_1}v_2, \cdots, \overset{-1}{v_1}v_{p+1}) +\sum_{i=1}^{p+1} (-1)^i\phi(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})$ for v_i in a sufficiently small open neighborhood of e in G. Then $H(V, V_*)$ is the cohomology ring of the graded ring C. Now let $X=G/G_*$ be the homogeneous space of a compact connected group G over a connected subgroup G_*. Let V be any open neighborhood of e in G and $V_*=V \cap G_*$. It is proved in this paper that the Alexander cohomology ring $H(X)$ with real coefficients is isomorphic with the ring $H(V, V_*)$. (Received April 30, 1951.)

420*t*. S. D. Liao: *On periodic transformations of homology spheres.*

Let Q be a compact Hausdorff space of finite dimension and T a periodic transformation of prime period p operating on Q. It was shown by Smith that if Q is a homology n_0-sphere over the group G_p of integers modulo p, then the fixed point set under T is a homology r_0-sphere over G_p, $-1 \leqq r_0 \leqq n_0$. The main aim of this paper is to establish the following theorem: In the Smith Theorem, if all the integral cohomology groups of Q are groups with a finite number of generators, then T preserves orientation or reverses orientation according as n_0-r_0 is even or odd. The meaning of orientation-preserving or orientation-reversing is as follows. If $p>2$, this is given in an obvious manner. In general, in the present theorem it can be seen that Q is also a homology n_0-sphere over the group of rational numbers, and hence the meaning of orientation-preserving or orientation-reversing can be settled. Moreover, for $p>2$, the two definitions will agree. The present theorem gives the positive solution of a problem proposed by Smith [Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, 1942, p. 373]. (Received April 23, 1951.)

J. W. GREEN,
Associate Secretary

FOURTH SYMPOSIUM ON APPLIED MATHEMATICS

The Fourth Symposium on Applied Mathematics of the American Mathematical Society was held at the University of Maryland, College Park, Maryland, on Friday and Saturday, June 22 and 23, 1951. The subject of the Symposium was *Fluid dynamics*. Attendance at each of the sessions was approximately 200. The following 270 persons, including 114 members of the Society, attended:

W. S. Amerst, M. W. Arsove, R. D. Baird, Ronald Bamford, H. C. Beaman, E. G. Begle, A. I. Bellin, F. D. Bennett, R. D. Bennett, D. Bershader, S. Bergman, Robert Betchov, E. E. Betz, Garrett Birkhoff, M. D. Bledsoe, M. J. Block, F. L. Boirtman, R. E. Bower, J. M. Boyer, Charlotte Brudno, R. W. Bryant, J. M. Burgers, R. S. Burington, A. Busemann, John Cangelosi, G. F. Carrier, I. Carstoiu, W. C. Carter, B. B. Cary, A. I. Chalfant, Subrahmanyan Chandrasekhar, C. C. Chang, Y. W. Chen, Peter Chiarulli, B. T. Chu, H. N. Chu, R. V. Churchill, H. G. Cohen, James Conlan, R. D. Cooper, G. M. Corcos, Stanley Corrsin, B. W. Corson, G. R. Costello, F. E. Cothran, C. M. Crenshaw, Tobias Dantzig, K. S. M. Davidson, C. R. DePrima, B. Des Clers De Beaumets, J. B. Diaz, Nicholas Dobi, C. L. Dolph, H. L. Dryden, P. Eisenberg, R. L. Eisenman, W. M. Ellsworth, J. L. Ericksen, Salaman Eskinazi, E. M. Fales, P. H. Fang, F. D. Faulkner, Michael Fekete, Morton Finston, J. W. Fischbach, W. J. Fitzgerald, A. Fiul, Charles Flagle, F. H. Fowler, F. N. Frenkiel, K. O. Friedrichs, A. S. Galbraith, P. R. Garabedian, C. D. Gardner, A. F. Gates, Abolghassen Ghaffari, J. H. Giese, David Gilbarg, L. S. Glover, Michael Goldberg, A. W. Goldstein, R. A. Good, Harold Grad, P. S. Granville, E. C. Gras, Wayland Griffith, R. A. Gross, Karl Guderly, J. M. Gwinn, W. L. Haberman, D. W. Hall, Ryosuke Hama, H. W. Handsfield, P. R. Hardesty, C. M. Hebbert, W. H. Hebrank, A. E. Heins, Aaron Hershfeld, I. R. Hershner, G. J. Higgins, O. H. Hoke, Ralph Hopkins, M. A. Hyman, P. Iribe, S. B. Jackson, S. J. Jacobs, H. T. Jones, M. L. Juncosa, I. Kanter, Carl Kaplan, B. Karlovitz, Theodor von Kármán, S. Karrer, Martin Katzin, C. J. Koch, I. I. Kolodner, Hans Kraft, G. Kuerti, S. H. Lachenbruch, O. E. Lancaster, Frank Lane, Lamar Layton, Patrick Leehey, Eugene Leimanis, Martin Lessen, W. W. Leutert, H. G. Lew, J. A. Lewis, J. V. Lewis, F. W. Light, C. C. Lin, R. K. Lobb, Joseph Logan, M. E. Long, S. C. Lowell, J. G. Loza, J. L. Luckler, G. S. S. Ludford, J. F. Ludloff, J. LuKasiewiez, B. W. McCormick, J. A. McFadden, D. G. McMahon, J. H. McMillen, J. C. MacCormick, H. M. MacNeille, M. S. Macovsky, R. C. Makino, W. R. Mann, P. T. Marshall, M. H. Martin, C. W. Matthews, J. S. Maybee, H. I. Meyer, R. E. Meyer, R. X. Meyer, Joseph Milkman, P. J. Miller, Knox Millsaps, A. J. Mitchell, A. K. Mitchell, G. M. Monroe, E. W. Montroll, F. K. Moore, T. W. Moore, W. E. Moore, M. J. Morkovin, C. R. Morris, F. J. Mulholland, H. Multhopp, C. H. Murphy, W. R. Murray, F. G. Myers, P. W. Nankey, H. E. Newell, O. M. Nikodým, C. O. Oakley, L. Onsager, Simon Ostrach, S. I. Pai, F. D. Parker, C. R. Paulson, J. Pavelka, G. W. Petrie, H. J. Pietrangeli, Y. K. Pieu, Karl Pohlhausen, Harry Polachek, I. Pollyn, H. L. Pond, C. D. Porter, I. R. Pounder, J. O. Fowers, William Prager, R. C. Prim, Mozelle Rankin, R. K. Reber, Mina Rees, Eric Reissner, H. S. Ribner, E. K. Ritter, R. C. Roberts, L. V. Robinson, W. G. Rose, Ray Ruetenik, H. C. Sanderson, F. J. Scheid, W. G. Scherberg, T. Schiffman, L. A. Schmidt, R. J. Seeger, O. Seidman, H. D. Seielstad, J. R. Sellars, J. B. Serrin. W. J. Sette, Daniel Shanks,

R. P. Shaw, A. W. Sherwood, J. T. Sinnette, R. Skalak, Leroy Smith, A. D. Solem, G. L. Spencer, J. R. Spreiter, J. A. Steketee, Joseph Stemberg, Marvin Stern, Joseph Sternberg, M. L. Storm, J. L. Synge, H. S. Tan, Olga Taussky, W. C. Taylor, Feodor Theilheimer, T. Theodorsen, W. R. Thickstun, T. Y. Thomas, R. N. Thurston, John Todd, Stephen Traugott, C. A. Truesdell, Maurice Tucker, J. L. Vanderslice, A. H. Van Tuyl, I. E. Vas, A. E. Von Doenhoff, H. V. Waldinger, J. L. Walsh, Chaio Wang, R. K. Warner, C. N. Warfield, Robert Wasserman, E. C. Watters, Jack Weener, P. P. Wegener, Herschel Weil, H. F. Weinberger, Alexander Weinstein, A. E. Weller, J. R. Weske, W. M. Whyburn, H. L. Wiser, F. H. Wright, Hsuan Yeh, Hideo Yoshihara, D. M. Young, G. B. W. Young, E. H. Zarantonello, P. W. Zettler-Seidel.

The United States Naval Ordnance Laboratory at White Oak, Maryland, served as co-host with the University of Maryland for the Symposium. University dormitories, lounges, and dining facilities were available to those attending the Symposium.

Symposium members and their families were guests at a tea in the Maryland Room of the Home Economics Building on Friday, June 22, 4–6 P.M. A dinner was arranged for those attending the Symposium and their guests on Friday evening. At this dinner greetings were extended by Dr. Ralph D. Bennett, Technical Director of the United States Naval Ordnance Laboratory, and Dean Ronald Bamford of the Graduate School of the University of Maryland. Professor Theodor von Kármán, Honorary President of the International Union on Theoretical and Applied Mechanics, spoke informally at the dinner.

The program of the Symposium consisted of 15 invited addresses, one of which was presented by title. The papers were presented in the following order, the last one, followed by the letter "*t*," being read by title:

1. Subrahmanyan Chandrasekhar: *Statistical theory of turbulence.*
2. C. C. Lin: *A critical discussion of the similarity concepts and their implications in the statistical theory of turbulence.*
3. A. Busemann: *The non-existence of transonic potential flow.*
4. R. E. Meyer: *Waves of finite amplitude in ducts of varying cross-section.*
5. T. Y. Thomas: *Separation of supersonic flow from curved profiles.*
6. G. F. Carrier: *On the construction of high speed flows.*
7. M. H. Martin and W. R. Thickstun: *An example of transonic flow for the Tricomi gas.*
8. L. Onsager: *Statistical hydrodynamics.*
9. A. E. Heins: *Gravity waves.*
10. J. M. Burgers: *Non-uniform propagation of shock waves.*

11. T. Theodorsen: *Theory of propellers.*

12. Garrett Birkhoff: *Numerical methods in conformal mapping.*

13. J. L. Synge: *Flow of viscous liquid through pipes and channels.*

14. Alexander Weinstein: *The method of singularities in the physical and in the hodograph plane.*

15*t*. S. R. De Groot: *Fluid dynamics and hydrodynamics.*

The presiding officers at the sessions were, respectively: Dr. H. L. Dryden, Director, National Advisory Committee for Aeronautics; Professor K. O. Friedrichs, New York University; Dr. Harry Polachek, United States Naval Ordnance Laboratory; and Professor William Prager, Brown University.

At the beginning of the Saturday afternoon session a resolution of thanks and appreciation for the hospitality extended was presented by Professor C. O. Oakley and unanimously adopted.

The papers presented at the Symposium will be published as volume four of the Proceedings of the Symposia in Applied Mathematics by the McGraw-Hill Book Company.

M. H. MARTIN,
Chairman of the
Committee on Arrangements

W. M. WHYBURN,
Associate Secretary
of the Society

GEORGE ABRAM MILLER
1863–1951

George Abram Miller was born on July 31, 1863, and died on February 10, 1951. He received the degrees of Bachelor of Arts (1887) and Master of Arts (1890) from Muhlenburg College, and of Doctor of Philosophy (1892) from Cumberland University. He was Professor of Mathematics (1888–1893) at Eureka College; Instructor at the University of Michigan (1893–1895) and Cornell University (1897–1901); Assistant and Associate Professor at Leland Stanford University (1901–1906); Associate Professor, Professor, and Professor Emeritus at the University of Illinois (1906–1951).

In October, 1891, Professor Miller was elected to membership in the New York Mathematical Society which had had a roster of twenty-three members at the beginning of the year. This Society became the American Mathematical Society in 1894. Later lists date Miller's membership from 1898; he became a life member in 1900. He was a member of the Council from 1901 to 1904. He was one of the founders of the San Francisco Section and its Secretary until 1906. He was Chairman of the Chicago Section from 1907 to 1909. He was Vice President of the Society in 1908.

He was a member of the London Mathematical Socety, the Deutschen Mathematiker Vereinigung, a corresponding member of the Sociedad Matematica Española, and an honorary life member of the Indian Mathematical Society. He was one of the organizers of the Mathematical Association of America at the end of 1915, a Vice President in 1916, and President in 1921. He was Secretary of Section A of the American Association for the Advancement of Science in 1899 and again from 1907 to 1912. He was a member of the National Academy of Sciences, and a fellow of the American Academy of Arts and Sciences. He was made an Honorary Doctor of Letters by Muhlenburg in 1936.

At Michigan in 1893 Miller lived in the home of F. N. Cole. Under Cole's direction and with his encouragement he started on the study of finite groups. His first publications were two short papers in volume 3 (1894) of the Bulletin of the New York Mathematical Society. In the next fifty-three years he contributed upward of 820 papers to the educational, scientific, and mathematical journals of eleven countries; about 450 of them made direct contributions to the theory of finite groups, the last in 1946. His course was set while he was at Michigan. He spent the years 1895 to 1897 at Leipzig and Paris where he attended the lectures of Sophus Lie and Camille Jordan. By the end of the year of his return, he had published 27 papers in the mathematical journals of America, England, France, and Germany. By 1900 he was considered to be the foremost student of groups in this country, and by 1910 as unsurpassed by any living student of the subject.

His first two papers completed the determination of the 200 substitution groups of degree eight and the 258 of degree nine. In each case he found two groups that had been missed by his predecessors. His immediate predecessors were Cayley, Cole, and Jordan. In the same year he published his own list of 994 intransitive groups of degree ten. None of these lists has been questioned since. In 1900 he was awarded a prize by the Academy of Sciences of Cracow for his work on groups of degree ten; this was the first award by a foreign academy to an American for work in pure mathematics. The fact that he was thirty years old and a Doctor of Philosophy when he started, and the rapidity with which his work progressed beyond any patterns he might have followed, arouses a curiosity about the years that went before.

He was born on a farm near Lynville, Pennsylvania, in a German-speaking family. He began teaching in the public schools at seventeen to earn money to prepare for and attend college, and for the first year after graduation was Principal in the public schools of Greeley, Kansas. At Muhlenburg the offerings in mathematics were calculus in the junior year, and astronomy and meteorology in the senior year. The Professor of Mathematics had been appointed to teach mathematics, physics, astronomy, and geology; at one time he taught the advanced classes in Latin also. When Miller went to Eureka in 1888, a knowledge of calculus was required for the bachelor's degree. Beginning in 1890, Miller offered courses for post-graduates leading to the degree of Doctor of Philosophy at the end of two years. In the meantime, he had spent part of the summer of 1889 at Johns Hopkins and the summer of 1890 at Michigan; neither university was in session during those summers. Bolza gave a series of lectures on groups at Johns Hopkins in 1889, but it is not known that Miller had any contact with the mathematicians there during that summer. He probably met Cole in the summer of 1890, but the steady expansion of his offerings at Eureka and the fact that a course in groups was not among them makes it quite certain that his attention had not turned to groups at that time. In 1891–1892 he was enrolled as a graduate student at Cumberland. Graduate work could be done by correspondence and his must have been done so, for he was teaching six hours a day at Eureka. The requirements for the doctor's degree at

Cumberland were the possession of a bachelor's degree, the completion of a number of courses specified by the text-books, and a thesis for which examinations in the required courses could be substituted. The Professor of Mathematics at Cumberland was a Civil Engineer who was also the engineering faculty and who, when not occupied by his duties at the university, was mapping Tennessee for the U. S. Coast and Geodetic Survey. This busy man offered to teach the advanced courses in mathematics by correspondence for $8 per month. Presumably Miller did not require the instruction, for the following year Cumberland added a year of residence to the requirements for the doctor's degree in mathematics.

It should be noted that in 1890, when Miller first offered courses in advanced mathematics at Eureka, the College of New Jersey offered three courses beyond the calculus, all open to seniors and all taught by Fine; the state universities of Illinois, Indiana, and Ohio offered nothing beyond the calculus; Michigan with Cole and Ziwet offered several advanced courses and the doctor's degree. Johns Hopkins had been offering instruction in advanced mathematics since 1876. In 1890 Harvard offered eight courses taught by Byerly, White, J. M. Peirce, and Dr. Osgood and offered research in four areas under the direction of the same four men; there was one graduate course offered, taught by B. O. Peirce, and a seminar on various topics meeting once a week and directed by Byerly, J. M. Peirce, and B. O. Peirce. There were fifteen graduate students including Bôcher who was on a fellowship at Göttingen where he received his doctor's degree at the end of that year. A new era in instruction in advanced mathematics at Harvard was said to have been inaugurated by Cole's lectures on groups in 1886.

It seems clear that Miller came to mathematics and to Michigan without benefit of benign pressures from without. If his taking up the study of groups upon his close association with Cole be interpreted as environment working upon Miller, the rest of his life exhibits Miller working on his environment. Lie's lectures on *Groups and invariants* apparently contained too little groups and too much invariants, for Miller's notes stop after the second lecture. He had a strong liking for Lie and he credits Lie with starting him on a systematic consideration of commutators and commutator subgroups. His attitude toward Lie's continuous groups was essentially the same as Lie's attitude toward finite groups—tolerance and respect but no enthusiasm. Jordan's lectures he followed through the year. They were concerned mostly with questions of primitivity and imprimitivity. Jordan had published in 1872 a list of primitive groups through degree seventeen; Miller had made one correction in 1894, he made the last in 1899. It seems likely that Miller would have made these corrections if he had never gone to Paris.

He published two books: *Determinants* (1892), written for the class in advanced algebra at Eureka; and *Historical introduction to mathematical literature* (1916). He wrote a part of each of two other books: *The algebraic equation*, pp. 211–260 of Monographs on Topics of Modern Mathematics (1911), edited by J. W. A. Young; and *Finite groups*, pp. 1–192 (1916), by Miller, Blichfeldt, and Dickson.

He wrote papers explaining groups to non-specialists, papers explaining mathematics to teachers of elementary mathematics and to non-mathematicians, a few papers on subjects of more general interest, and papers on the history of mathematics. His expository papers on groups began with about forty pages in fourteen installments in volumes 2 and 3 (1895–1896) of the American Mathematical Monthly; his last paper of this type was published in 1942. He urged the use of groups in elementary number theory, and in trigonometry. His most important paper of a more general nature was probably *Some thoughts on modern mathematical research* (1912). This was published in Science, in the Journal of the Indian Mathematical Society, and republished in Reports of the Smithsonian Institution as one of the most significant scientific publications of the year.

His interest in the history of mathematics dates from his days at Eureka. His first publication on a historical subject was in 1901, and his last in 1947. In his *Historical introduction* he says "The caliber of a mathematician can probably be judged just as accurately from the errors to which he pays attention as from the new results which he announces. In both cases, he can devote himself to trivialities or to big things." We might add that a man may concern himself so much with little things as to distract attention from bigger things he has done. In his papers on history he gave attention to too many things of questionable importance, like the anomaly of two first International Congresses of Mathematicians, or the authenticity of a dialogue between Euclid and a student used to dramatize the fact that the Greeks had an interest in mathematics that was not utilitarian. One could wish that he had not gone on to pillory the seventy times seventh errors in a popular history of mathematics. His own *Historical introduction*, as well as a history of elementary mathematics which he left in manuscript, is better than that.

His technical papers were all directed at the fundamental problem of finite groups, namely, the determination of what groups exist and methods of distinguishing one group from another. He carried the determination of substitution groups through the transitive groups of degree seventeen, listing only the primitive groups of degrees fifteen and sixteen. Jordan's list of 1872 was far from complete, containing, for example, only twelve of the twenty primitive groups of degree sixteen.

His publications on abstract groups began in 1896 when he gave all the groups of orders smaller than 48. This paper was the first to determine the 15 groups of order 24 and the 51 groups of order 32. Cayley had proved that there are only two groups of each of the orders four and six in 1854 and later that there are five groups of each of the orders eight and twelve. The fourteen groups of order 16 had been determined previously by Kirkman and by J. W. A. Young. Early in 1896 Le Vavasseur had announced that he had found 75 groups of order 32 and had not completed his investigation; later he stated that the number was 50, but finally agreed that 51 was correct. In his last long technical paper, in 1930, Miller determined the 294 groups of order 64.

He was the first to answer many of the questions that every student of finite groups must ask. For example, he was the first to determine that the order of the product of two elements is in no way limited by the orders of the two elements; the first example of a commutator subgroup containing elements which are not commutators was given by Fite in a dissertation prepared under Miller's direction, and Miller later gave an infinite system of such groups; he was the first to prove the simplicity of the multiply transitive Mathieu groups of degrees 22, 23, and 24.

He was always looking for a key that would unlock the problem of the groups of order n. In 1902 he gave a method for the determination of all the groups of order p^m, and that is the method he used on the groups of order 64. He tried the attack of determining the orders n for which a specified small number of groups exist. A large number of his papers determine the groups which have what look like arbitrary queer sets of properties; all are motivated by the desire to find out what sorts of questions one should ask of a group in order to be able to identify it.

One would have to be intrepid to attempt to extend Miller's work on groups by his methods; on the other hand, very little of it has been extended by any other methods. His determination of substitution groups of low degree could be carried on. Thirty-four years elapsed between his determination of the groups of order 32 and those of order 64 and in the meantime only Potron's incorrect determination of the latter was published. His proof in 1900 that there is no simple group of odd composite order that can be written transitively on fewer than fifty-one letters is the last definitive word on such groups.

He wrote in 1935 that he was fortunate to have begun the study of groups at the right time. He was fortunate also in his marriage in 1901 to Miss Cassandra Boggs, who was a charming lady and a sympathetic helpmeet for forty years. He was fortunate in his vigorous constitution, which enabled him to write ten papers in 1944 when he was eighty-one and to go to his office regularly through the first five months of his eighty-eighth year. He was fortunate in his mental and moral endowments, which led him to success in a worthy career.

H. R. Brahana

BOOK REVIEWS

Statistical decision functions. By A. Wald. New York, Wiley, 1950. 9+179 pp. $5.00.

This book, published shortly before Wald's death in an airplane crash in India, is an outgrowth and generalization of the author's papers during the period 1939–1949 on the general theory of statistical decision. It represents a remarkable application to statistical theory of the methods and spirit of modern mathematics.

The general statistical decision problem as formulated by Wald can be described in the following terms. There is given a stochastic process $X=\{X_i\}$ and a class Ω of distributions which is known to contain the true distribution F of X as an element. Any rule δ for sampling from X and arriving at a terminal decision d^t belonging to a given class D^t is called a *decision function.* For any F and d^t there is given a function $W(F, d^t) \geqq 0$ representing the loss involved in taking the decision d^t when F is the true distribution of X. All these preliminaries go to define the *risk function* $r(F, \delta)$, which is the expected value of the loss plus that of the cost of sampling, given the decision function δ and true distribution F. The object of the statistician is to choose δ so as to minimize $r(F, \delta)$. However, the statistician controls only δ while F is, so to speak, controlled by Nature. The problem is therefore analogous to that of the zero-sum two-person game of von Neumann, with the difference that, whereas the statistician wishes to minimize $r(F, \delta)$, we can hardly say that Nature wishes to maximize it.

The problem in choosing δ arises from the fact that no δ will simultaneously minimize $r(F, \delta)$ for all F in Ω. Two alternatives are considered by Wald. (1) If F has a known prior probability distribution ξ in Ω (that is, if Nature's mixed strategy is known to the statistician), then the decision problem is solved by that δ which minimizes the average risk

$$\int_\Omega r(F, \delta) d\xi; \tag{1}$$

such a δ is called a *Bayes solution* corresponding to ξ. (2) If a prior distribution ξ of F does not exist or is not known, the statistician who, in the spirit of the theory of games, regards Nature as an opponent out to maximize $r(F, \delta)$, should adopt a *minimax solution* δ for which $\sup r(F, \delta)$ with respect to F is as small as possible. Wald himself feels that this attitude is "perhaps not unreasonable."

Alternatives (1) and (2) represent extremes of optimism and pessimim respectively concerning the statistician's attitude toward Nature. Most statisticians will probably find themselves somewhere between the two extremes, and faced with the problem of how to utilize their rather vague feelings about the frequency with which various possible F's can be expected to occur. For this reason, this reviewer suspects that minimax solutions as such are likely to be of little interest in statistics. For example, on p. 142 Wald cites the minimax point estimate of the mean θ of a binomial variate; the corresponding risk function is a constant $r_0 = [2(1+N^{1/2})]^{-2}$. The traditional (non-minimax) estimate has risk function $\theta(1-\theta)/N$. For large N the ratio of this to r_0 is near zero except in a small interval about $\theta = 1/2$, where it is slightly greater than 1. It is very hard to believe in the superiority of the minimax estimate in this case, which is by no means unusual in its nature.

To those who are indifferent to minimax solutions the principal interest of the book will lie in the main theorem that, under very general conditions, the class $\mathcal{B}$ of all Bayes solutions is *essentially complete* in the following sense: for any decision function δ there exists a δ^* in $\mathcal{B}$ such that $r(F, \delta^*) \leqq r(F, \delta)$ for all F in Ω. (Mathematically, this theorem represents a highly nontrivial extension of the method of Lagrange multipliers in the calculus of variations.) There is obviously no loss involved in restricting the choice of a decision function to any essentially complete class, in particular to $\mathcal{B}$. But even a minimal essentially complete class will usually be so large that further reduction is necessary before the statistician can turn the problem of selecting a decision rule over to the experimenter. One criterion for reduction, the minimax principle, has already been mentioned. Other criteria exist (unbiasedness, invariance, and so on) but are not dealt with in the present volume.

The book makes effective use of the modern theory of measure and integration, and operates at a high level of rigor and abstraction. For this reason few statisticians will be prepared to read it, yet its ultimate liberating effect on statistical theory will be great. It is to be hoped that so rich and stimulating a book as this will reach an audience among mathematicians.

HERBERT ROBBINS

Introduction to the theory of algebraic functions of one variable. By C. Chevalley. (Mathematical Surveys, no. 6.) New York, American Mathematical Society, 1951. 12+188 pp. $4.00.

Here is algebra with a vengeance; algebraic austerity could go no further. "We have not tried to hide (says the author) our partiality to the algebraic attitude . . . "; he has not indeed; and, if it were not for a few hints in the introduction and one casual remark at the end of Chapter IV, one might never suspect him of having ever heard of algebraic curves or of taking any interest in them. Fields and only fields are the object of his study. A field is given, or rather two fields: one, the function-field R; the other, the field K of constants; K is algebraically closed in R; and R is finitely generated and of degree of transcendency 1 over K. Everything must be "intrinsic," i.e. must be born from these by some standard operations. Later on the family circle is enlarged by the appearance of another function-field S containing R, with a field of constants L containing K, and a large portion of the book is devoted to the mutual relations between R and S; but nowhere except in one or two lemmas is any element allowed to appear unless it is contained in those fields or canonically generated from them.

The contents of the book are as follows. Valuations are introduced and the basic existence theorem on valuations is proved in the standard manner (th. 1, p. 6), by the use of Zorn's lemma: this is the theorem according to which every "specialization" of a subring $\mathfrak{o}$ of a field R (i.e. every homomorphic mapping of $\mathfrak{o}$ into a field) can be extended to a "valuation" of R (i.e. a specialization of a subring $\mathfrak{O}$ of R such that $R=\mathfrak{O}\cup\mathfrak{O}^{-1}$); the theorem, however, is not stated in its full generality. One might observe here that, in a function-field of dimension 1, every valuation-ring is finitely generated over the field of constants, and therefore, if a slightly different arrangement had been adopted, the use of Zorn's lemma (or of Zermelo's axiom) could have been avoided altogether; since Theorem 1 is formulated only for such fields, this treatment would have been more consistent, and the distinctive features of dimension 1 would have appeared more clearly.

Places are defined as being in one-to-one correspondence with the non-trivial valuation-rings of R, i.e. with those proper subrings $\mathfrak{o}$ of R which contain the field of constants K of R and satisfy $R=\mathfrak{o}\cup\mathfrak{o}^{-1}$. In Zariski's terminology, on the other hand, a place is a homomorphic mapping of a valuation-ring $\mathfrak{o}$ into a field; in consequence, if $\mathfrak{o}$ is a non-trivial valuation-ring of R, and $\mathfrak{p}$ the ideal of non-units in $\mathfrak{o}$ (the "place" in Chevalley's sense), there will be as many "places" belonging to $\mathfrak{o}$ and $\mathfrak{p}$, with values in a given "universal domain" Ω, as there are isomorphisms of the "residue-field" $\Sigma=\mathfrak{o}/\mathfrak{p}$ into Ω; their number is equal to the degree d_s over K of the maximal separable extension Σ_s of K contained in Σ. According to Chevalley's selfimposed taboos, however, only the field $\mathfrak{o}/\mathfrak{p}$ is allowed to exist, and the "place" determined by $\mathfrak{o}$ and $\mathfrak{p}$ (or by either of them) must be unique. This has far-reaching consequences: while otherwise sums (e.g. the sum of the residues of a differential) could be extended over all the d_s places belonging to $\mathfrak{o}$ and $\mathfrak{p}$, here the d_s terms belonging to such a sum can never be separated from each other. It is true that traces (of elements of Σ_s over K) are adequate substitutes for such sums; but it may well be doubted whether the constant use of traces is not an unnecessary complication, and whether it helps a beginner to understand the subject.

Chapter I then brings, as usual, proofs for the existence of a uniformizing variable at a place (i.e. a $t \in \mathfrak{p}$ such that $\mathfrak{p} = t\mathfrak{o}$), for the independence of valuations (or of "places"), and for the existence of the divisor of a function; it ends up with the theorem that the degree of the divisor of zeros of $x \in R$ is equal to $[R:K(x)]$. Chapter II follows, with the definition of differentials and the proof of the Riemann-Roch theorem due to A. Weil. The genus is defined by means of Riemann's theorem. A "repartition" is defined as a function assigning to each place $\mathfrak{p}$ an element $x(\mathfrak{p})$ of R (or, later, an element $x(\mathfrak{p})$ of the $\mathfrak{p}$-adic completion $\overline{R}_{\mathfrak{p}}$ of R at $\mathfrak{p}$), so that those places $\mathfrak{p}$ for which $x(\mathfrak{p})$ has a pole at $\mathfrak{p}$ are in finite number; then a differential is a linear function on the space of repartitions, continuous in a suitable sense, which vanishes on the subspace of "principal" repartitions (those for which $x(\mathfrak{p}) = x$ for all $\mathfrak{p}$, with $x \in R$). This rather abstract concept of differential is of course what makes possible such a brief proof of the Riemann-Roch theorem; while this is very convenient for many purposes, one should not forget that eventually (in the case where R is separably generated over K) differentials have to be identified with the expressions ydx, or, what amounts roughly to the same thing, it must be shown that the sum of the residues of ydx is 0; for this, in the present volume, one has to wait until p. 117.

Chapter III introduces the local or $\mathfrak{p}$-adic completions of the function-field K by means of the usual definitions and of Hensel's lemma; Σ_s being defined as before, it is shown that Σ_s can be canonically identified with a subfield of the completion $\overline{K}$ of R at $\mathfrak{p}$, and that, if $\Sigma = \Sigma_s$, this completion is essentially the ring $\Sigma((t))$ of power-series in t with coefficients in Σ, where t is any uniformizing variable at $\mathfrak{p}$; the structure of $\overline{R}$ when Σ is not separable over K is not further discussed. The last § of Chapter III brings the concept of residue of a differential, in terms of the values of the differential at certain "repartitions"; it then becomes trivial that the sum, not of the residues, but of the traces of the residues of a differential is 0.

So far only one function-field has been considered. Now another one, S, is introduced, with the field of constants L, such that $S \supset R$ and $K = L \cap R$; the next three chapters (nearly half the book) are devoted to the simultaneous study of the two fields R, S, under various assumptions. Some of the questions raised here by the author had never been treated before; unfortunately, as he treads new ground, his footsteps become more uncertain, and to follow in them is at times no easy task. In the language of algebraic geometry, the passage from R to S consists partly in enlarging the field of constants of a given curve, partly in considering the mutual relations of two curves in a $(1, m)$-correspondence with each other; the author tries to treat both problems by the same methods; however tempting this idea may appear to the algebraist, it is not altogether successful, and may well have caused some blurring of the picture.

Chapter IV is chiefly devoted to the case where S is of finite degree over R, and to the behavior of the places of R and S with respect to one another; these questions are fairly familiar, at least in the parallel case of number-fields, and no surprises are to be expected here. Because of the too special formulation which has been given of the theorem on the extension of specializations in Chapter I, the existence of a place of S lying over a given place $\mathfrak{p}$ of R is made to depend, strangely enough, upon Riemann's theorem. The ramification indexes e_λ and the relative degrees d_λ of the places $\mathfrak{P}_\lambda$ of S lying over $\mathfrak{p}$ are defined, and it is proved that $\sum_\lambda d_\lambda e_\lambda = [S:R]$; this might well have been postponed until it is shown that $d_\lambda e_\lambda$ is the degree of the $\mathfrak{P}_\lambda$-adic completion $\bar{S}_\lambda$ of S over the $\mathfrak{p}$-adic completion $\bar{R}$ of R, and a basis is explicitly given for the former over the latter (Theorems 4 and 5, p. 60–61); in between those results are inserted some remarks on the case of normal extensions, the proof for the existence of a base of S, integral at $\mathfrak{p}$, and auxiliary definitions and results on the Kronecker product of fields or commutative algebras, the latter being necessary in order to show that the direct sum of the $\bar{S}_\lambda$ is no other than the algebra over $\bar{R}$ obtained by considering S as an algebra over R and extending its ground-field to $\bar{R}$.

Then norms and conorms, traces and cotraces are defined for divisors and repartitions in S and in R; norms and traces are defined as usual; the conorm and cotrace are the dual operations, i.e. consist in "lifting" divisors, repartitions, etc., from R to S; the consistent use of these terms (rather than the more usual identification of divisors, etc., in R with the corresponding ones in S) is perhaps cumbersome, but is very helpful in keeping apart essentially distinct concepts while their main properties are being developed. The different is then defined, and its basic properties are given.

Chapter V discusses the extension of the field of constants; in the absence of a universal domain, such extensions have to be generated by the clumsy device of the tensor-product of fields; in the inseparable case, one has then to face the disagreeable appearance of radicals, whose rather arbitrary dismissal follows at once, not without the intervention of minimal ideals. More than half of the chapter is spent in such awkward discussions, beginning with the definition of separable (not necessarily algebraic) extensions by the non-existence of nilpotent elements in certain tensor-products and leading up to Theorem 3 (p. 92) which expresses, again in terms of such products, the effect on a place of R of an extension of the field of constants. There is hardly any connection between the foregoing and the basic Theorem 4; according to the latter, if the field of constants K of R is extended to a field L, separable over K, and if $\mathfrak{a}$ is a divisor of R, every element y of the extended function-field which is a multiple of $\mathfrak{a}$ (more accurately, of the "conorm" of $\mathfrak{a}$) is a linear combination, with coefficients in L, of elements of R which are multiples of $\mathfrak{a}$ (cf. A. Weil's *Foundations*, Chapter VIII, th. 10); from this it is deduced that the genus is not altered by the extension of the field of constants from K to L if L is separable over K, and that it can only decrease by an arbitrary extension.

Chapter VI takes up the behavior of differentials under an extension of the function-field; one of its main objectives is to identify the differentials in R with the symbols ydx, provided R is separable (i.e., separably generated) over K. This is done by means of a general theory for the "lifting" of a differential from a field R to a field S under suitable conditions; this operation is called the "cotrace." An explicit definition being given for a certain differential, called dx, in a purely transcendental extension $K(x)$ of the field of constants, dx is then "lifted" from $K(x)$ to R for every non-constant x in R. A drawback of this method is, of course, that, as dx and dy are lifted from different fields, there is no obvious connection between them, and $d(x+y)=dx+dy$ becomes a deep theorem; perhaps a more satisfactory arrangement would have been provided by a definition similar to that adopted for meromorphic differentials in Chapter VII. However that may be, after a preliminary discussion of the field $K(x)$, the fields K, R, L, S are again considered; the trace of a differential of S is defined, in the case where S is of finite degree over R; and the cotrace of a differential of R is defined, but merely for the case $K=L$; the behavior of residues under the operations of trace and cotrace, and other elementary properties, are established. The different (which had disappeared during the whole of Chapter V) turns up again, and it is shown that, when a differential is lifted from R to S, its divisor is multiplied by the different of S over R; it is a pity that this § is separated from the § on the different, as both could just as easily have been put together, either here or, even better, in Chapter IV. One then comes back to dx, which can now be lifted from $K(x)$ to R; among other results, its divisor is calculated; and it is shown that, if x is a uniformizing variable at a place $\mathfrak{p}$ of degree 1, the residue of ydx at $\mathfrak{p}$ is the coefficient of x^{-1} in the power-series expressing y in terms of x in the completion of R at $\mathfrak{p}$. The investigation is again interrupted, this time in order to introduce the general concept of derivation in fields, algebraic function-fields and power-series fields; it is resumed for the proof of the decisive Theorem 9, according to which, for a given x, dy/dx is the derivation D_xy of R which vanishes on K and has the value 1 at x; this is ingeniously proved by showing that the differential $dy-(D_xy)dx$ has infinitely many zeros.

The concept of cotrace is then extended to the case $K\neq L$, provided R is separable; since in that case the differentials of R can be written as ydx, with x, y in R, these same expressions can be used to lift them into S; in particular, under an extension of the field of constants of R, it is shown that the residues of a differential remain the same, that the divisor of a differential is unchanged if the genus is unchanged, that otherwise it is divided by an integral divisor. A further section (the purpose of which remains unexplained) discusses the effect on differentials and their residues of a derivation of the field of constants; and the chapter ends up, rather disappointingly, with a theory of differentials of the second kind confined to characteristic 0, in which case it is an easy application of the Riemann-Roch theorem.

Maybe the appearance of characteristic 0 at the end of Chapter VI was meant as a transition to the extensive Chapter VII (more than 50 pages), which treats the "classical" case, i.e. the case where the field of constants is the complex number-field, with its topology; this is almost a different book. It is hard to say what knowledge is assumed of the reader in this chapter; while it is tediously proved that meromorphic functions in an open set form a field, and one full page is devoted to the calculation of $\int dx/x$ on a circle surrounding the origin in the complex plane (the value being found to be $\pm 2\pi(-1)^{1/2}$), Schwarz's lemma suddenly turns up from nowhere (p. 152) in order to prove that holomorphic mappings preserve the orientation, a fact for which, fortunately, a more reasonable justification is given later (p. 181). The reader is further required to take for granted the validity of all the "axioms" of Eilenberg and Steenrod for the singular homology theory in arbitrary topological spaces, a statement of which is given in §3; thanks to this, says the author, "we have avoided the cumbersome decomposition of the Riemann surface into triangles." The truth is that this triangulation is a quite trivial matter; and, while the reduction of a triangulation to standard form (as done e.g. in Seifert-Threlfall) is a somewhat clumsy process, the canonical dissection of the Riemann surface which is so obtained has immense advantages over a purely homological theory; it shows that all Riemann surfaces of a given genus are homeomorphic; it gives the structure of the fundamental group, which, even to the pure algebraist, is of prime importance in determining the nature of the non-abelian extensions of the given function-field: such advantages seem to be more than enough to outweigh those of the more algebraic (and "intrinsic") procedure adopted by the author.

The chapter begins with the definition of the Riemann surface, i.e. of the set of places of the given field, as a topological space; unfortunately, its definition as an analytic manifold is given only much later, so that orientation is defined twice, and various special cases of Stokes' formula have to be proved separately. Meromorphic functions and differentials on open subsets of the Riemann surface are defined; it is shown in the usual manner that the meromorphic functions on the Riemann surface are the elements of the function-field. Periods of differentials are defined, essentially by analytic continuation (not by integration, since the 1-chains are not assumed to be differentiable), so that their definition virtually depends upon the concept of fundamental group, which however is carefully avoided.

We come now to one of the most interesting and original features of the whole book. With the author, let us denote by S the Riemann surface, by P and Q two mutually disjoint finite subsets of S. Then $H_1(S-P, Q)$ is the "relative" homology group of the open set $S-P$ modulo Q; in other words, it is the group of classes of 1-chains lying in $S-P$, with boundary in Q (the "relative cycles" in $S-P$ mod. Q), such a chain being homologous to 0 if it bounds in $S-P$. If γ is such a relative cycle, and γ' is a relative cycle in $S-Q$ mod. P, then, as the boundary of each cycle is disjoint from the other, the intersection-number or Kronecker index $I(\gamma, \gamma')$ is defined, it depends only upon the homology classes of γ, γ'; and it determines a duality between $H_1(S-P, Q)$ and $H_1(S-Q, P)$, in the sense that $I(\gamma, \gamma')$ cannot be 0 for all γ unless γ' is homologous to 0, and that there is a γ' such that $I(\gamma, \gamma')$ is equal to an arbitrarily given integral-valued linear function on $H_1(S-P, Q)$. These groups, and the duality between them, can now be translated into algebraic terms by means of the following concepts. Let $E(P, Q)$ be the set of differentials on S with no poles at the points of Q and no residue $\neq 0$ at any point outside P. Take a canonical dissection of S by means of curves, not going through the points of P and Q; S is then represented as a canonical polygon of $4g$ sides (where g is the genus, which we assume to be $\neq 0$, the case $g=0$ being similar but simpler), all the vertices corresponding to one and the same point of S, and the sides occuring in the order $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$; join the origin O of a_1 (the extremity of b_g^{-1}) to the points P_μ of P, and to the points Q_ν of Q, by mutually disjoint simple arcs p_μ resp. q_ν, interior (except for their common origin O) to the fundamental polygon. In the polygon, cut along the arcs q_ν, an element ω of $E(Q, P)$ is the differential $\omega = d\phi$ of a one-valued function ϕ; similarly, in the polygon, cut along the arcs p_μ, an element ω' of $E(P, Q)$ is the differential $\omega' = d\phi'$ of a one-valued function ϕ'; we may assume that, at the vertex O, $\phi = \phi' = 0$. The integral of $\phi d\phi'$, or that of $-\phi' d\phi$, along the contour of the canonical polygon is equal to

$$\begin{aligned} I(\omega, \omega') &= \int \phi d\phi' = -\int \phi' d\phi \\ &= \sum_{\lambda=1}^{g} \left(\int_{a_\lambda} \omega \int_{b_\lambda} \omega' - \int_{b_\lambda} \omega \int_{a_\lambda} \omega' \right). \end{aligned} \tag{1}$$

Apply now Cauchy's theorem, either to the differential $\phi\omega' = \phi d\phi'$ and to the polygon cut along the arcs q_ν, or to the differential $-\phi'\omega = -\phi' d\phi$ and to the polygon cut along the arcs p_μ. We get

$$\begin{aligned} \frac{1}{2\pi i} I(\omega, \omega') + \sum_\nu \phi'(Q_\nu) \cdot \mathrm{Res}_{Q_\nu}\, \omega + \sum_\nu \mathrm{Res}_{Q_\nu} \left\{ [(\phi' - \phi'(Q_\nu)]\omega \right\} \\ = \sum_\mu \mathrm{Res}_{P_\mu}\, (\phi\omega') + \sum_\rho \mathrm{Res}_{R_\rho}\, (\phi\omega') \end{aligned}$$

where the R_ρ are all the poles of ω or of ω', other than the P_μ and Q_ν, and therefore:

$$\begin{aligned} j(\omega, \omega') &= \sum_\mu \mathrm{Res}_{P_\mu} \left\{ [\phi - \phi(P_\mu)]\omega' \right\} - \sum_\nu \mathrm{Res}_{Q_\nu} \left\{ [\phi' - \phi'(Q_\nu)]\omega \right\} \\ &\quad + \sum_\rho \mathrm{Res}_{R_\rho}\, (\phi\omega') \\ &= \frac{1}{2\pi i} I(\omega, \omega') + \sum_\nu \phi'(Q_\nu)\, \mathrm{Res}_{Q_\nu}\, \omega - \sum_\mu \phi(P_\mu)\, \mathrm{Res}_{P_\mu}\, \omega'. \end{aligned} \tag{2}$$

Here $j(\omega, \omega')$ is an alternating bilinear form, defined for $\omega \in E(Q, P)$,

$\omega' \in E(P, Q)$. Of the two expressions for it given by (2), the first one depends only upon the power-series expansions of ω, ω' at the points P_μ, Q_ν, R_ρ; in fact, $\phi-\phi(P_\mu)$ is the function vanishing at P_μ with the differential ω; $\phi'-\phi'(Q_\nu)$ is the function vanishing at Q_ν with the differential ω'; and at a point R_ρ we may, in calculating $\mathrm{Res}_{R_\rho}(\phi\omega')$, take for ϕ any function with the differential ω (this being meromorphic there since $\mathrm{Res}_{R_\rho}\,\omega=0$), the residue of $\phi\omega'$ being independent of the choice of the additive constant in ϕ since $\mathrm{Res}_{R_\rho}\,\omega'$ is 0; and we have $\mathrm{Res}_{R_\rho}(\phi\omega') = -\mathrm{Res}_{R_\rho}(\phi'\omega)$. Put $j(\omega, \omega')=j_\omega(\omega')$; then $\omega\to j_\omega$ maps $E(Q, P)$ into the space of linear functions on $E(P, Q)$; the second expression for $j(\omega, \omega')$ in (2) shows at once that the kernel of this mapping consists of the differentials $\omega=df$ of the meromorphic functions f on S which are 0 at the P_μ; if $F(P)$ is this kernel, then that same expression shows that $E(Q, P)/F(P)$, $E(P, Q)/F(Q)$ are two vector-spaces of finite dimension equal to $2g+(p-1)^+ + (q-1)^+$ (where p, q are the numbers of points in P, Q respectively, and $a^+=\max(a, 0)$), and that $j(\omega, \omega')$ establishes a duality between them.

Now take $\omega'\in E(P, Q)$, and $c\in H_1(S-P, Q)$; let γ be a relative cycle in $S-P$ mod. Q belonging to the homology class c and disjoint from the poles of ω'; it is easy to see that $\int_\gamma\omega'$ depends only upon c and upon the class of ω' mod. $F(Q)$; therefore there is an $\omega_c\in E(Q, P)$ such that $\int_\gamma\omega'=j(\omega_c, \omega')$ for all ω', and the class of ω_c mod. $F(P)$ depends only upon c. Similarly, if $c'\in H_1(S-Q, P)$, one will attach to it an element $\omega'_{c'}$, of $E(P, Q)$, well defined mod. $F(Q)$.

It is easy to determine, in terms of the canonical dissection, the structure of $H_1(S-P, Q)$; this is generated by the a_λ, b_λ, by small circles γ_μ surrounding the P_μ and positively oriented, and by the linear combinations $\sum_\nu m_\nu q_\nu$ of the arcs q_ν, with integral coefficients m_ν satisfying $\sum_\nu m_\nu=0$; the only relation between these generators is $\sum_\mu \gamma_\mu \backsim 0$; $H_1(S-P, Q)$ is therefore a free abelian group of rank $2g+(p-1)^+ + (q-1)^+$. Also the intersection-numbers of cycles in $H_1(S-P, Q)$ with cycles in $H_1(S-Q, P)$ are then obvious. On the other hand, if one uses the second expression in (2) for $j(\omega, \omega')$, one obtains the conditions which $\omega=\omega_c$ has to satisfy, for a given c, in terms of the periods of ω and of the $\phi(P_\mu)$; proceeding similarly for $\omega'_{c'}$, one finds that $j(\omega_c, \omega'_{c'})$ *is equal to the intersection-number of* c, c' (i.e. of two cycles belonging to these homology classes).

While these are some of the main results of the author, he proceeds in an entirely different way. He first gives the algebraic definition of $j(\omega, \omega')$, and shows in a purely algebraic manner that this is a bilinear function on the spaces $E(Q, P)/F(P)$, $E(P, Q)/F(Q)$ and establishes a duality between them. He next defines ω_c, $\omega'_{c'}$ as above; and he calls $j(\omega_c, \omega'_{c'})$ the intersection-number of c, c'! He then (without condescending to say so) goes on to show that this intersection-number has the properties which characterize it from the topological point of view, viz., that it is an integer, that $I(\gamma, \gamma')=0$ if γ, γ' are disjoint, and that $I(\gamma, \gamma')=1$ if γ, γ' are arcs inside a small circle, with extremities on that circle, the cyclic order of these extremities being suitably related to the orientation; the proof for the second one of these facts (pp. 158–161) is a singularly difficult and tortuous one, appeals to a theorem (attributed to Montel) on so-called "normal families," and also (without any reference) to the fact that a continuous function of two complex variables, separately holomorphic in each, is holomorphic in both. All this could easily have been shown by means of the canonical dissection, even if one did not want merely to verify it *a posteriori* in the manner sketched above. The author now proceeds to the determination of the homology groups, which is far from easy and requires all the combined resources of topology and of algebraic function-theory. As this gives him the technical equivalent of the tools ordinarily provided by the canonical polygon and integration along its contour, he can then prove Abel's theorem and Riemann's bilinear inequalities; even at that stage, he needs two pages to prove Stokes' formula for the complement of the union of finitely many small circles on S, and two more pages, involving the use of differentials of the second kind, for the proof of the bilinear inequalities for periods of differentials of the first kind. On the other hand, a very simple and direct proof is given for the fact that the group of divisor-classes of degree 0 is isomorphic to the torus-group of real dimension $2g$.

We have not yet mentioned some illustrative sections in this and the earlier chapters, on fields of genus 0 and 1, on fields of elliptic functions, and one (Chapter IV, §9) on hyperelliptic fields. Except for the latter which is a kind of *tour de force* (hyperelliptic fields over an arbitrary field of constants had probably never been discussed before), these are elementary and could well have been given in the form of exercises or series of exercises; and it is greatly to be regretted that the author has not added many more in that form, out of his rich stock of knowledge on such subjects; he could thus have greatly enhanced the usefulness of the book at the cost of a very moderate increase in size. As it is, one will not even find in it a calculation of the genus of a field $k(x, y)$ defined by $y^2=P(x)$, where $P(x)$ is a polynomial. The book is also without any bibliography beyond a small number of references in the brief introduction, a few lines of which comprise all that the reader is told about the history of the subject; the name of Riemann occurs only as a label for some theorems and in "Riemann surface." Chapters I–VI are without a single reference; it is true that they are almost entirely self-sufficient, but even at places where a reference could help the reader (as e.g. in the section on elliptic functions) none is given; of the four references in Chapter VII, two (those to Montel and to Bourbaki) are irrelevant to the author's main purposes. There is nothing to indicate to the reader which results should be considered important, which ones could be further extended. As to the style, it is that of the modern algebraic or formalistic school; the resources of the English vocabulary and syntax could not be cut down any further. Definitions are either not motivated, or else the patronizing tone in which this is done indicates that it is mere condescension to human weaknesses of which the author does not approve. At the same time, as one could expect of him, he achieves everywhere the utmost precision; there is never one vague word to mislead the unwary, perplex the novice, or let loose the fancy of the imaginative. The reader is not to look forward to a conducted tour through a picturesque countryside; he is on a bus which runs to a schedule. Why should he want to look out of the window?

Enough has been said to indicate that, in spite of some shortcomings which it was our duty to point out, this is a valuable and useful book, and also a timely one. While it is not as attractively written as the classical paper of Dedekind and Weber, or as H. Weyl's *Idee der Riemannschen Fläche*, it covers far more ground than the former, and, even in its final chapter, has little in common with the latter. It was highly desirable that the principles of the theory of algebraic functions should be treated at least once in their full generality by purely algebraic methods; this is what the author has done as perhaps no one but he could do it, and for this he has a right to expect the gratitude of the mathematical community. His attitude towards his subject has been professedly one-sided; but his work should be of value, not only to those who will always prefer the algebraic methods for their own sake, but also to those who wish to ascertain both their scope and their limitations. Indeed some conclusions already seem to emerge from it, and will now briefly be set forth.

This branch of mathematics, says the author (p. v), has an "algebraico-arithmetical" and a geometric aspect; it is surprising that he should not even mention the function-theoretic method, which was that of Riemann, and is still in many ways the most powerful one of all; alone it supplies the proof for Riemann's existence theorems, and is therefore the only source of our present knowledge about the structure of the non-abelian extensions of a function-field; in higher-dimensional problems, it leads directly to harmonic integrals and the theory of Kähler manifolds, which has achieved such striking successes in the last 20 years. Even now its advantages are such that one who is chiefly interested in characteristic p will frequently begin by investigating the "classical" case and will do so by using function-theory, topology and harmonic integrals. However, as the author points out, there are valid reasons for considering other fields of constants than the complex numbers; if the characteristic is 0, it is still possible, by "Lefschetz' principle," to apply to them many of the results obtained in the classical case by function-theory; but finite fields of constants are becoming increasingly important, both for their own sake and because of possible applications to analytic number-theory; in particular, the so-called "singular series" depend upon numbers of solutions of equations over finite fields. Therefore, if for no other reason, one must be able to deal at any rate with fairly general fields of constants; and while in substance the means for doing so must be of a purely algebraic nature, one has to choose here between the language and technique of algebraic geometry and another language, originating in the theory of fields of algebraic numbers, which takes the function-field as the primary object of its study. "Whichever method is adopted," says the author, "the main results to be established are the same"; his book is good evidence to the contrary. On the one hand, the algebraic method which he follows emphasizes the analogies with algebraic numbers; one of its main advantages, in fact, is that many questions concerning algebraic numbers and functions of one variable can be so treated simultaneously, as has been shown by Artin and Whaples, and more recently by Hasse in his *Zahlentheorie*; in the book we are reviewing, the greater part of Chapters I, III and IV applies with little change to number-fields, and it is perhaps unfortunate that this is not pointed out there. For the same reasons, class-field theory can best be treated, at least at present, from that point of view; it is true that a treatment of class-field theory over function-fields of dimension 1 by the methods of algebraic geometry is greatly to be desired, for its own sake and as a preparation for the same theory in higher dimensions; but this would not, at least for some time to come, deprive the algebraic method of the advantages which it now seems to possess.

It also appears that the algebraic method is as yet the only one to deal with certain phenomena connected with inseparability. The algebraic geometer, in order to study a function-field, must assume, in the case of dimension 1, that it possesses a model without multiple points (a curve whose points are all absolutely simple in the sense of Zariski). But perhaps not much is lost by that assumption. It is always fulfilled in the case of a finite ground-field, since such fields are perfect; and the curves which arise from the application of the Picard method to varieties of higher dimension are without singular points, provided one starts from a "normal" variety. The fields which have no non-singular model may therefore, at the present moment, be regarded as pathological beings; these are the fields whose genus decreases under a suitable extension of the ground-field, and they are necessarily inseparable. If the author had excluded them, he could have spared himself some of the complications inherent in Chapters IV, V and VI; and the reader would not be left with the uncomfortable feeling of being told (sometimes without proof) that certain unpleasant things can happen, but not how and when they may happen. Thus, by imposing upon himself the task of working with an entirely unrestricted field of constants, the author, perhaps unintentionally, has overemphasized certain more or less pathological features connected with characteristic p, because he had to devise procedures which do not exclude them; on the other hand. far more interesting facts on characteristic p, such as Witt's residue-theorem for "Witt's vectors" (Crelles J., 176 (1936), p. 140), are not included, perhaps because the author considers them as pertaining to class-field theory.

The algebraic method begins to show its weakness when it comes to dealing with extensions of the field of constants. Here also a new language and new techniques had to be invented by the author, chiefly in order to show the invariance of the most important properties of a function-field under such extensions; in his introduction, he acknowledges the considerable effort which this has cost him, and, strangely enough, finds no better justification for it than a reference to a notably unsuccessful paper of Deuring on correspondence-theory, where the latter rediscovered rather clumsily a few of Severi's more elementary results on the same subject. Undoubtedly, whatever method is adopted, there are some crucial proofs (e.g. that of Theorem 4, Chapter V, p. 96) which cannot be avoided; but most readers of this book will feel the need for a language by which those properties and results which are invariant under an extension of the ground-field can be expressed and proved in a manner independent of that field; this is what algebra does not do, and what algebraic geometry does without any effort.

All this would not be decisive; as long as the geometer is exploring curves and nothing else than curves, the algebraist can keep pace with him; he will sometimes be in front, and at worst not far behind. What is decisive is that algebra stops short of higher-dimensional problems; and, even in the theory of curves, these cannot be avoided. To begin with, there are times when curves have to be embedded into projective spaces; even the author could not refrain at least once (at the end of Chapter IV) from interpreting in this manner a statement on the differentials of a non-hyperelliptic field. But the crucial test is supplied by the theory of correspondences, which is the theory of the product of two curves, and by that of the jacobian variety of a curve; there it would be impossible to take function-fields as the primary object, since one has to deal with properties of surfaces and varieties which depend upon the use of a particular model, and are not invariant under arbitrary birational transformations. It is therefore no accident that in the present book the group of divisor-classes of degree 0, which is nothing else than the jacobian, is discussed only in the "classical case" and by topological methods (v. Theorem 16 of Chapter VII, p. 176, and its corollary); and it is no accident that the algebraists who attacked those problems by their own methods failed to obtain any significant results.

Thus it appears that the author has somewhat overstated his claims, and has been too partial to the method dearest to his algebraic heart. Who would throw the first stone at him? It is rather with relief that one observes such signs of human frailty in this severely dehumanized book. And it would only remain for us to congratulate him on the service he has rendered to the mathematical public, if it were not necessary to devote some of our attention to typographical matters.

The book is generally well printed, and fairly free from misprints, much more so indeed than most previous publications of the same author; here are a few which might embarrass the reader: pp. 98–99, the references to Theorem 5 are really to Theorem 4; p. 111, l. 6–7, instead of "for the elements . . . to all be" read "for all the elements . . . to be"; p. 121, l. 6 from below, for "ramification index" read "differential exponent"; p. 123, l. 14, for "Theorem 5, V, §5" read "Theorem 4, V, §6"; p. 132, last line, for "such that" read "such that $\omega\equiv 0$ (mod. $\mathfrak{a}^{-2}$) and that"; p. 165, l. 22, for 2-chain, read 2-cycle. It is regrettable that the running title at the top of each page does not include the indication of the Chapter and §, as this makes references unnecessarily hard to find. But the point upon which we wish to draw attention is a far more serious one, and one which affects not merely this volume, but all modern mathematical printing in America. No typesetter would separate the word "and" into "a-" and "nd" at the end of a line. Yet on p. 169 of this book, $H_1(S-P, Q)$ is broken into $H_1(S-$ and $P, Q)$; on p. 149–150, a similar formula is similarly broken between one page and the next; on p. 121, the two factors of a product occur, one on line 22 and the other on line 23; several dozens of such instances could easily be given. It is difficult enough to follow such a text in detail without having constantly to reconstruct in one's mind what has been separated on paper; and, apart from all aesthetical considerations, such practices, which in this country are fast becoming the rule rather than the exception, may soon make many of our mathematical texts intolerably hard to read. It is high time that a reaction should set in against the tendency to cram as much text as possible into each page at the lowest possible cost, regardless of the effect on the reader; this will require a coördinated effort on the part of authors, editors and the printing-presses. The authors, who undoubtedly bear some responsibility for the present situation, should be more mindful of such matters in the preparation of their manuscripts; editors and editorial assistants should cooperate with them to a greater extent than sometimes happens now. As to the typesetters, who are doing an extraordinarily good job of setting the most complicated formulas, they could very easily be trained to avoid broken formulas, if their attention were drawn to it by the presses; they could well be trusted to use their judgment in displaying some long formulas, even in the absence of an indication from the author or editor; as to short formulas, all that is mostly required is some adjustment in the spacing of words; this might sometimes take more time than mechanically running along, but would still be far less expensive than later corrections which may affect a whole paragraph of type. Possibly, at least in the transitional period until typesetters acquire experience in such matters, the average cost of the printed page in mathematical texts would increase slightly; possibly the number of pages to be printed every year by mathematical journals would have to be somewhat cut down. Maybe the gain would be greater than the loss.

A. WEIL

Projektive Differentialgeometrie. Part I. By G. Bol. (Studia Mathematica, vol. 4.) Göttingen, Vandenhoeck and Ruprecht, 1950. 8+365 pp. 20 DM.

The present book is the first part of a treatise on projective differential geometry. This volume is divided into four sections: I. *Plane curves*; II. *Introduction to space geometry*; III. *Space curves*; IV. *Surface strips* (*Flächenstreifen*). The second (and last) volume will deal more specifically with the theory of surfaces. As the author points out in his preface, the main difference between his treatise and preceding ones on projective differential geometry is in the method employed. The general idea is that of choosing the arbitrary elements in the analytic representation of the quantities involved (proportionality factor in the homogeneous projective coordinates of a point; change of parameter) so that the formulas are as simple as possible. As an example of this process, let us denote a point in the projective plane by a vector x, so that a (regular) arc of a curve is obtained by giving the (three) components of x (defined, of course, only up to a factor different from zero) as functions of a parameter t (with all the regularity conditions required by the subsequent developments). If the arc under consideration has no inflexion points, then the determinant $D=(x, x', x'')$ (the primes indicating differentiation with respect to t) is different from zero, and we can always normalize the vector x so that along our arc: (1) $D=\text{const.} \neq 0$. For a transformation of parameter: $t^*=t^*(t)$, (1) is not preserved. The author associates any transformation of parameter with such a normalization of the vector x that (1) still holds. Analogous methods are followed for space curves and surfaces. Also the duality principle is extensively used to simplify the formulas.

In the first section, after the above mentioned specification for the representation of a plane curve, we find the usual developments about local projective frames, local expansions, projective arc and projective curvature. The last section is devoted to questions of projective differential geometry in the large; for example, the author proves that on any closed curve without inflexions, and cut by any straight lines in no more than 2 points, there are at least 6 sextactic points (that is, points where the osculating conic is stationary).

In the second section, the author presents an introduction to space geometry; we begin with curves, ruled and developable surfaces. A number of paragraphs are devoted to the geometry of the line in R_5, and its applications to ruled surfaces, linear complexes, null systems, and so on. The following paragraphs contain the first formulas of the theory of surfaces. This subject will be dealt with in more detail in the second volume; however, we find here, in addition to the ideas of asymptotic lines and of asymptotic parameters, the quadrics of Lie and Darboux, the tangents of Darboux and Segre, the most important results about conjugate nets, and the successive Laplace-transforms of a given surface, with the cases in which a Laplace-chain breaks down; here some extensions to hyperspaces are also given. After presenting the main theorems of the theory of line congruences, the section ends with a very accurate analysis of the ideas of analytic and geometric contact between curves or surfaces; the fact that the idea of geometric contact is independent of the analytic representation, as well as its topological rather than projective character, are presented in the clearest way.

In section III, the author's method of treating questions of normalization yields its best results, making possible the introduction, with a minimum of effort, of the most important configuration attached to a space curve. By attaching a suitable projective frame to any point of the curve, we arrive quite easily at the C^3's of Fubini (in particular, the harmonic C^3), the osculating linear complex, and the characterization of the curves belonging to a linear complex, to a quadric, and so on, by means of the ordinary differential equation of which the components of the vector x (associated with a point of our curve) are a set of independent solutions.

Section IV is the most original and presents in detail many of the author's recent results about the geometry of surface strips. Here too a suitable projective frame is associated with every point of the strip-curve, and by using the duality principle the author succeeds in giving a particularly simple form to most of his results. Among the many results of this section, we mention the ideas of osculating conic, cone, and quadric, as well as the surfaces generated or enveloped by them, the ruled surfaces belonging to a strip, in particular those for which the strip-curve is flecnodal, and so on. The section ends with some results about the strips of the second order (defined by assigning for every point of the strip curve, and in every strip plane, the asymptotic tangents of any surface containing the strip), and their connections with pangeodesic strips.

As one may see from the preceding survey, the book is mainly concerned with the projective differential geometry of ordinary space, although some topics of the projective differential geometry of hyperspaces are dealt with; this limitation is, in the reviewer's opinion, the only adverse criticism to be made of this excellent book. The reviewer would have liked very much, for instance, an exposition of the connection between curves in a projective n-space and linear ordinary differential equations. The proofs are very clear and rigorous; the geometrical point of view is never lost, and this will certainly help to make Dr. Bol's book very welcome. Very little previous knowledge is required from the reader, who will find after the last section a summary of the most important results and theorems which were needed in the text. In addition, at the end of every paragraph the author presents a collection of exercises and additions.

The book concludes with a very complete bibliography; this is intended to be the continuation of the one appearing at the end of the *Introduction à la géométrie projective différentielle* (Paris, 1931), by Fubini and Čech; the author's list includes almost all papers and books on this subject from 1931 to 1950, and will certainly be of the greatest aid to everyone interested in projective differential geometry.

V. DALLA VOLTA

Στοιχεῖα θεωρητικῆς γεομετρίας. (*Elements of theoretical geometry*). By N. Sakellariou. Athens, Pountza, 1950. Vol. I, 224 pp.; vol. II, 208 pp.; vol. III, 208 pp.

One approaches the task of reviewing this work with more than usual interest since here at the middle of the twentieth century appears a new *Elements* of geometry written some twenty-two and a half centuries after the *Elements* of Euclid (c. 300 B.C.) and in the same language. Although this work appears in modern Greek, one cannot help but marvel at the fact that most of it can be read with the help of the same dictionary that unlocks the treasures of Euclid.

The author of the present work is a Professor of Mathematics at the University of Athens; Euclid was the head of the mathematics group at the Museum in Alexandria. Both works cover essentially the same material although the modern *Elements* includes many theorems discovered in the centuries since Euclid, such as the circle of Lemoine, the triangle and circle of Brocard, Tucker's circle, and similar complicated constructions. It is also worthy of observation that students who purchased the original *Elements* paid for it in coins of the same denomination as those used to buy the modern text, namely, drachmas, although the values have slipped a bit during the centuries; for while the original might have been purchased, perhaps, for two or three drachmas (3 drachmas = one bushel of wheat) the publisher's announcement states that the first volume of the new *Elements* will cost around 20,000 drachmas.

It is also an interesting matter to compare the contents of the present work with that of Euclid. One difference is the frequency of historical comment which illuminates the text since such material obviously was not available to the father of the subject, although modern historians would have welcomed information on the part of Euclid as to the origin of much of his material, his debt to Pythagoras and Eudoxus among others. The modern work contains pictures of Euclid, Pythagoras, Archimedes, Pierre de Fermat, and K. F. Gauss, with brief biographical sketches of these mathematicians. The first book begins with a short history of geometry starting with Thales

(643–548 B.C.), and many more historical allusions are scattered through the book than are to be found in comparable American texts.

The first volume, of the three which comprise the complete work, deals with traditional material about lines, angles, triangles, and circles and contains 352 exercises, some of them quite elaborate. The second volume continues with theorems about triangles, polygons, and circles. Several proofs of the theorem of Pythagoras, including the traditional one, are given. A five-decimal approximation to π is computed, using the perimeter of an inscribed polygon of 768 sides, and some history of the modern computation and the nature of π is included. Midway through the second volume one finds in succession proofs of the theorems of Menelaus, Desargues, and Ceva, theorems which are usually met with by American students in college courses. This volume concludes with an account of what is sometimes called the modern geometry of the triangle and the circle, the constructions of Lemoine, Brocard, Tucker, and other members of this school which flourished in the nineteenth century. The second volume contains 501 exercises, many of them of a difficult nature.

The third volume is devoted to solid geometry (stereometry) and does not appear in its 208 pages to go much beyond the material usually included in American text books on the subject. One does find, however, a proof of Euler's proposition that in convex polyhedra the sum of the vertices and the sides exceeds by two the number of edges. This volume contains 517 exercises.

The work as a whole forms an excellent treatment of Euclidean geometry and contains material that is usually difficult to find. The volumes are well printed and the figures are clear and well drawn.

H. T. DAVIS

NOTES

Dr. R. A. Rankin of Cambridge University has been appointed to a professorship at The University, Birmingham, England.

Mr. R. J. Arthur has accepted a position as aerodynamicist at Bell Aircraft Corporation, Buffalo, New York.

Mr. Sholom Arzt of New York University has accepted a position as mathematician at the Applied Physics Laboratory of Johns Hopkins University, Silver Spring, Maryland.

Mr. D. Y. Barrer of Northwestern University has accepted a position as staff member with the Operations Evaluation Group in the Office of the Chief of Naval Operations, Washington, D. C.

Mr. R. W. Bryant of Miami University has accepted a position as examiner at the Naval Research Laboratory, Washington, D. C.

Dr. K. L. Chung has been appointed to a visiting associate professorship at Cornell University.

Mr. Carl Cohen of Cambridge Junior College has been appointed a lecturer at Wheaton College, Norton, Massachusetts.

Mr. M. L. Curtis of Northwestern University has been appointed a member of the Institute for Advanced Study.

Mr. R. B. Davis of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor C. R. DePrima of the California Institute of Technology has been promoted to an associate professorship. He is on leave of absence and has accepted a position at the Office of Naval Research, Washington, D. C.

Assistant Professor N. J. Divinsky of Ripon College has been appointed to an assistant professorship at the University of Manitoba.

Mr. F. T. Dresser of the University of Virginia has accepted a position as analyst in the Department of Defense, Washington, D. C.

Mr. E. L. Eagle of the University of Tennessee has accepted a position as electro-mechanical engineer at the Glenn L. Martin Aircraft Company, Baltimore, Maryland.

Mr. J. L. Ericksen has accepted a position as mathematician with the Naval Research Laboratories, Washington, D. C.

Miss Sarah G. Fleming has accepted a position as engineers' assistant at the General Electric Company, Schenectady, New York.

Mr. W. H. Fleming has accepted a position as mathematician at the Rand Corporation, Santa Monica, California.

Mr. I. M. Garfunkel has accepted a position as mathematician with the Naval Ordnance Test Station, Pasadena, California.

Mr. Daniel Gorenstein has been appointed to an assistant professorship at Clark University.

Dr. R. N. Goss of the University of Tulsa has accepted a position as mathematician at the U. S. Navy Electronics Laboratory, San Diego, California.

Dr. E. J. Gumbel has been appointed a consultant in the Applied Mathematics and Statistical Laboratory at Stanford University.

Mr. Ralph Hafner has accepted a position as mathematician at the Naval Ordnance Plant, Indianapolis, Indiana.

Associate Professor Edwin Hewitt of the University of Washington is on leave of absence and has been appointed a visiting lecturer at the University of Uppsala.

Dr. B. E. Howard has accepted a position as senior mathematician at the Air Weapons Research Center of the University of Chicago.

Assistant Professor Lamar Layton of James Millikin University has accepted a position at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Mrs. J. E. Maxfield has accepted a position as mathematician at the Naval Ordnance Test Station, China Lake, California.

Mr. F. D. Miller has accepted a position as mathematician at Bell Aircraft Corporation, Buffalo, New York.

Dr. C. E. Mullan has accepted a position as engineer at Bell Aircraft Corporation, Buffalo, New York.

Dr. G. G. Murray has accepted a position as mathematician at the Applied Physics Laboratory of Johns Hopkins University, Silver Spring, Maryland.

Dr. J. E. L. Peck of Brown University has been appointed a senior lecturer at the University of Natal.

Associate Professor P. M. Pepper of Ohio State University has been appointed research coordinator of the Ohio State University Research Foundation.

Assistant Professor J. W. Ponds of West Virginia State College has accepted a position as mathematician with the U. S. Air Force, Washington, D. C.

Assistant Professor M. H. Protter of Syracuse University has been appointed a member of the Institute for Advanced Study.

Mr. H. G. Rice of Syracuse University has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor L. G. Riggs of San Diego State College has been appointed to an assistant professorship at Ohio University.

Dr. Jane C. Rothe has accepted a position as mathematician at the

Air Force Cambridge Research Laboratories, Cambridge, Massachusetts.

Mr. W. T. Sandlin has accepted a position as manager and advertising director at C. E. Sandlin and Son, Richmond, Kentucky.

Mr. L. L. Scott has been appointed to an assistant professorship at the University of Mississippi.

Mr. R. G. Selfridge has accepted a position as mathematician at the Naval Ordnance Test Station, China Lake, California.

Assistant Professor W. T. Sharp of the University of Maryland has accepted a position as assistant research officer with the National Research Council, Deep River, Canada.

Mr. Maurice Sion has accepted a position as mathematician in the Ordnance Development Division of the National Bureau of Standards, Washington, D. C.

Mr. George Stephenson has accepted a position with the Armed Forces Security Agency in the Department of Defense, Washington, D. C.

Assistant Professor W. M. Stone of Oregon State College has accepted a position as engineer at Boeing Airplane Company, Seattle, Washington.

Mr. Irwin Stoner of the University of Minnesota has accepted a position as assistant scientist at the Rosemount Research Center, Minneapolis, Minnesota.

Mr. R. B. Talmadge has accepted a position as research engineer at North American Aviation, Downey, California.

Mr. Michael Tikson has accepted a position with the Office of Air Research, Dayton, Ohio.

Mr. C. H. Tyler of Lewis College has accepted a position as engineer at the Bell Aircraft Corporation, Buffalo, New York.

Mr. J. W. Weihe has been appointed lecturer at the University of California, Berkeley, California.

Dr. J. G. Wendel of Yale University has accepted a position at the Rand Corporation, Santa Monica, California.

Mr. J. G. Wozencraft has accepted a position as mathematician at the Naval Ordnance Test Station, China Lake, California.

The following promotions are announced:

J. J. Andrews, St. Louis University, to an assistant professorship.

H. A. Antosiewicz, Montana State College, to an associate professorship.

Winifred A. Asprey, Vassar College, to an associate professorship.

May M. Beenken, Immaculate Heart College, to a professorship.

F. M. Carpenter, Colorado School of Mines, to an associate professorship.

Abraham Charnes, Carnegie Institute of Technology, to an associate professorship.

R. A. Clark, Case Institute of Technology, to an assistant professorship.

S. H. Crandall, Massachusetts Institute of Technology, to an associate professorship.

D. B. Dekker, University of Washington, to an assistant professorship.

R. M. Exner, Syracuse University, to an assistant professorship.

Herbert Federer, Brown University, to a professorship.

Alvaro Ferlini, St. Joseph College, to an assistant professorship.

A. D. Fialkow, Polytechnic Institute of Brooklyn, to a professorship.

G. H. Handelman, Carnegie Institute of Technology, to an associate professorship.

A. E. Heins, Carnegie Institute of Technology, to a professorship.

L. Aileen Hostinsky, Temple University, to an assistant professorship.

J. G. Kemeny, Princeton University, to an assistant professorship.

E. H. Lee, Brown University, to a professorship.

Eugene Leimanis, University of British Columbia, to an associate professorship.

Howard Levene, Columbia University, to an assistant professorship.

H. D. Lipsich, University of Cincinnati, to an assistant professorship.

Henry Lisman, Yeshiva University, to an associate professorship.

Albert Newhouse, University of Houston, to a professorship.

Emma J. Olson, Kent State University, to a professorship.

Paul Olum, Cornell University, to an associate professorship.

K. S. Phelan, St. John's University, to an assistant professorship.

F. M. Pulliam, U. S. Naval Postgraduate School, Annapolis, Maryland, to an associate professorship.

C. E. Rickart, Yale University, to an associate professorship.

J. P. Russell, Polytechnic Institute of Brooklyn, to an associate professorship.

F. J. Scheid, Boston University, to an assistant professorship.

S. S. Shu, Illinois Institute of Technology, to an associate professorship.

H. W. Syer, Boston University, to an associate professorship.

C. J. Tremblay, Bard College, to an assistant professorship.

Robert Ullman, Polytechnic Institute of Brooklyn, to an assistant professorship.

G. W. Whitehead, Massachusetts Institute of Technology, to an associate professorship.

Louise A. Wolf, University of Wisconsin in Milwaukee, to an associate professorship.

H. E. Wolfe, Indiana University, to a professorship.

The following appointments to instructorships are announced: University of California, Los Angeles: Dr. S. S. Walters; University of Chicago: Mr. Erwin Kleinfeld; Columbia University: Mr. E. K. Blum; University of Illinois: Dr. K. T. Chen; Massachusetts Institute of Technology: Dr. J. F. Nash; University of New Hampshire: Mr. Frederic Cunningham; Swarthmore College: Dr. Joanne Elliott; Western Reserve University: Dr. A. S. Littell; Woman's College, University of North Carolina: Miss Frances L. Wolfe.

Sister M. Bertrand Walton of Marywood College died in November 1950. She had been a member of the Society for twenty-three years.

Reverend Joseph Wilczewski died at the age of seventy-three. He had been a member of the Society for twenty-seven years.

NEW PUBLICATIONS

Advances in applied mechanics. Vol. 2. Ed. by R. von Mises and T. von Kármán. New York, Academic Press, 1951. 10+233 pp. $6.50.

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Scott, E. J. See Banach, S.

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THOMSON, W. T. *Laplace transformation. Theory and engineering applications.* New York, Prentice-Hall, 1950. 9+230 pp. $3.75.

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WEISE, K. H. See KÖNIG, R.

WHEELER, D. J. See WILKES, M. V.

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The initiation fee and annual dues of members of the Society (see this BULLETIN, November, 1950, pp. 567–568) are payable to the Treasurer of the Society, A. E Meder, Jr., 80 Waterman Street, Providence 6, R I.

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ON THE EXISTENCE OF PLANE CURVES WITH PRESCRIBED SINGULARITIES

GERALD B. HUFF

1. Introduction. It is the purpose of this lecture to present three problems on the existence of plane curves with prescribed singularities and to give some indication of the present state of these questions. The geometry is the classical algebraic geometry of the plane over the field of complex numbers.

If n, m, δ, κ, and i are the order, class, number of double points, number of cusps, and numbers of inflections of an irreducible plane curve $f=0$, these numbers are Pluecker characteristics of $f=0$, and satisfy Pluecker's equations:

$$\text{(P)} \qquad \begin{aligned} m &= n(n-1) - 2\delta - 3\kappa, \\ i &= 3n(n-2) - 6\delta - 8\kappa. \end{aligned}$$

It is natural to raise the question:

I. *Given a solution of* (P) *in non-negative integers, does there exist an irreducible plane curve for which these integers are the Pluecker characteristics?*

A solution of (P) with this geometrical interpretation will be said to be *proper*.

The second question arises in the theory of linear systems of plane curves. A *linear system* Σ is given by

$$\lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_d f_d = 0,$$

where the λ's are parameters and $f_0, f_1, \cdots, f_d$ are ternary forms of order x_0. If $f_0, f_1, \cdots, f_d$ are linearly independent, then d is the *dimension* of the linear system Σ. If P_i is a point such that all the curves of Σ are on P_i, then P_i is a base point of Σ. If the general curve of Σ has multiplicity x_i at P_i, then Σ is said to have multiplicity x_i at P_i. Let $P_1, P_2, \cdots, P_\rho$ be base points of Σ with multiplicities $x_1, x_2, \cdots, x_\rho$. If Σ contains all curves of order x_0 with these multiplicities at these points, then Σ is said to be *complete* (with respect to these multiplicities at these base points). If the linear conditions imposed by asking that a curve of order x_0 have these multiplicities at these points are independent, then Σ is said to be *regular* (with

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respect to these multiplicities at these points). The set of non-negative whole numbers $\{x_0; x_1, x_2, \cdots, x_\rho\}$ is called the *characteristic* x of Σ. If the general curve of the system Σ is irreducible and is of genus p, then the system is of *genus* p. If x is the characteristic of a complete and regular linear system of dimension d and genus p, and for which the general curve is irreducible and has no singularities except those prescribed at the base points, then these integers satisfy Cremona's equations [5, 3]:[1]

$$\text{(C)}\qquad \begin{aligned} x_0^2 - x_1^2 - x_2^2 - \cdots - x_\rho^2 &= d + p - 1, \\ 3x_0 - x_1 - x_2 - \cdots - x_\rho &= d - p + 1. \end{aligned}$$

(If $d>0$, the assumption that the general curve of Σ is irreducible implies that Σ has no singularities except at the base points.) It is customary to call any set of integers, $x=\{x_0; x_1, x_2, \cdots, x_\rho\}$, a *characteristic*, whether or not it arises from a complete and regular linear system. Our second question may be phrased as follows:

II. *If* $x=\{x_0; x_1, x_2, \cdots, x_\rho\}$ *is a characteristic composed of non-negative integers which determine non-negative* p *and* d *in* (C), *does there necessarily exist a linear system* Σ *for which* x *is the characteristic?*

A characteristic x which is associated with a linear system in this way is said to be *proper* (or geometric).

Cremona and his followers were interested in this question in the case $p=0$ and $d=2$. The interest in these linear systems—homaloidal nets—arises from the following fact. If a homaloidal net in a plane Π is put into projective correspondence with the (homaloidal) net of lines in a plane Π', there is induced a point transformation between the planes which is birational and single-valued both ways with the exception of a finite number of points in each plane. Under this Cremona transformation a linear system Σ with characteristic x at ρ general points $F_1, F_2, \cdots, F_\rho$ has an image Σ' with characteristic x' at a congruent [1] set of points in Π'. In fact, x' is the image of x under a linear substitution with integer coefficients:

$$S:\quad \begin{aligned} x_0' &= n x_0 - s_1 x_1 - \cdots - s_\rho x_\rho, \\ x_1' &= r_1 x_0 - a_{11} x_1 - \cdots - a_{1\rho} x_\rho, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ x_\rho' &= r_\rho x_0 - a_{1\rho} x_1 - \cdots - a_{\rho\rho} x_\rho. \end{aligned}$$

Here $\{n; s_1, \cdots, s_\rho\}$ is the characteristic of the homaloidal net which determines the transformation. The integers n, s_i, r_j, a_{ij} are non-negative and the forms $Q=x_0^2-x_1^2-x_2^2-\cdots-x_\rho^2$ and $L=3x_0-x_1-x_2-\cdots-x_\rho$ are invariant under S. The form of the last question is now clear.

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[1] Numbers in brackets refer to the references at the end of the paper.

III. *If S is a linear substitution of the form above in which the letters are non-negative integers and which leaves Q and L invariant, is this S necessarily associated with a planar Cremona transformation in the way described above?*

A linear substitution S which is asociated with a Cremona transformation is said to be *geometric.*

It is known that the answer to all of these questions is *no.* For each question, our procedure will be to give a simple example to show that the answer is in the negative and to examine modifications of the question suggested by the example.

2. **Improper solutions of Pluecker's equations.** In Salmon-Fiedler [18], mention is made of a curve with $n=7$, $\delta=0$, and $\kappa=13$, which makes $m=3$ and $i=1$ in the equations (P). Recall now that the Pluecker characteristics of a curve also satisfy the dual equations:

$$\text{(P*)}\qquad \begin{aligned} n &= m(m-1)-2t-3i,\\ \kappa &= 3m(m-2)-6t-8i, \end{aligned}$$

where t designates the number of bitangents and the remaining letters have the meaning described above. Substituting the numbers above into the first of these dual equations, we see that (P*) yields $t=-2$ and we conclude that there is no curve of order 7 with no double points and thirteen cusps.

While it is necessary to require that all the values of n, m, δ, κ, i, t must be non-negative to insure the existence of a curve for which any subset are Pluecker characteristics, it is still easy to construct an example showing that this is also not sufficient. A simple example is $n=m=4$, $\delta=t=4$, and $\kappa=i=0$. When an irreducible curve does exist, its genus p is given by

$$p=(n-1)(n-2)/2-\delta-\kappa.$$

Thus we are led to a modification of the original equation which has received considerable attention.

I$_a$. *Given a solution of* (P) *and* (P*) *in non-negative integers such that $p=(n-1)(n-2)/2-\delta-\kappa$ is also non-negative, does there exist an irreducible plane curve for which these integers are the Pluecker characteristics?*

In his well-known article on the principle of projection and section, Veronese [20] showed that the answer to I$_a$ is yes for $p=0$. Later, Lefschetz, in a much quoted paper [**15**], showed that the answer to I_a is in the affirmative for $p=0$, 1, and 2. He further showed that the same would be true for $p=3$ if it were possible to construct a curve for $n=7$, $\delta=1$, and $\kappa=11$. In 1931, while studying problems about algebraic surfaces [**21**], Zariski obtained a curious theorem about the cusps of a plane curve. Let a plane curve of order n have cusps and let j be an integer such that $6j<n$. The theorem asserts that the linear system of curves of order $n-3-j$ with simple points at the cusps is a regular linear system. Now suppose that a curve of order 7 with 11 cusps can exist. It then follows that there exists a regular linear system of cubics on eleven points and this makes the dimension -2! This theorem thus shows that even the modified question I_a must also be answered in the negative.

Lefschetz and Zariski made use of the fact that if n, p, κ are given, then the remaining characteristics are determined as integers. They studied the arithmetic conditions imposed on n, p, and κ by the fact that these remaining integers must not be negative in a proper solution. Finally, Zariski [**22**] gave a set of inequalities satisfied by n, p, and κ which were necessary and sufficient conditions that the remaining integers be non-negative. To these he adjoined one more inequality which insures that the solution corresponding to n, p, κ cannot be shown to be improper by his theorem. The final form of the first question may be put as follows:

I_Z. *Is there an irreducible plane curve corresponding to every* n, p, κ *which satisfies the Zariski inequalities?*

This is at present an open question.

3. Improper solutions of Cremona's equations. Even in the classical case, $p=0$, $d=2$ studied by Cremona and his followers [**5**; **12**; **14**], there are non-negative solutions of equations (C) which are not associated with homaloidal nets. It is customary here to refer to the fact that Ruffini [17] erroneously gave $\{10; 6, 4, 3^5, 1^2\}$ as the characteristic of a homaloidal net. For our purposes we shall consider the simplest nongeometric solution,

$$(0) \quad x = \{5; 3, 3, 1, 1, 1, 1, 1, 1\}.$$

To see that no homaloidal net corresponds to this solution we may reason as follows. If there were a homaloidal net of this description, it would determine a linear substitution S with $\{5; 3, 3, 1, 1, 1, 1, 1, 1\}$ in the first row and with Q, L as invariants. Now one may make a diophantine argument to show that the remaining coefficients can not all be integers. This shows that $\{5; 3, 3, 1, 1, 1, 1, 1, 1\}$ is nongeometric and suggests also the following modification of question II for $p=0$, $d=2$.

II_a. Is there a homaloidal net associated with every solution of (C) for $p=0$, $d=2$ which appears in the first row of a linear subsitution S which has Q, L as invariants? This conjecture has been repeated in the literature [1; 2] and is easily seen to be equivalent to the original question III. Although one may find incomplete arguments in the literature [4; 13] which claim to show that the answer is in the affirmative, we shall see in a moment that it is easy to construct an example showing that the answer is no.

It should first be pointed out that there is a simpler way of showing that $\{5; 3, 3, 1, 1, 1, 1, 1, 1\}$ is not a geometric solution of equations (C). If there were indeed a linear system of quintics with an irreducible general curve, then the line through the two triple base points would meet this curve in at least six points and this asserts that the curve is reducible. This shows completely that the example is not geometric and suggests a further necessary condition for proper solution. If x is proper and c is the characteristic of a curve of $c_0 < x_0$, then

$$c_0x_0 - c_1x_1 - c_2x_2 - \cdots - c_px_p \geqq 0. \tag{L}$$

In an hour address to the society over a quarter of a century ago, Coble [2] suggested that any solution of (C) for $p=0$, $d=2$ which also satisfied all possible inequalities of the form (L) must then be proper. Here we shall modify our second question to:

II_b. *Is there a linear system with irreducible general curve associated with every solution of equations* (C) *which also satisfy inequalities* (L)?

4. The invariant theory of Cremona's equations. There does exist a reasonably simple theory of characteristics which permits a systematic attack on these problems. If, for a fixed ρ, S and S^* are two integer substitutions which are geometric, then it may be shown that SS^* is also geometric. This is the decisive step in showing that the collection of all geometric S for a given ρ form a group, which we shall designate by G_ρ.

Associated with a quadratic transformation there is the substitution:

$$A_{123}: \quad x_i' = x_i + a_i(a_0x_0 - a_1x_1 - \cdots - a_\rho x_\rho), \qquad i = 0, 1, \cdots, \rho,$$

where a is the vector $\{1; 1, 1, 1, 0, \cdots, 0\}$. For any $\rho \geqq 3$, $A_{123} \in G_\rho$. It is further true that any substitution which represents a permutation of $x_1, x_2, \cdots, x_\rho$ is in G_ρ [2]. We shall designate the group generated by A_{123} and the permutations by (A_{123}, permutations). It is certainly a subgroup of G_ρ.

If T is a linear substitution which leaves Q unchanged, then it follows—exactly as for orthogonal substitutions—that the determinant of T is ± 1. Thus the set of all linear substitutions with integer coefficients and with Q and L as invariants must be a group, which we shall call $G_\rho(I)$. The set of those elements of $G_\rho(I)$ which have n, s_i, r_j, a_{ij} non-negative in the form (S) form a complex of $G_\rho(I)$ which will be designated by $H_\rho(I^+)$. It is clear from these definitions that

$$(A_{123}, \text{permutations}) \subseteq G_\rho \subseteq H_\rho(I^+) \subseteq G_\rho(I).$$

Several of the significant results in this connection can now be stated in terms of whether or not these inclusions are proper inclusions.

For $\rho \leqq 8$, these sets are all finite and are all the same. Du Val has studied the groups in detail [6] and has identified them with certain crystallographic groups. For $\rho = 9$, the sets are all infinite, but even here, all these sets are the same [19]. Notice that our original question III, in this notation, becomes: Is $G_\rho = H_\rho(I^+)$? For $\rho \leqq 9$, questions III and II_a must be answered in the affirmative.

For $\rho > 9$, the situation is different. We should first mention that in a celebrated paper [16] M. Noether showed that

(1) *For all* ρ, (A_{123}, permutations) $= G_\rho$. *That is*, G_ρ *is generated by* A_{123} *and the permutations of* $x_1, x_2, \cdots, x_\rho$.

(Actually, Noether announced a more general result at that time, although this is exactly what his argument demonstrated.) In 1934 Coble constructed an example, for $\rho = 10$, of an element of $G_\rho(I)$ not in $H_\rho(I^+)$ [3]:

(2) *For* $\rho \geqq 10$, $H_\rho(I^+)) \subset G_\rho(I)$.

Much earlier, S. Kantor, in his prize memoir crowned by the Naples Academy [13; 4], had claimed that $G_\rho = H_\rho(I^+)$ and had supported the assertion by an argument containing an incomplete induction. If we pause here to see how Coble constructed his example to prove (2), it will be clear that the faulty argument gave an incorrect answer. Coble thought of $Q = 0$ and $L = 0$ as quadric and hyperplane in a projective space of ρ dimensions and reasoned that the harmonic perspectivity in $L = 0$ and its pole would leave these invariant. He found that these could be written with integer coefficients for $\rho = 7$, 8, 10, and 11. The substitution for $\rho = 10$ was Coble's example. By similar geometric reasoning it may be shown that if $a = \{a_0;\ a_1, \cdots, c_\rho\}$ is an integer point in $L = 0$, and such that

$a_0^2 - a_1^2 - \cdots - a_\rho^2 = -2$, then the linear substitution of the form of A_{123} for this a has Q and L as invariants. Three more ones adjoined to the counter example (0) gives $a = \{5; 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$ as an a satisfying these conditions and an element of $H_{11}(I^+)$, which is obviously not in G_{11} [7]:

(3) *For* $\rho \geqq 11$, $G_\rho \subset H_\rho(I^+)$, *and the answer to question* III *is in the negative.*

A study of the invariants of G_ρ is the setting in which we shall consider the remaining question II_b. It is an almost direct consequence of the definition that:

(4) *If* x *is a proper characteristic and* $g \in G_\rho$, *then* $x' = gx$ *is also proper. Properness is an invariant of* G_ρ.

This is true only if one adopts the rather curious convention that $\{0; -1, 0, \cdots, 0\}$ is a proper characteristic for $p = d = 0$, and is a consequence of the fact that this is the image of the proper characteristic $\{1; 0, 1, 1, 0, \cdots, 0\}$ under A_{123}. Even with this exceptional proper characteristic, x_0, p, and d are non-negative for all proper characteristics. Let A_ρ be the set of all characteristics such that x_0, p, and d are non-negative. Then by using the fact (1) that the generators of G_ρ are known, it may be shown that [8]:

(5) *If* $x \in A_\rho$ *and* $g \in G_\rho$, *then* $x' = gx$ *is in* A_ρ. *The property of being in* A_ρ *is an invariant of* G_ρ.

It is not known if the inequalities (L) are invariant under G_ρ, but the following is a result of this sort. Let E_ρ be the set of proper solutions of (C) for $p = d = 0$. Then if x is a proper solution of (C), $e \in E_\rho$ and $e_0 < x_0$, it follows as before that:

$$\text{(PI)} \qquad e_0x_0 - e_1x_1 - \cdots - e_\rho x_\rho \geqq 0, \qquad e \in E_\rho \text{ and } e_0 < x_0.$$

These are called the *properness inequalities*. It may be shown that [9]:

(6) *If* $x \in A$ *satisfies* (PI) *and* $g \in G_\rho$, *then* $x' = gx$ *also satisfies* (PI). *Satisfaction of the properness inequalities is an invariant of* G_ρ.

Now let $x \in A$ and let $G_\rho(x)$ be the set of images of x under G_ρ. In accordance with (5), if $y \in G_\rho(x)$, then $y_0 \geqq 0$, and there is an element x^* in $G(x)$ such that x^* is of minimal order; that is, x_0^* is a minimum. The result we need here is that [10]

(7) *A characteristic* x *is of minimal order if and only if* $x_0 - x_i - x_j - x_k \geqq 0$, *where* x_i, x_j, x_k *are three of the largest among* $x_1, \cdots, x_\rho$.

We may now ask if all solutions of (C) and (PI) are indeed proper. If x is a solution of (C) and (PI), then $G_\rho(x)$ contains a minimal order element x^* satisfying (C) and (PI) and the condition of (7). Since properness is an invariant, x is proper if and only if x^* is. But it is possible to tabulate all solutions of (C), (PI), and (7) for $p = 0$ and

1. This leads to the conclusion [10]:

(8) *For* $p=0$ *and any* d, *all solutions of* (C) *and* (PI) *are proper. For* $p=1$ *and* d *positive, all solutions of* (C) *and* (PI) *are proper. For* $p=1$ *and* $d=0$, *any solution of* (C) *and* (PI) *of* g.c.d. $=1$ *is proper.*

Since for any x, the inequalities (PI) are included among the inequalities (L), this means that Coble's conjecture is settled: *any characteristic* x *which gives* $p=0$ *and* $d=2$ *in* (C) *and also satisfies the inequalities* (L) *must be proper.* It also gives the answer to the more general question II_b for $p=0$ and 1. For arbitrary values of p, II_b remains an open question.

On the other hand, the problem of giving an arithmetic description of geometric linear substitutions may be regarded as settled. The result (5) states that a geometric substitution S must send elements of A_p into elements of A_p. It may be shown that (8) implies that this is also sufficient [11]:

(9) *A linear substitution with integer coefficients which leaves* Q *and* L *invariant and sends elements of* A_p *into elements of* A_p *must be geometric.*

It should be stated at this point that in restricting our attention to regular linear systems, we have neglected substantial results of J. A. Todd, Nollet, and Jongmans on the existence of linear systems with base points in special position. We close by listing several problems suggested by the ideas we have considered. Is there an appropriate group associated with Pluecker's equations which will permit arguments like those made for Cremona's equations? Are there results like (8) for any values of p and d? What is the most general abstract setting in which these methods have application?

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THE UNIVERSITY OF GEORGIA

THE ASYMPTOTIC DENSITY OF SEQUENCES

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1. Introduction. Our purpose is to outline the recent work on the asymptotic or limit density of sets of positive integers, and to give further details of some recently announced results [45; 46].[1] §§2–7 are concerned with the first objective, and §§8–12 with the second. The related concept of Schnirelmann density is touched upon, but we mention only the high spots of work on this topic, including basic sequences and essential components.

In the case of many sequences, much more than the mere density is known. The prime number theorem, for example, implies that the set of primes has density zero, but it tells much more. However, asymptotic estimates are not available for many sequences, and for these it is of interest to know the density. In selecting examples of sequences of density zero in §11, as applications of the results of §8, we have tried to choose sets for which analytic estimates are not at present known. We are indebted to H. S. Zuckerman for valuable suggestions concerning the formulations in §8.

2. Definitions. The sequence A of positive integers $a_1 < a_2 < \cdots$ has lower density $\delta_1(A)$ and upper density $\delta_2(A)$ defined by

$$(1) \qquad \delta_1(A) = \liminf_{n\to\infty} \frac{A(n)}{n}, \qquad \delta_2(A) = \limsup_{n\to\infty} \frac{A(n)}{n},$$

where $A(n)$ denotes the number of integers of A which are not greater than n. The value $\delta_1(A)$ has been referred to variously as the asymptotic density, limit density, or density of A. In this paper, however, we shall say that A has a density $\delta(A)$ only if $\delta_1(A) = \delta_2(A)$, in which case we can write

$$(2) \qquad \delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}.$$

This is sometimes called the natural density of A. The assertion that almost all positive integers have a certain property P means that $\delta(A) = 1$, where A is the set of integers having the property P. It can be established that for infinite sequences $\delta_1(A) = \liminf n/a_n$,

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[1] Numbers in brackets refer to the references at the end of the paper.

$\delta_2(A) = \lim\sup n/a_n$, so that $\delta(A) = \lim n/a_n$ if this limit exists.

The Schnirelmann density $d(A)$ of the sequence A is defined by

$$d(A) = \text{g.l.b. } \frac{A(n)}{n}. \tag{3}$$

It can be easily proved [53, p. 202] that $d(A) > 0$ if and only if $\delta_1(A) > 0$ and $1 \in A$.

We shall not concern ourselves here with other definitions of density, such as the variation of Schnirelmann density employed by Besicovitch [6], or the maximum and minimum densities of Pólya [51].

3. **The sum of sets.** The sum $A+B$ of two sequences is defined as the set of all integers of the form a, b, or $a+b$ where $a \in A$ and $b \in B$. The conjecture of Khintchine [36] that $d(A+B) \geqq d(A)+d(B)$ or $d(A+B)=1$ was proved by Mann [41]; in fact Mann proved somewhat more than this. Other formulations of Mann's proof and generalizations of his result have been given by Artin and Scherk [1], Dyson [18], and van der Corput [13]. Cheo [10] has given an example to show that the extension of the Mann theorem to higher dimensions (for example, the two-dimensional Gaussian integers) fails. Details concerning the history of this so-called $\alpha+\beta$ problem, and references to earlier work, can be found in various expository papers [30; 48; 53].

It is readily proved [53, p. 206] that if $d(A)+d(B) \geqq 1$ then $d(A+B)=1$; the analogous theorem [53, p. 207] for lower density is that if $\delta_1(A)+\delta_1(B)=1$, then there exists an integer m such that $n \in A+B$ if $n > m$. Suppose that we are given non-negative real numbers α, β, and γ satisfying $0 \leqq \alpha+\beta \leqq \gamma \leqq 1$; the question arises whether there exist sets A and B such that $d(A)=\alpha$, $d(B)=\beta$, and $d(A+B)=\gamma$. This has been answered affirmatively in recent work [9; 39; 50]. As yet unsolved is the corresponding problem for lower density, which can be formulated as follows: for sets A of lower density α, B of lower density β, what is the range of possible values of $\delta_1(A+B)$? Certain inequalities have been obtained, as we now indicate.

The analogue of the $\alpha+\beta$-theorem of Mann for lower density is false. We formulate the known results in terms of the set C of all elements of the form $a+b$ with $a \in A$, $b \in B$. Admit 0 as a possible element of any set A, but do not count it in $A(n)$. Thus $C=A+B$ if $0 \in A$ and $0 \in B$. Erdös [27] proved that $\delta_1(C) \geqq \delta_1(A)+\delta_1(B)/2$ under the hypotheses $0 \in A$, $0, 1 \in B$, $\delta_1(B) \leqq \delta_1(A)$, and $\delta_1(A)+\delta_1(B) \leqq 1$, generalizing an earlier result [26] for the case $A=B$. Shapiro [61] removed the hypothesis $\delta_1(B) \leqq \delta_1(A)$ by use of the Mann theorem. Ostmann [49] also removed this asymmetric hypothesis, and, replacing the hypothesis $0, 1 \in B$ by the assumption that B contain k consecutive integers, established the generalization $\delta_1(C) \geqq \delta_1(A) + (1-k^{-1})\delta_1(B)$. Mann [42] has recently obtained an inequality connecting the lower densities of B and C and a modified Schnirelmann density of A.

4. **Basic sequences, essential components, and the primes.** A basic sequence A is one for which there exists a fixed integer h such that every positive integer is a sum of at most h elements of A. A sequence A is called an essential component if, for every sequence B with $1 > d(B) = \beta > 0$, $d(A+B) \geqq \beta + \phi(\beta)$ where $\phi(\beta)$ depends only on β. Erdös [22; or 38, Theorem 96] has proved that every basic sequence is an essential component. The converse of this is false, by an example of Linnik [40]. Schnirelmann [58; or 38, Theorem 91] has proved that the sequence P consisting of 1 and all primes is basic, in fact that $d(P+P)>0$.

Let K be the collection of kth powers of the positive integers for a fixed integer $k>1$, so that $d(K)=0$. Romanoff [54; or 38, Theorem 103] proved that $d(K+P)>0$, and Davenport and Heilbronn [17] showed that $\delta(K+P)=1$. Next let H be the set of non-negative powers of a fixed integer h. Romanoff [54; or 38, Theorem 106] established that $d(H+P)>0$. Write $r(h)$ for $\delta_1(H+P)$; Landau [37] proved that $\lim_{h\to\infty} r(h) \log h = 1$. In the case $h=2$, van der Corput [14] showed that $\delta_2(H+P)<1$.

5. **Sequences without progressions.** Various writers have discussed sequences no one of whose members is divisible by any other; let $G=\{g_i\}$ denote such a set. Behrend [3] obtained an asymptotic bound for the partial sums of $\sum g_i^{-1}$. It had been conjectured that any such set would have density zero, but Besicovitch [5] constructed a set G_1 with $\delta_2(G_1)>1/4$ and $\delta_1(G_1)=0$. Erdös [24] generalized a result of Besicovitch, and proved [21] that any G sequence must have lower density zero, that is, $\delta_1(G)=0$. More recently Erdös [28] established a necessary and sufficient condition on the elements of G such that $\delta(H)$ exists, H being the set of all multiples of the integers of G.

Let J be a sequence no three of whose terms are in arithmetic progression. It had been conjectured that $J(n)=C(n^{\alpha})$ for some $\alpha<1$, but Salem and Spencer [56] disproved this, later adding a sequel [57]. Their inequality on $J(n)$ was improved by Behrend [4], who constructed a sequence J for which

$$J(n) > n^{1-c/(\log n)^{1/2}} \quad \text{as } n \to \infty.$$

Erdös and Turán [31] proved that any J sequence must satisfy $J(n) < (3/8+\epsilon)n$ for $n > n(\epsilon)$, and Moser [44] improved this to $J(n) < 5+4n/11$ for all n. Whether every J sequence must have density zero, or even lower density zero, is an unsolved problem.

6. **Other special sequences.** Let A_1 be the set of abundant numbers, that is, positive integers n satisfying $\sigma(n) \geqq 2n$, $\sigma(n)$ being the sum of all divisors of n. Behrend [2] showed that $.241 < \delta_1(A_1) \leqq \delta_2(A_1) < .332$. Davenport [15] proved that $\delta(A_1)$ exists, as did also Chowla [11] and Erdös [19]. The integer n is said to be k-abundant if $\sigma(n) \geqq kn$. Davenport [15] established that the density of k-abundant numbers exists and is a continuous function of the real number $k \geqq 1$.

Denoting the number of divisors of m by $\sigma_0(m)$, let A_2 be the sequence of integers m for which $\sigma_0(m+1) > \sigma_0(m)$. Erdös [23] proved Chowla's conjecture that $\delta(A_2) = 1/2$, and that integers of the form $p_1^2+p_2^2-p_3^2-p_4^2$ have positive lower density [25], the p_i being primes.

The set A_3 of integers not divisible by any of the nonzero integers $a_1, a_2, \cdots, a_m$ has density

$$\delta(A_3) = 1 - \sum_i \frac{1}{a_i} + \sum_{i<j} \frac{1}{[a_i, a_j]} - \sum_{i<j<k} \frac{1}{[a_i, a_j, a_k]} + \cdots + \frac{(-1)^m}{[a_1, a_2, \cdots, a_m]},$$

where the brackets denote least common multiple. Heilbronn [34] and Rohrbach [52] proved that

$$\delta(A_3) \geqq \prod_{i=1}^{m} (1 - a_i^{-1}).$$

Consider a fixed integer $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ with every $\alpha_i > 1$, the p_i being different primes. Let B_m denote the sequence of integers n such that n/m is a square-free integer that is relatively prime to m. Define B_1 to be the square-free integers. Schoenberg [59], studying the Fourier transforms of distribution functions, proved that

$$\delta(B_m) = \frac{6}{m\pi^2} \prod_{i=1}^{r} \left(1 + \frac{1}{p_i}\right)^{-1},$$

and indicated that this gave another method to obtain a result of

Feller and Tornier [32], that the density of integers which are divisible by an even number of squares of different primes is

$$\frac{1}{2}+\frac{1}{2}\prod_{i=1}^{\infty}(1-2p_i^{-2}),$$

the product being over all primes. The Schoenberg result for $\delta(B_1)$ is the well known value $6/\pi^2$ for the density of the square-free integers. Details and references concerning density theorems obtained by analytic study of distribution problems of additive arithmetic functions can be found in an expository paper by Kac [35].

Fogels [33] proved that the (rational) integers which are uniquely factorable (apart from units and the order of the factors) in the quadratic field $R((-5)^{1/2})$, have density zero. A natural density can be defined for a set of quadratic integers of $R((-5)^{1/2})$, using the norm to get a counting device. Then the uniquely factorable integers of this particular quadratic field have zero density.

For nonzero integers $a_1, a_2, \cdots, a_m$, say that $n \in A_4$ provided that all of $n^2+a_1, n^2+a_2, \cdots, n^2+a_m$ are square-free. Mirsky [43] proved that either A_4 is empty or $\delta_1(A_4)>0$. S. Selberg [60] showed that integers of the form $ax^2+by^2+c^{z^2}$ have positive lower density, where a, b, and $c>1$ are positive.

Let $p(n)$ denote the greatest prime factor of n. Chowla and Todd [12] have discussed the sequence A_5 of integers n for which $p(n)<2n^{1/2}$, and more generally the sequence A_6 for which $p(n)>cn^{\alpha}$ with $c>1$ and $1/2\leqq\alpha<1$. They prove that $\delta(A_5)=1-\log 2$, and indicate that the same technique yields $\delta(A_6)=-\log\alpha$.

Davenport and Erdös [16] have related the lower density and the logarithmic density $\lim_{n\to\infty}\{\sum_{a_i\leq n}a_i^{-1}\}/\log n$ of a sequence whose terms are divisible by at least one integer from a fixed sequence. Erdös [20] gave some results on the existence of natural densities of certain sequences defined in terms of additive and multiplicative functions.

Salem and Spencer [55] proved theorems concerning the density of sequences having gaps defined by certain analytic conditions.

7. **Measure.** None of the densities δ, δ_1, or δ_2 is a measure. Disjoint sequences A and B can be readily constructed so that $\delta_1(A\cup B)\neq\delta_1(A)+\delta_1(B)$ and $\delta_2(A\cup B)\neq\delta_2(A)+\delta_2(B)$; for example let A be any set not having a density, and let B be the complement of A. As to the density δ, sequences A and B can be constructed [7, p. 571] so that $\delta(A)$ and $\delta(B)$ exist but $\delta(A\cup B)$ and $\delta(A\cap B)$ do not.

R. C. Buck [7] has formulated a measure and a class of measurable sequences, and later added a sequel [8] with E. F. Buck. Let $\mathcal{D}_0$ denote the class of sequences which are unions of a finite number of arithmetic progressions, or which differ from some such union by a finite set. Clearly any set in $\mathcal{D}_0$ has a density. For any sequence S, define $\mu(S)=\lim\inf \delta(A)$ for all $A\in\mathcal{D}_0$ such that $A\supset S$. The class $\mathcal{D}_\mu$ of measurable sets is defined as all sets S such that $\mu(S)+\mu(S')=1$, where S' denotes the complement of S. The class $\mathcal{D}_0$ is a proper subset of $\mathcal{D}_\mu$, which in turn is a proper subset of the class of sequences possessing a density. The measure of any set is its natural density: if $S\in\mathcal{D}_\mu$, then $\mu(S)=\delta(S)$.

Buck proved that $\mathcal{D}_0$ is a proper subset of $\mathcal{D}_\mu$ by exhibiting infinite sequences, for example $\{n^2\}$, which have measure zero but which do not belong to $\mathcal{D}_0$. From a number theoretic point of view, it would be of interest to know whether some of the nontrivial sets having positive density are members of $\mathcal{D}_\mu$.

8. **Upper density.** For any prime p let A_p denote the subset of integers n of A for which $p\mid n$ but $p^2\nmid n$.

THEOREM 1. *If* $\{p_i\}$ *is a set of primes such that* $\sum p_i^{-1}=\infty$, *then* $\delta_2(A)\leqq \sum\delta_2(A_{p_i})$ *for any sequence* A.

PROOF. Let I denote the set of all positive integers, and let B^r be the complement of the set

$$I_{p_1}\cup I_{p_2}\cup\cdots\cup I_{p_r}.$$

Thus B^r is the set of all integers except those n for which $p_i\mid n$ but $p_i^2\nmid n$ for at least one of the primes $p_1, p_2, \cdots, p_r$. Now the number $B^r(x)$ of integers in B^r which do not exceed x can be seen by induction on r to be

$$\sum(-1)^{\alpha_1+\alpha_2+\cdots+\alpha_r}\left[\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}}\right],$$

where the sum extends over the 3^r terms obtained by taking each α_i equal to 0, 1, or 2. The removal of the brackets changes the value of each term by something less than 1, so we have

$$B^r(x)<3^r+\sum(-1)^{\alpha_1+\alpha_2+\cdots+\alpha_r}\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}},$$

or

$$\frac{B^r(x)}{x}<\frac{3^r}{x}+\prod_{i=1}^{r}(1-p_i^{-1}+p_i^{-2}). \tag{5}$$

Now the divergence of $\sum p_i^{-1}$ implies the divergence of $\sum(p_i^{-1}-p_i^{-2})$, and thus we can choose r sufficiently large so that

$$\prod_{i=1}^{r}(1-p_i^{-1}+p_i^{-2})<\frac{\epsilon}{2} \tag{6}$$

for any specified $\epsilon>0$. Next choose N sufficiently large that $3^r/x<\epsilon/2$ for $x\geqq N$. Hence (4), (5), and (6) imply that

$$\frac{A(x)}{x}\leqq\frac{B^r(x)}{x}+\sum_{i=1}^{r}\frac{A_{p_i}(x)}{x}<\epsilon+\sum_{i=1}^{r}\frac{A_{p_i}(x)}{x},$$

from which the theorem follows.

COROLLARY 1. *If for a set of primes* $\{p_i\}$ *we have* $\delta(A_{p_i})=0$ *for every* i, *and if* $\sum p_i^{-1}=\infty$, *then* $\delta(A)=0$.

Feller and Tornier [32, p. 216] have proved the related result that if $\sum p_i^{-1}=\infty$, then the sequence of integers divisible by at most k of these primes has density zero, for fixed k. This is implied by Corollary 1; in fact, we have the following slightly stronger formulation.

COROLLARY 2. *For any fixed* k; *if* $\{p_i\}$ *is a set of primes for which* $\sum p_i^{-1}=\infty$, *and if* A *is any sequence whose members are divisible by at most* k *of these primes to the first degree, then* $\delta(A)=0$.

PROOF. We use induction on k, assuming the truth of the result in the case $k-1$. Denote the set $\{p_i\}$ by P. Then the set A_{p_i} has density zero because its members are divisible by at most $k-1$ primes to the first degree chosen from the set $P-p_i$ (set theoretic subtraction). Thus $\delta(A)=0$ by Corollary 1.

COROLLARY 3. *Let* $\{p_i\}$ *be a set of primes for which* $\sum p_i^{-1}=\infty$. *If* $\delta_2(A)\neq 0$, *then* $\sum_{i=1}^{\infty}\delta_2(A_{p_i})=\infty$.

PROOF. If $\sum_{i=1}^{\infty}\delta_2(A_{p_i})\neq\infty$, then for any given $\epsilon>0$ there exists an m such that $\sum_{i=m}^{\infty}\delta_2(A_{p_i})<\epsilon$. But $\sum_{i=m}^{\infty}p_i^{-1}$ diverges, so by Theorem 1 we have $\delta_2(A)<\epsilon$. Hence $\delta_2(A)=0$, contrary to hypothesis.

9. **Lower density.** We now show that the results of the last section are false for lower density, by exhibiting a set A for which $\delta_1(A)=1/2$ but $\delta_1(A_p)=0$ for every prime p. Let p_k denote the kth prime, $k=1, 2, \cdots$. Now any positive integer n can be located between successive binomial coefficients as follows: $C(m, 2)\leqq n<C(m+1, 2)$. Define the positive integer $f=f(n)=n+1-C(m,2)$, so that f is a single-valued function of n, whereas to each value of f there correspond infinitely many values of n. For any positive integer n, define S^n as

the set of integers x satisfying the inequality $n!\leqq x<(n+1)!$. Define T^n as the set S^n with any integer deleted if it is divisible by p_f, where $f=f(n)$ as just defined. Finally, define A as the set obtained by taking the union of the T^n, thus

$$A = T^1 \cup T^2 \cup T^3 \cup \cdots .$$

To evaluate $\delta_1(A)$, we note that S^n is a finite set of $(n+1)!-n!$ integers, that not more than half of these integers are deleted from S^n to obtain T^n, so that T^n contains at least $\{(n+1)!-n!\}/2$ integers. Also the S^n are disjoint and exhaust I, so that $\delta_1(A)\geqq 1/2$. In case n is of the form $C(m, 2)$ we have $f=f(n)=1$, and T^n has exactly $\{(n+1)!-n!\}/2$ integers. Thus $\delta_1(A)=1/2$.

On the other hand, for any prime p_f we consider the sets T^n for each of the infinite number of values of n for which $f=f(n)$ as defined above. Such sets T^n consist of those integers x in the range $n!\leqq x<(n+1)!$ for which $p_f\nmid x$. Hence we have

$$A_{p_f}((n+1)!) - A_{p_f}(n!) = 0,$$

$$\frac{A_{p_f}((n+1)!)}{(n+1)!} = \frac{A_{p_f}(n!)}{n!}\cdot\frac{1}{n+1},$$

from which it follows that $\delta_1(A_{p_f})=0$.

10. The hypothesis of Theorem 1. For any set of primes $\{p_i\}$ for which $\sum p_i^{-1}$ converges, we now exhibit a set A such that $\delta(A_{p_i})=0$ for all i, but $\delta_2(A)>0$, so that $\delta(A)>0$ if it exists. Thus the hypothesis $\sum p_i^{-1}=\infty$ is essential in Theorem 1 and its corollaries. Define A as the sequence of all integers except those divisible by at least one of the set of primes $\{p_i\}$, so that $\delta(A_{p_i})=0$ for all i. We may presume that the set $\{p_i\}$ is infinite, since otherwise it is clear that $\delta_2(A)>0$.

Since $\sum p_i^{-1}$converges, so the product $\prod(1-p_i^{-1})$ converges, say to a value $k>0$. Choose r sufficiently large so that $\sum_{i=r+1}^{\infty} p_i^{-1}<k/4$ and then choose m sufficiently large so that $2^r/m<k/4$. Then for any $n\geqq m$ we have

$$A(n) \geqq \sum (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_r}\left[\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}}\right] - \sum_{i=r+1}^{\infty}\left[\frac{n}{p_i}\right],$$

where the first sum ranges over the 2^r terms obtained by taking each α_i equal to 0 or 1. Thus we have, on removal of brackets,

$$A(n) > -2^r + \sum (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_r}\frac{n}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}} - \sum_{i=r+1}^{\infty}\frac{n}{p_i},$$

$$\frac{A(n)}{n} > -\frac{2^r}{n} + \prod_{i=1}^{r}(1 - p_i^{-1}) - \sum_{i=r+1}^{\infty} p_i^{-1}$$
$$> -\frac{k}{4} + k - \frac{k}{4} = \frac{k}{2},$$

for all $n \geqq m$. Thus $\delta_2(A) > k/2 > 0$.

11. Sets of density zero.

THEOREM 2. *The multinomial coefficients*

$$\frac{n!}{a_1!a_2! \cdots a_r!}, \tag{7}$$

where $n = \sum_{j=1}^{r} a_j$ *and each* $a_j < n-1$, *have density zero.*

PROOF. The restriction $a_j < n-1$ is needed, for otherwise any integer n can be put in form (7) by choosing $r=2$, $a_1=1$, $a_2=n-1$. There is no loss of generality in assuming that

$$a_1 \leqq a_2 \leqq \cdots \leqq a_r.$$

We give two lemmas, in which ρ_m denotes the mth prime and $C(n, r)$ the binomial coefficients.

LEMMA 1. *If* $2 \leqq r \leqq n-r$, *there exist positive constants* N *and* $c<1$ *such that*

$$C(n, r) > c\rho_r\rho_{n-r} \quad \textit{for all } n, \qquad C(n, r) > \rho_r\rho_{n-r} \quad \textit{for } n > N. \tag{8}$$

PROOF. We prove that $C(n, r) > \rho_r\rho_{n-r}$ for n sufficiently large. Let k be a constant such that $\rho_m < km \log m$ for all m. In case $3 \leqq r \leqq n-r$ we have for n sufficiently large,

$$C(n, r) \geqq n(n-1)(n-2)/6 > n(n-1)(k \log n)^2$$
$$> kr \log r \cdot k(n-r) \log (n-r) > \rho_r\rho_{n-r}.$$

In case $r=2$ we have for n sufficiently large,

$$C(n, r) = 3n(n-1)/6 > 3n(k \log n) > 3\rho_n > \rho_2\rho_{n-2},$$

which completes the proof of the lemma.

LEMMA 2. *If* $a_1 = a_2 = \cdots = a_{s-1} = 1$ *and* $a_s > 1$, *then* (7) *is greater than*

$$c^N \rho_1^{s-1} \prod_{j=s}^{r}{}' \rho_{a_j}^2, \tag{9}$$

where the dash on the product indicates that the first and last terms, ρ_{a_s}

and ρ_{a_r}, *are not squared.*

Proof. In case $a_1>1$ so that $s=1$, we write (7) in the form

$$C(n, a_r)C(n - a_r, a_{r-1})C(n - a_r - a_{r-1}, a_{r-2}) \cdots C\left(n - \sum_{j=3}^{r} a_j, a_2\right).$$

By Lemma 1 this exceeds

$$\rho_{a_r}\rho_{n-a_r}\rho_{a_{r-1}}\rho_{n-a_r-a_{r-1}} \cdots \rho_{a_2}\rho_{n-\Sigma_2^r a_j}\, c^N,$$

since the constant c enters in at most N cases. This can be seen to be larger than (9), by replacing ρ_{n-a_r} by $\rho_{a_{r-1}}$, $\rho_{n-a_r-a_{r-1}}$ by $\rho_{a_{r-2}}$, and so on.

In case $s>1$, we write (7) in the form

$$n(n-1)\cdots(n-s+2)\frac{(n-s+1)!}{\prod_{j=s}^{r} a_j!},$$

which by the previous case exceeds

$$n(n-1)\cdots(n-s+2)c^N\prod_{j=s}^{r}{}' \rho_{a_j}^2,$$

and this is larger than (9), and so the lemma is proved.

We now establish a one-to-one correspondence between the integers (7) and the integers

$$\rho_1^{s-1}\prod_{j=s}^{r}{}' \rho_{a_j}^2, \tag{10}$$

with the conventions of notation as in Lemma 2. Different sets of values $(a_1, a_2, \cdots, a_r)$ in (7) yield different integers (10), and the integers (10) have density zero by Corollary 2. Thus by Lemma 2 we complete the proof of Theorem 2, since the c^N factor in (9) is irrelevant, being independent of (7).

Theorem 3. *The set A of integers $\sigma_s(n)$ for all $s\geqq 1$ and all $n\geqq 1$ has density zero, $\sigma_s(n)$ being the sum of the sth powers of all divisors of n.*

Proof. We separate A into two possibly overlapping parts B and C, $\sigma_s(n)$ being put in B if n has more than k distinct prime factors, otherwise in C. We shall specify k in a moment. For any given $\epsilon>0$ we prove that $A(m)<\epsilon m$ for all sufficiently large m.

Any member of the set B satisfies the inequality

$$\sigma_s(n) \geqq n^s \prod_{j=1}^{k} (1 + \rho_j^{-s}) = n^s c_s,$$

ρ_j being the jth prime. The last equality defines c_s, a function of s and k, and we choose k so that $\epsilon c_1 > 4$. Thus for fixed s the number of $\sigma_s(m)$ counted by $B(m)$ is not more than $(m/c_s)^{1/s}$, and hence

$$B(m) \leqq m/c_1 + (m/c_2)^{1/2} + \cdots + (m/c_r)^{1/r},$$

where $r \leqq \log_2 m$ because for any larger value of r, $\sigma_r(2) > m$. Now $(m/c_s)^{1/s} < m^{1/2}$ for $s = 2, 3, \cdots, r$, and so we have

$$B(m) < m/c_1 + m^{1/2} \log_2 m. \tag{11}$$

Turning now to the set C, we observe that $\sigma_s(n)$ is counted by $C(m)$ provided that n has k or fewer prime factors, and that $\sigma_s(n) \leqq m$. Since $\sigma_s(n) \geqq n^s$, this implies that $n^s \leqq m$ or $n \leqq m^{1/s}$. Let F designate the sequence of integers having k or fewer prime factors. Thus we have

$$C(m) \leqq \sum_{s=1}^{r} F(m^{1/s}),$$

again with $r \leqq \log_2 m$. Now F is a set of density zero (by Corollary 2, for example) so that there exists an n_1 such that $F(m) < \epsilon m/2$ for $m > n_1$. Also $F(m^{1/s}) \leqq m^{1/2}$ for $s = 2, 3, \cdots, r$, and consequently

$$C(m) < \epsilon m/2 + m^{1/2} \log_2 m.$$

This with (11) gives

$$A(m) \leqq B(m) + C(m) < m/c_1 + \epsilon m/2 + 2m^{1/2} \log_2 m,$$

and $A(m)/m < c_1^{-1} + \epsilon/2 + 2m^{-1/2} \log_2 m$. The last factor is less than $\epsilon/4$ for $m > n_2$, say, so that for $m > \max(n_1, n_2)$ we have, recalling that $\epsilon c_1 > 4$,

$$\frac{A(m)}{m} < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon.$$

But ϵ is an arbitrary positive number, so A has density zero.

THEOREM 4. *For fixed k, the set A of integers n such that $(n, \phi(n)) \leqq k$ has density zero, where $\phi(n)$ is the Euler function.*

PROOF. It clearly suffices to prove that the set B of integers n such that $(n, \phi(n)) = k$ has density zero, since the union of a finite number of sets of density zero also has density zero. Let $\{p_i\}$ be the set of primes not dividing k, so that $\sum p_i^{-1}=\infty$. Now any integer in B_{p_i} has the form mp_i, with $(m, p_i)=1$, and furthermore $(\phi(m), p_i)=1$ since otherwise $p_i | k$. Hence m has no divisor of the form yp_i+1. But the primes $\{q_j\}$ of this form have the property $\sum q_j^{-1}=\infty$, so that by Corollary 1 the set of permissible values of m has density zero, so that $\delta(B_{p_i})=0$. Applying Corollary 1 again, we get $\delta(B)=0$.

For $k=1$, the set A of Theorem 4 consists of those integers n with $(n, \phi(n))=1$, and has been studied by Erdös [29], who proved that

$$A(n) = (1+o(1))ne^{-\gamma}(\log\log\log n)^{-1}.$$

THEOREM 5. *For fixed m, the set A of integers a for which $x^2\equiv m \pmod a$ is solvable has density zero if and only if m is not a perfect square.*

PROOF. If m is a perfect square, say $m=k^2$, then $x\equiv k \pmod a$ is a solution of the quadratic congruence. If m is not a perfect square, then m must be a quadratic residue of any prime which divides a. But the Gaussian reciprocity law and the Dirichlét theorem imply the existence of infinitely many primes $\{p_i\}$ in an arithmetic progression, each prime having m as a quadratic nonresidue. In fact $\sum p_i^{-1}$ diverges, so that $\delta(A)=0$ by Corollary 1.

12. **A binary operation on sequences.** If A and B denote sequences of integers $\{a_i\}$ and $\{b_i\}$ respectively, we define the "product" AB to be the set $\{b_{a_i}\}$. Under this operation, sets of integers form an associative, noncommutative groupoid.

THEOREM 6. $\delta_1(AB)\geqq\delta_1(A)\delta_1(B)$.

PROOF. We have

$$\delta_1(AB) = \liminf n/b_{a_n} \geqq \liminf n/a_n\cdot\liminf a_n/b_{a_n} \geqq \delta_1(A)\delta_1(B).$$

In case the densities of A and B exist, then the above proof holds with equality signs throughout, so that $\delta(AB)=\delta(A)\cdot\delta(B)$. If $\delta(A)=0$, then $\delta(AB)=0$ for any set B; likewise if $\delta(B)=0$, then $\delta(AB)=0$ for any set A.

THEOREM 7. *If $\delta(AB)$ exists, and if either $\delta(A)$ or $\delta(B)$ exists and is positive, then the other one exists.*

PROOF. If $\delta(B)$ exists and is positive, then $\lim b_{a_n}/a_n$ exists as $n\to\infty$, and by hypothesis $\lim n/b_{a_n}$ exists, and so $\lim n/a_n$ exists. On the other hand, if $\delta(A)$ exists and is positive, we denote AB by C and note that $C(b_n)=A(n)$. Thus $n/b_n=\{n/A(n)\}\cdot\{C(b_n)/b_n\}$, and so $\lim n/b_n$ exists.

THEOREM 8. *For Schnirelmann density,* $d(AB) \geqq d(A)d(B)$.

PROOF. Denote $d(A)$ by α, $d(B)$ by β, and AB by C. We shall establish that $C(x) \geqq \alpha\beta x$ for all integers x such that $x+1 \in C$, and this implies the theorem. Thus we write

$$\begin{aligned} \alpha\beta(b_{a_n} - 1) &\leqq \alpha B(b_{a_n} - 1) = \alpha(a_n - 1) \\ &\leqq A(a_n - 1) = n - 1 = C(b_{a_n} - 1), \end{aligned}$$

and the proof is complete.

Finally, we cite an example to show that inequalities cannot be removed in Theorems 6 and 8. Let A consist of 1 and all integers x satisfying $2^n \leqq x \leqq 2^n + 2^{n-1} - 1$ for $n \geqq 2$. Let B consist of 1 and all x satisfying $2^n + 2^{n-1} \leqq x \leqq 2^{n+1} - 1$ for $n \geqq 2$. Then AB consists of 1 and all x such that $3 \cdot 2^n \leqq x \leqq 3 \cdot 2^n + 2^{n-1} - 1$ for $n \geqq 2$, and BA consists of 1 and all x such that $5 \cdot 2^n \leqq x \leqq 5 \cdot 2^n + 2^n - 1$ for $n \geqq 1$. Also $\delta_1(A) = 1/2$, $\delta_1(B) = 1/3$. $\delta_1(AB) = 1/6$, $\delta_1(BA) = 1/5$, $d(A) = 1/3$, $d(B) = 1/5$, $d(AB) = 1/11$, and $d(BA) = 1/9$.

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ISOMORPHISMS OF INFINITE-DIMENSIONAL ANALOGUES OF THE CLASSICAL GROUPS

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1. Introduction. The classical groups referred to here are the *full linear*, *symplectic*, *orthogonal*, and *unitary* groups. These are groups of linear transformations operating on a finite-dimensional linear vector space over a (commutative) field. In order to save time, we shall not repeat the usual definitions of the groups but shall proceed directly to definitions[1] which are meaningful for the infinite- as well as the finite-dimensional case. At the same time, the condition that the coefficient domain be commutative will be relaxed and semi-linear as well as linear transformations will be admitted.

The main objectives in this address are to describe the structure of isomorphisms between two such generalized classical groups and to outline some of the methods used in studying these isomorphisms. It turns out that the isomorphisms are, roughly speaking, induced by isomorphisms between the underlying vector spaces on which the transformations act.

In the case of the full linear group in finite dimensions, this problem (for automorphisms) has been considered by Schreier and van der Waerden [12][2] when the coefficient domain is commutative, and by Dieudonné [2] when the coefficient domain is not commutative. Dieudonné has also considered the other classical groups in the finite-dimensional case. Mackey [9] has considered the problem for the multiplicative group of all bounded linear transformations with bounded inverses on an infinite-dimensional real normed linear space. The case studied by Mackey is included in our generalization of the full linear group.

The method of attack employed here is the standard one of investigating the way in which involutions are transformed by the isomorphisms plus an application of the fundamental theorem of projective geometry. However the special methods used in the investigation are refinements of methods introduced by Mackey in the paper mentioned above. The main results outlined are discussed in detail for linear transformations in [10] for the full linear case

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[1] These definitions are given in the next section.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

and in [11] for the other cases.[3] The extension to semi-linear transformations is sketched here for the first time.

2. Definitions. We consider first the notion of dual linear vector spaces as introduced by Dieudonné [4] and Jacobson [8]. Let $\mathfrak{X}$ and $\mathfrak{X}^*$ be left and right linear vector spaces respectively over a division ring (skew field or sfield) $\mathfrak{D}$. Assume given a bilinear functional (x, x^*) defined on $\mathfrak{X}\times\mathfrak{X}^*$ to $\mathfrak{D}$ which is nondegenerate in the sense that $(x, x^*)=0$ for all x (resp. all x^*) implies $x^*=0$ (resp. $x=0$); then $\mathfrak{X}$ and $\mathfrak{X}^*$ are said to be *dual* relative to (x, x^*). If $\bar{\mathfrak{X}}$ denotes the right vector space of all linear functionals on $\mathfrak{X}$ and if $(x, \bar{x})$ denotes the value of the functional $\bar{x}$ at x, then $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ are evidently dual relative to $(x, \bar{x})$. In general, if $\mathfrak{X}$ and $\mathfrak{X}^*$ are dual relative to (x, x^*), then (x, x^*), for fixed x^*, defines a linear functional on $\mathfrak{X}$. Hence $\mathfrak{X}^*$ can be regarded as a subspace of $\bar{\mathfrak{X}}$. Similarly, $\mathfrak{X}$ can be regarded as a subspace of $\bar{\mathfrak{X}}^*$. In case $\mathfrak{X}$ is finite-dimensional, we necessarily have $\mathfrak{X}^*=\bar{\mathfrak{X}}$.

Let $\mathfrak{X}$ be a left linear vector space over $\mathfrak{D}$ and let T be a transformation defined on $\mathfrak{X}$ to $\mathfrak{X}$. The transformation T is said to be *semi-linear* provided it is additive and there exists an automorphism $\alpha\to\alpha^\tau$ of $\mathfrak{D}$ such that $(\alpha x)T=\alpha^\tau(xT)$ for all $\alpha\in\mathfrak{D}$ and $x\in\mathfrak{X}$. Now, if $\mathfrak{X}^*$ is dual to $\mathfrak{X}$ relative to (x, x^*), then $(xT, x^*)^{\tau^{-1}}$ defines, for fixed x^*, a linear functional on $\mathfrak{X}$. If for every x^* there exists y^* in $\mathfrak{X}^*$ such that $(xT, x^*)^{\tau^{-1}}=(x, y^*)$, then $x^*T^*=y^*$ defines a semi-linear transformation T^* on $\mathfrak{X}^*$ with associated automorphism τ^{-1} of $\mathfrak{D}$. When T^* exists, it is uniquely determined and is called the *adjoint* of T. Similarly, T is called the adjoint of T^* and we write $T=(T^*)^*$. If T and S are semi-linear on $\mathfrak{X}$ with associated automorphisms τ and σ of $\mathfrak{D}$, then TS is a semi-linear transformation on $\mathfrak{X}$ with automorphism $\tau\sigma$. Also, if T^* and S^* exist, then $(TS)^*$ exists and is equal to S^*T^*. If T^* exists and T has an inverse T^{-1}, then $(T^{-1})^*$ exists and is equal to $(T^*)^{-1}$. We can therefore consider the multiplicative group $\mathcal{L}(\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^*)$ of all semi-linear transformations on $\mathfrak{X}$ which have adjoints as well as multiplicative inverses.[4] These groups and certain subgroups of them provide us with our desired generalizations of the full linear group. Included here will be the group of all linear transformations in $\mathcal{L}(\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^*)$. The latter group obviously reduces to the classical full linear group if $\mathfrak{X}$ is finite-dimensional and $\mathcal{D}$ is a field. For want of a better terminology we shall refer to the groups considered here as "full linear groups," although they may contain semi-linear transformations and need not exhaust $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$.

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[3] In the papers [10, 11], the group operation is taken as the circle operation, $A \circ B=A+B-AB$, instead of multiplication as is done here. However, the two groups obtained are isomorphic under the mapping $A\to I-A$, where I is the identity transformation.

[4] Strictly speaking, the notation $\mathcal{L}(\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^*)$ should exhibit the bilinear functional (x, x^*).

Next we consider the notion of a *self-dual* linear vector space. Again let $\mathfrak{X}$ be a left linear vector space over the division ring $\mathcal{D}$. Furthermore, assume given an involution $\alpha \rightarrow \alpha^*$ (that is, an anti-automorphism of period two) in $\mathcal{D}$ and a functional (x, y) defined on $\mathfrak{X} \times \mathfrak{X}$ to $\mathcal{D}$ and called a *scalar product* with the following properties: (1) (x, y) is linear in x for each y. (2) $(x, y)=0$ for all y implies $x=0$. (3) $(x, y)=\epsilon(y, x)^*$, where $\epsilon=\pm 1$ is a constant independent of x and y. Under these conditions $\mathfrak{X}$ is said to be *self-dual*. In fact, if $\mathfrak{X}^*$ denotes a right linear vector space over $\mathcal{D}$ which is identical with $\mathfrak{X}$ except with right multiplication by scalars defined by $x\alpha=\alpha^* x$, then $\mathfrak{X}$ and $\mathfrak{X}^*$ are dual relative to (x, y). Two vectors x, y in $\mathfrak{X}$ are said to be *orthogonal*, written $x \perp y$, provided $(x, y)=0$. If $\mathfrak{M}$ is a subspace of $\mathfrak{X}$, then $\mathfrak{M}^{\perp}$ will denote the set of all vectors in $\mathfrak{X}$ each of which is orthogonal to every vector in $\mathfrak{M}$. Evidently $\mathfrak{M}^{\perp}$ is a linear subspace of $\mathfrak{X}$ and is called the *orthogonal complement* of $\mathfrak{M}$. A subspace $\mathfrak{M}$ is said to be *isotropic* provided $\mathfrak{M} \cap \mathfrak{M}^{\perp} \neq (0)$. If $\mathfrak{M} \subseteq \mathfrak{M}^{\perp}$, then $\mathfrak{M}$ is said to be *totally isotropic*. The maximum possible dimension for a totally isotropic subspace of $\mathfrak{X}$ is called the *index* of $\mathfrak{X}$. A vector x such that $(x, x)=0$ is called an *isotropic vector*. In general, even when a subspace $\mathfrak{M}$ is nonisotropic (that is, $\mathfrak{M} \cap \mathfrak{M}^{\perp}=(0)$), we do not have $\mathfrak{X}=\mathfrak{M} \oplus \mathfrak{M}^{\perp}$. On the other hand, if $\mathfrak{M}$ is finite-dimensional and nonisotropic, it is true that $\mathfrak{X}=\mathfrak{M} \oplus \mathfrak{M}^{\perp}$ [11, Lemma 1.1].

Self-dual spaces fall naturally into two classes. In the first, every vector is isotropic, which implies that $\mathcal{D}$ must be a field with identity mapping as involution and $\epsilon=-1$. These spaces are called *symplectic*. On the other hand, if there exist nonisotropic vectors in the space, a trivial modification of the involution and scalar product yields $\epsilon=1$. In this case the space is called *unitary*. It will be assumed hereafter that the self-dual spaces considered are either symplectic or unitary (that is, if $\mathfrak{X}$ is not symplectic, then $\epsilon=1$).

Now let T be a semi-linear transformation on the self-dual space $\mathfrak{X}$ with τ as the associated automorphism of $\mathcal{D}$. It is natural to call a second transformation T^* on $\mathfrak{X}$ the *adjoint* of T provided $(xT, y) = (x, yT^*)^{\tau}$ for all x and y [7]. If T^* exists, then it is unique and is a semi-linear transformation with $\alpha \rightarrow \alpha^{*\tau^{-1}*}$ as the associated automorphism of $\mathcal{D}$. Suppose in addition that T has the property $T^*=T^{-1}$. Since T^{-1} is a semi-linear transformation with automorphism τ^{-1}, it follows that $\tau^{-1}={}^*\tau^{-1*}$, or what amounts to the same thing, τ commutes with the involution ($\alpha^{\tau *} = \alpha^{*\tau}$, for all α). Furthermore, $(xT, yT) = (x, y)^{\tau}$ for all x and y. The set of all semi-linear transformations on $\mathfrak{X}$ such that $T^* = T^{-1}$ constitutes a group under multiplication which will be denoted by $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$, in the general self-dual case, and by $\mathcal{L}_S^*(\mathfrak{X}, \mathcal{D})$ or $\mathcal{L}_U^*(\mathfrak{X}, \mathcal{D})$ according as $\mathfrak{X}$ is symplectic or unitary. These groups and certain subgroups of them constitute the desired generalizations of the classical symplectic and unitary groups. In the unitary case when $\mathcal{D}$ is a field with involution $\alpha^* \equiv \alpha$, we obtain generalizations of the classical orthogonal groups which may be denoted by $\mathcal{L}_O^*(\mathfrak{X}, \mathcal{D})$. Thus, for our purposes, the orthogonal groups are included among the unitary groups so do not require separate treatment.

In order to simplify our discussion, we shall assume throughout that the vector spaces involved are infinite-dimensional, although everything goes through without change for sufficiently large finite dimensions. It will also be necessary for us to restrict attention to characteristic different from two and, in the unitary case when the index is not zero, to avoid the finite fields $GF(3)$ and $GF(9)$ as coefficients. If the groups considered contain only linear transformations, then the exclusion of the field $GF(9)$ in the unitary case can be dropped. These are the only restrictions imposed in the discussion which follows.

3. Involutions. Let $\mathfrak{X}$ be a left linear vector space over the division ring $\mathcal{D}$ and let T be a semi-linear transformation on $\mathfrak{X}$ for which $T^2 = I$. Such transformations are called *involutions* and play an important rôle in the proofs of the isomorphism theorems to be stated below. Observe that the automorphism τ of $\mathcal{D}$ associated with an involution T is of period two ($\tau^2 = 1$). Since $\mathfrak{X}$ is not of characteristic two, every involution T determines a unique direct sum decomposition $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}$ of the additive group of $\mathfrak{X}$ such that $xT = x$ for x in $\mathfrak{M}$ and $xT = -x$ for x in $\mathfrak{N}$; that is, T coincides with the identity transformation I on $\mathfrak{M}$ and with $-I$ on $\mathfrak{N}$. The components $\mathfrak{M}$ and $\mathfrak{N}$ are called the *subspaces* of T although they are linear subspaces in $\mathfrak{X}$ if, and only if, T is a linear transformation. Observe that $\mathfrak{M} = \mathfrak{X}(I+T)$ and $\mathfrak{N} = \mathfrak{X}(I-T)$ and, for x in $\mathfrak{X}$, the decomposition $x = m+n$ with m in $\mathfrak{M}$ and n in $\mathfrak{N}$ is given by $m = (x+xT)/2$ and $n = (x-xT)/2$. It is easy to verify that an arbitrary additive transformation Z commutes with the involution T if, and only if, $\mathfrak{M}Z \subseteq \mathfrak{M}$ and $\mathfrak{N}Z \subseteq \mathfrak{N}$.

Let T be a linear involution with subspaces $\mathfrak{M}$ and $\mathfrak{N}$. Then T is called *minimal* if $\mathfrak{N}$ is one-dimensional, *maximal* if $\mathfrak{M}$ is one-dimensional, and *extremal* if either $\mathfrak{M}$ or $\mathfrak{N}$ is one-dimensional. Note that T is maximal if, and only if, $-T$ is minimal. A linear involution belongs to $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$ and is minimal if, and only if, it has the form $xT = x - 2(x, v^*)u$, where $u \in \mathfrak{X}$, $v^* \in \mathfrak{X}^*$, and $(u, v^*) = 1$. The one-dimensional subspace[6] of T is $[u]$ and the other subspace of T is a maximal linear subspace of $\mathfrak{X}$ equal to the zero manifold of the functional v^* and is thus determined uniquely by the one-dimensional subspace $[v^*]$ of $\mathfrak{X}^*$. In this way there is associated with each extremal involution in $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$ a pair of one-dimensional subspaces $[u]$ in $\mathfrak{X}$ and $[v^*]$ in $\mathfrak{X}^*$ such that $(u, v^*) = 1$. Conversely, to each such pair $[u]$, $[v^*]$ there corresponds two extremal involutions (a minimal involution and its negative) in $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$ having $[u]$ and the zero manifold of v^* as subspaces. Evidently there are many extremal involutions having a common subspace; however two such involutions commute if, and only if, they have the same subspaces and are thus equal except for sign.

If $\mathfrak{X}$ is a self-dual space and T is a linear involution such that $T^* = T$, then T is in $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$ and the corresponding decomposition of $\mathfrak{X}$ has the form $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{M}^\perp$. Conversely, if $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{M}^\perp$ and T is an involution with subspaces $\mathfrak{M}$ and $\mathfrak{M}^\perp$, then T is in $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$. It is obvious that the subspaces associated with a linear involution in $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$ must be nonisotropic. An involution in $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$ is minimal if, and only if, it has the form $xT = x - 2(x, u)(u, u)^{-1}u$, where u is a fixed element of $\mathfrak{X}$. Hence $\mathcal{L}^*_S(\mathfrak{X}, \mathcal{D})$ contains no extremal involutions according to our definition. On the other hand, although a symplectic space contains no one-dimensional nonisotropic subspaces, it is true that there exist many two-dimensional subspaces which are nonisotropic. In fact, every isotropic vector in an arbitrary self-dual space can be embedded in a nonisotropic two-dimensional subspace. Since $\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, for $\mathfrak{M}$ a finite-dimensional nonisotropic subspace of $\mathfrak{X}$, it follows that $\mathcal{L}^*_S(\mathfrak{X}, \mathcal{D})$ contains linear involutions with two-dimensional subspaces. Therefore, if T is a linear involution in $\mathcal{L}^*_S(\mathfrak{X}, \mathcal{D})$ with subspaces $\mathfrak{M}$ and $\mathfrak{M}^\perp$ ($xT = x$ for $x \in \mathfrak{M}$ and $xT = -x$ for $x \in \mathfrak{M}^\perp$), then it is appropriate to define T to be *maximal* or *minimal* according as $\mathfrak{M}$ or $\mathfrak{M}^\perp$ is two-dimensional and *extremal* in either case.

In order to handle nonlinear involutions a reduction, which is suggested by an argument used by Dieudonné [2, p. 9] in a slightly different situation, is required. Let $\mathfrak{X}$ be a left linear vector space over $\mathcal{D}$ and let T be a nonlinear involution in $\mathfrak{X}$ with associated automorphism τ of $\mathcal{D}$; then $\tau \neq 1$ and $\tau^2 = 1$. Define $\mathcal{D}_\tau$ as the set of all ξ in $\mathcal{D}$ such that $\xi^\tau = \xi$ and $\mathcal{D}_\tau'$ as the set of all η in $\mathcal{D}$ such that $\eta^\tau = -\eta$. It is obvious that $\mathfrak{D}_\tau$ is a division subring of $\mathfrak{D}$ and that $\mathfrak{D}_\tau \cap \mathfrak{D}_\tau' = (0)$. Also $\mathfrak{D} = \mathfrak{D}_\tau \oplus \mathfrak{D}_\tau'$; in fact, for every α in $\mathfrak{D}$, $\alpha = \xi + \eta$, where $\xi = (\alpha + \alpha^\tau)/2$, $\eta = (\alpha - \alpha^\tau)/2$, ξ is in $\mathfrak{D}_\tau$, and η is in $\mathfrak{D}_\tau'$. If η is an arbitrary nonzero element of $\mathfrak{D}_\tau'$, then it is easy to see that $\mathfrak{D}_\tau' = \eta\mathfrak{D}_\tau = \mathfrak{D}_\tau\eta$ which shows that $\mathfrak{D}$ has (right and left) dimension two over $\mathfrak{D}_\tau$. Now $\mathfrak{X}$ can be regarded as a linear vector space over $\mathfrak{D}_\tau$ in which T becomes a linear involution. Hence the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of T are linear subspaces of $\mathfrak{X}$ over $\mathfrak{D}_\tau$. Let η be a fixed nonzero element of $\mathfrak{D}_\tau'$. Then the mapping $x \rightarrow \eta x$ is a one-to-one mapping of $\mathfrak{M}$ onto $\mathfrak{N}$ which is semi-linear with respect to $\mathfrak{D}_\tau$. Thus $\mathfrak{M}$ and $\mathfrak{N}$ have the same dimension over $\mathfrak{D}_\tau$, which is equal to the dimension of $\mathfrak{X}$ over $\mathfrak{D}$. If $\mathfrak{X}$ is self-dual and T is in $\mathcal{L}^*(\mathfrak{X}, \mathfrak{D})$, then $\alpha^{*\tau} = \alpha^{\tau *}$, for all α in $\mathfrak{D}$, so that $\mathfrak{D}_\tau^* = \mathfrak{D}_\tau$. In other words, $\xi \rightarrow \xi^*$ is an involution in $\mathfrak{D}_\tau$. Moreover $\mathfrak{X}$, as a linear vector space over $\mathfrak{D}_\tau$, is self-dual relative to the new scalar product $[x, y] = [(x, y) + (x, y)^\tau]/2$. Clearly $T^*(=T)$ is the adjoint of T with respect to the scalar product $[x, y]$, which implies that $\mathfrak{N}$ is the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{X}$ with respect to $[x, y]$. Since $(xT, yT) = (x, y)^\tau$ for all x, y in $\mathfrak{X}$, it follows that $(x, y)^\tau = (x, y)$ for x, y both in either $\mathfrak{M}$ or $\mathfrak{N}$. Hence the two scalar products (x, y) and $[x, y]$ coincide in $\mathfrak{M}$ and $\mathfrak{N}$. The above discussion, in a sense, reduces the study of nonlinear involutions to the study of linear involutions and makes available methods based on linearity of the subspaces of the involution.

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[6] The one-dimensional subspace of $\mathfrak{X}$ which contains a nonzero vector u will be denoted by $[u]$.

The next problem here is to give a group theoretical characterization of extremal involutions. First let $\mathfrak{G}$ denote an arbitrary set of involutions in the group under consideration and denote by $c(\mathfrak{G})$ the set of all those involutions in the group which commute with each element of $\mathfrak{G}$. Now let T be an arbitrary involution in the group and denote by ν_T the maximum number of elements which can occur in $c(c(T, T'))$ where T' is any involution which commutes with T. Also denote by ν the maximum value which ν_T can have. The following lemma, whose proof will be omitted, enables one to evaluate the quantities ν and ν_T.

LEMMA 1. *Let T_i be an involution with subspaces $\mathfrak{M}_i$, $\mathfrak{N}_i$ $(i=1, 2)$ and assume that T_1 and T_2 commute. Set $\mathfrak{P}_1 = \mathfrak{M}_1 \cap \mathfrak{M}_2$, $\mathfrak{P}_2 = \mathfrak{M}_1 \cap \mathfrak{N}_2$, $\mathfrak{P}_3 = \mathfrak{N}_1 \cap \mathfrak{M}_2$, $\mathfrak{P}_4 = \mathfrak{N}_1 \cap \mathfrak{N}_2$. Observe that $\mathfrak{X} = \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{P}_3 \oplus \mathfrak{P}_4$ and denote by P_i the projection of $\mathfrak{X}$ onto $\mathfrak{P}_i$ determined by this decomposition of $\mathfrak{X}$. Then every T in $c(c(T_1, T_2))$ is of the form $T = \sum \epsilon_i P_i$ where each $\epsilon_i = \pm 1$.*

If the T_1 and T_2 in Lemma 1 are linear involutions, then each $\mathfrak{P}_i$ is a linear subspace of $\mathfrak{X}$ and $\sum \epsilon_i P_i$ is a linear involution which belongs to the group under consideration for every choice of $\epsilon_i = \pm 1$. If none of the P_i is zero, it follows in this case that the number of distinct elements in $c(c(T_1, T_2))$ is equal to 16. It is not difficult to construct linear involutions T_1 and T_2 in the group such that no P_i is zero; hence we obtain $\nu = 16$. If either T_1 or T_2 is not linear, then in order for $\sum \epsilon_i P_i$ to be a semi-linear transformation the coefficients ϵ_i can no longer be independently chosen; in fact, it turns out that $c(c(T_1, T_2))$ contains only the elements $\pm I$, $\pm T_1$, $\pm T_2$, $\pm T_1 T_2$. This same result holds if either T_1 or T_2 is extremal or if either T_1 or T_2 is equal to $\pm I$. These remarks lead to the following lemma.

LEMMA 2. (i) *T linear but not extremal is equivalent to* $\nu_T = \nu$. (ii) *T either extremal or nonlinear is equivalent to* $\nu_T = \nu/2$. (iii) *T equal to $\pm I$ is equivalent to* $\nu_T = \nu/4$.

The above lemma gives a group-theoretic characterization of nontrivial linear involutions which are not extremal but does not distinguish between extremal involutions and nonlinear involutions. On the other hand, it is not difficult to see that any extremal involution can be written as a product of two commuting linear non-extremal involutions. This remark plus Lemma 2 gives the desired characterization of extremal involutions.

LEMMA 3. *An involution T is extremal if, and only if,* $\nu_T = \nu/2$ *and there exist involutions T_1 and T_2, with* $\nu_{T_1} = \nu_{T_2} = \nu$, *such that* $T = T_1 T_2 = T_2 T_1$.

This criterion for extremal involutions is valid for any subgroups of $\mathcal{L}(\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^*)$ or $\mathcal{L}^*(\mathfrak{X}, \mathfrak{D})$ provided only that the subgroups contain all minimal involutions. Such subgroups need not contain $-I$ in which case it turns out that $\nu = 8$ instead of 16. Included here are the groups of linear transformations for which the above discussion becomes considerably simpler. The criterion also applies for finite dimensions at least equal to ten in the symplectic case and to five in the other cases. However, it breaks down for lower dimensions since an extremal involution can no longer be written as a product of non-extremal linear involutions. On the other hand, if the groups contain only linear transformations, then Lemma 2 gives a criterion for extremal involutions which applies for dimensions greater than six in the symplectic case and greater than three in the other cases. For lower dimensions, the linear involutions are all extremal.

We pass now to the isomorphism theorems and consider the full linear case first.

4. The full linear case. Consider two groups of type $\mathcal{L}(\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^*)$

and $\mathcal{L}(\mathfrak{Y}, \mathcal{E}, \mathfrak{Y}^*)$ and let $\mathcal{G}$, $\mathcal{H}$ be subgroups of $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$, $\mathcal{L}(\mathfrak{Y}, \mathcal{E}, \mathfrak{Y}^*)$ respectively which contain all minimal involutions in each case. Also assume given an isomorphic mapping $G \rightarrow \mathfrak{g}(G)$ of $\mathcal{G}$ onto $\mathcal{H}$ and denote by $\mathcal{E}_0$ the multiplicative group consisting of the nonzero elements of $\mathcal{E}$. The following theorem gives the form of $\mathfrak{g}$ in this situation. In order to state the theorem, some definitions are required. First, a one-to-one mapping Φ of $\mathfrak{X}$ onto $\mathfrak{Y}$ is called an *isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{Y}$ if it is additive and there exists an isomorphism $\alpha \rightarrow \alpha^\phi$ of $\mathcal{D}$ onto $\mathcal{E}$ such that $(\alpha x)\Phi = \alpha^\phi(x\Phi)$ for all α in $\mathcal{D}$ and x in $\mathfrak{X}$. Similarly, a one-to-one mapping Φ of $\mathfrak{X}$ onto $\mathfrak{Y}^*$ is called an *isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{Y}^*$ if it is additive and there exists an anti-isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{E}$ such that $(\alpha x)\Phi = (x\Phi)\alpha^\phi$. Next, a mapping $G \rightarrow \chi(G)$ of $\mathcal{G}$ into $\mathcal{E}_0$ is called a *crossed character of* $\mathcal{G}$ *in* $\mathcal{E}_0$ [1] if there exists a homomorphism $G \rightarrow \sigma(G)$ of $\mathcal{G}$ into the group of automorphisms of $\mathcal{E}_0$ such that $\chi(G_1G_2) = \chi(G_1)^{\sigma(G_2)}\chi(G_2)$ for all G_1, G_2 in $\mathcal{G}$. If $\sigma(G) = 1$ for all G, then $\chi(G)$ is called a *character of* $\mathcal{G}$ *in* $\mathcal{E}_0$.

THEOREM I. *The isomorphism* $\mathfrak{g}$ *has, for all* G *in* $\mathcal{G}$, *one of the two forms*: (i) $\mathfrak{g}(G) = \chi(G)\Phi^{-1}G\Phi$, (ii) $\mathfrak{g}(G) = (\Phi^{-1}G\Phi\chi(G))^*$, *where* Φ *in case* (i) *is an isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{Y}$ *and in case* (ii) *is an isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{Y}^*$ *and* $\chi(G)$ *is a crossed character*[6] *of* $\mathcal{G}$ *in* $\mathcal{E}_0$.

COROLLARY. *If the groups consist of linear transformations, then* $\chi(G)$ *is a character of* $\mathcal{G}$ *in the center of* $\mathcal{E}_0$.

The proof of Theorem I is too long to be given in detail here;[7] however it is possible to outline the main ideas which rely heavily on methods introduced by Mackey [9].

By the group-theoretic characterization of extremal involutions given in Lemma 3 of the preceding section, it follows that $\mathfrak{g}$ establishes a one-to-one correspondence between extremal involutions in $\mathcal{G}$ and in $\mathcal{H}$. In order to use this fact it is necessary to formulate a group-theoretic criterion for two (noncommutative) extremal involutions to have a common subspace. This is given in the next lemma, whose proof will be omitted.

LEMMA 4. *Two noncommutative extremal involutions* T_1 *and* T_2 *have a common subspace if and only if,* $c(c(T_1, T_2)) = c(c(U_1, U_2))$ *for every pair* U_1, U_2 *of noncommutative extremal involutions in* $c(c(T_1, T_2))$.

Now let $[x]$ be an arbitrary one-dimensional subspace of $\mathfrak{X}$ and consider two noncommutative extremal involutions T_1, T_2 which have $[x]$ as a common subspace. Then, by Lemma 4, the extremal involutions $\mathfrak{g}(T_1)$, $\mathfrak{g}(T_2)$ also have a common subspace which, however, may or may not be one-dimensional. On the other hand, it is possible to show that the subspace common to $\mathfrak{g}(T_1)$ and $\mathfrak{g}(T_2)$ is independent of the choice of T_1, T_2 and is, or is not, one-dimensional independently of $[x]$. In case the subspace common to $\mathfrak{g}(T_1)$ and $\mathfrak{g}(T_2)$ is one-dimensional, we denote it by $[x]'$ and so obtain a one-to-one correspondence $[x]\leftrightarrow[x]'$ between the one-dimensional subspaces of $\mathfrak{X}$ and those of $\mathfrak{Y}$. In the other case, the subspace common to $\mathfrak{g}(T_1)$ and $\mathfrak{g}(T_2)$ is a maximal subspace of $\mathfrak{Y}$ which is the zero manifold of a linear functional y^* in $\mathfrak{Y}^*$. In this case we define $[x]'=[y^*]$ and obtain a one-to-one correspondence between the one-dimensional subspaces of $\mathfrak{X}$ and those of $\mathfrak{Y}^*$. Now an application of the fundamental theorem of projective geometry gives an isomorphism Φ, of $\mathfrak{X}$ onto $\mathfrak{Y}$ in the first case and of $\mathfrak{X}$ onto $\mathfrak{Y}^*$ in the second case, such that $[x\Phi]=[x]'$ for all x in $\mathfrak{X}$. The isomorphism Φ has the additional property that it maps the subspaces of an extremal involution T in $\mathcal{G}$ onto the subspaces of $\mathfrak{g}(T)$ or $\mathfrak{g}(T)^*$ according as Φ maps $\mathfrak{X}$ onto $\mathfrak{Y}$ or $\mathfrak{Y}^*$. It follows that if T is an extremal involution, then $\mathfrak{g}(T) = \pm\Phi^{-1}T\Phi$ in the first case and $\mathfrak{g}(T)=\pm(\Phi^{-1}T\Phi)^*$ in the second. Thus $\mathfrak{g}(T)$ has the indicated form for extremal involutions. That $\mathfrak{g}(G)$ has the desired form for all G is a consequence of the obvious fact that the minimal involutions constitute an invariant subset of $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$. This can be seen in the following way for the case in which Φ maps $\mathfrak{X}$ onto $\mathfrak{Y}$. Define $G^\sigma=\Phi\mathfrak{g}(G)\Phi^{-1}$. Then $G\rightarrow G^\sigma$ is an isomorphism of $\mathcal{G}$ onto another subgroup of $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$ such that $T^\sigma=\pm T$ when T is a minimal involution. Furthermore, if T is minimal, then so also is GTG^{-1} for arbitrary G in $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$. Hence $G^\sigma T(G^\sigma)^{-1}=GTG^{-1}$ for all G in $\mathcal{G}$. It follows that $xGT=xG$ if, and only if, $xG^\sigma T=xG^\sigma$. On the other hand, if x_1 and x_2 are linearly independent elements of $\mathfrak{X}$, then there exists a minimal involution T such that $x_1T=x_1$ and $x_2T\neq x_2$. Therefore xG^σ and xG are linearly dependent for all x. This implies that G^σ is equal to a scalar multiple of G. Returning to $\mathfrak{g}(G)$, we obtain case (i) of the theorem. That $\chi(G)$ is a crossed character in $\mathcal{E}_0$ is immediate from the fact that $G\rightarrow\mathfrak{g}(G)$ is a group isomorphism. The case in which Φ maps $\mathfrak{X}$ onto $\mathfrak{Y}^*$ is treated similarly and leads to case (ii) of the theorem.

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[6] The homomorphism $G \rightarrow \sigma(G)$ of $\mathcal{G}$ into the group of automorphisms of $\mathcal{E}_0$ is given by $\sigma(G) = \gamma'$ where γ' is the automorphism of $\mathcal{E}$ associated with the semi-linear transformation $\mathfrak{g}(G)$.

[7] See [10] for a detailed proof in the case of groups of linear transformations.

The above theorem is true and the proof outlined here applies also in the finite-dimensional case provided only that the vector spaces have dimension at least equal to five. If the groups contain only linear transformations, then the minimum dimension five can be replaced by three. The finite-dimensional case for groups of linear transformations is due essentially to Dieudonné [2, pp. 7, 15] who considered automorphisms rather than isomorphisms. Dieudonné also disposes of the case when the characteristic is equal to two. Hua [6, p. 101] extended the Dieudonné results to dimension two. When the coefficient domain is a (commutative) field, these cases follow from results of Schreier and van der Waerden [12] whose methods are quite different from those used here. The following infinite-dimensional case has been considered by Mackey [9]. Let $\mathfrak{X}$ be an infinite-dimensional real normed linear space and take the dual space $\mathfrak{X}^*$ as the space of all bounded linear functionals on $\mathfrak{X}$. Then $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$ is the group of all bounded linear transformations on $\mathfrak{X}$ which have inverses. Mackey [9] proved that isomorphism of two groups of this type implies isomorphism of the vector spaces but did not obtain the form of the group isomorphism in terms of the vector space isomorphism. Construction of the isomorphism of the vector space is, of course, the main part of the proof of Theorem I and the methods introduced by Mackey carry over without essential modification to the general case. As a matter of fact, Mackey had already observed [9] that his discussion could be applied to the group of all linear transformations in $\mathcal{L}(\mathfrak{X}, \mathcal{D}, \mathfrak{X}^*)$ when $\mathcal{D}$ is the real numbers. He calls the dual pair $\mathfrak{X}$, $\mathfrak{X}^*$ a linear system.

5. **The symplectic and unitary cases.** Consider two groups of type $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$ and $\mathcal{L}^*(\mathfrak{Y}, \mathcal{E})$ both of which are either symplectic or unitary and let $\mathcal{G}$, $\mathcal{H}$ be subgroups of $\mathcal{L}^*(\mathfrak{X}, \mathcal{D})$, $\mathcal{L}^*(\mathfrak{Y}, \mathcal{E})$ respectively which contain all of the minimal involutions in each case. Also assume given an isomorphic mapping $G \rightarrow \mathfrak{g}(G)$ of $\mathcal{G}$ onto $\mathcal{H}$ and denote by $\mathcal{E}_0^*$ the multiplicative group of elements μ in $\mathcal{E}$ such that $\mu\mu^* = 1$. The following theorem gives the form of $\mathfrak{g}$ in this situation.

THEOREM II. *The isomorphism* $\mathfrak{g}$ *has the form* $\mathfrak{g}(G) = \chi(G)\Phi^{-1}G\Phi$, *where* Φ *is an isomorphism of* $\mathfrak{X}$ *onto* $\mathfrak{Y}$ *and* $\chi(G)$ *is a crossed character*[6] *of* $\mathcal{G}$ *in* $\mathcal{E}_0^*$. *If* ϕ *denotes the isomorphism of* $\mathcal{D}$ *onto* $\mathcal{E}$ *associated with* Φ, *then there exists a constant* ρ *in* $\mathcal{E}$ *such that* $\rho^* = \rho$, $\alpha^{\phi *} = \rho^{-1}\alpha^{*\phi}\rho$ *for all* α *in* $\mathcal{D}$, *and* $(x_1\Phi, x_2\Phi) = (x_1, x_2)^{\phi}\rho$ *for all* x_1, x_2 *in* $\mathfrak{X}$.

COROLLARY. *In the symplectic and orthogonal cases* $\chi(G) = \pm 1$ *for each* G. *If* $\mathcal{G}$ *and* $\mathcal{H}$ *consist only of linear transformations, then* $\chi(G)$ *is in the center of* $\mathcal{E}$ *so that in case* $\mathcal{G}$ *also coincides with its commutator subgroup*,[8] *we have* $\chi(G) \equiv 1$.

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[8] This is so, for example, for the finite-dimensional symplectic group of linear transformations [3, p. 12].

We outline[9] the proof of Theorem II which parallels that of Theorem I. The first step is to construct a one-to-one corrrespondence between the one-dimensional subspaces of $\mathfrak{X}$ and of $\mathfrak{Y}$. Since $\mathfrak{g}$ establishes a one-to-one correspondence between extremal involutions in $\mathcal{G}$ and $\mathcal{K}$, one obtains immediately (in the unitary case) a one-to-one correspondence $[x]\leftrightarrow[x]'$ between the nonisotropic one-dimensional subspaces of $\mathfrak{X}$ and of $\mathfrak{Y}$. When the index is zero, the first step is thus completed. On the other hand, if $[x]$ is isotropic, choose nonisotropic two-dimensional subspaces $\mathfrak{M}_1$, $\mathfrak{M}_2$ such that $\mathfrak{M}_1\cap\mathfrak{M}_2=[x]$ and let T_1, T_2 be involutions in $\mathcal{G}$ with $\mathfrak{M}_1$, $\mathfrak{M}_2$ as subspaces. In the symplectic case, T_1 and T_2 are extremal so that $\mathfrak{g}(T_1)$ and $\mathfrak{g}(T_2)$ are also extremal. Moreover the two-dimensional subspaces of $\mathfrak{g}(T_1)$ and $\mathfrak{g}(T_2)$ intersect in a one-dimensional subspace $[x]'$. This fact is ensured by the following lemma.

LEMMA 5. *Let T_1 and T_2 be extremal symplectic involutions. Then a necessary and sufficient condition for the two-dimensional subspaces of T_1 and T_2 to intersect in a one-dimensional subspace is that $c(c(T_1, T_2))=c(c(U_1, U_2))$ for every pair U_1 and U_2 of noncommutative extremal involutions in $c(c(T_1, T_2))$.*

It turns out that $[x]'$ is independent of the choice of T_1, T_2 so that $[x]\leftrightarrow[x]'$ gives the desired one-to-one correspondence in this case. In the unitary case, T_1 and T_2 are not extremal but each can be written as a product of two extremal involutions. This implies that $\mathfrak{g}(T_1)$ and $\mathfrak{g}(T_2)$ each has a two-dimensional subspace and an additional argument shows that these two-dimensional subspaces intersect in an isotropic one-dimensional subspace $[x]'$ which is independent of the choice of $\mathfrak{M}_1$ and $\mathfrak{M}_2$. The resulting correspondence between isotropic one-dimensional subspaces plus that already obtained for the nonisotropic one-dimensional subspaces provides us with the desired one-to-one correspondence $[x]\leftrightarrow[x]'$ between one-dimensional subspaces of $\mathfrak{X}$ and of $\mathfrak{Y}$. In both the symplectic and unitary cases it turns out that the correspondence $[x]\leftrightarrow[x]'$ preserves orthogonality. The fundamental theorem of projective geometry thus gives an orthogonality preserving isomorphism Φ of $\mathfrak{X}$ onto $\mathfrak{Y}$ such that $[x\Phi]=[x]'$. The fact that Φ preserves orthogonality implies the properties mentioned in the last statement of the theorem. The representation of $\mathfrak{g}(G)$ is obtained by an argument similar to that given in the full linear case.

The above theorem is true and the proof outlined applies to the finite-dimensional case provided that the vector spaces have dimensions at least equal to ten. In the unitary case with zero index, it is enough to have the dimensions at least equal to five. If the groups contain only linear transformations, then these minimum dimensions can be replaced by six and three respectively. For the finite-dimensional symplectic case and automorphisms of groups of linear transformations, Hua [5] has proved the above theorem for dimensions at least equal to two. Hua's result was extended by Dieudonné [2, p. 39] to include the case of characteristic two. Dieudonné [2, pp. 51, 79, 82] has also considered the finite-dimensional orthogonal and unitary cases (characteristic not equal to two) for automorphisms of groups of linear transformations when the dimension is at least equal to three and the index is different from zero. However, in the unitary case, he restricts the coefficient domain to the following two special instances: (1) the generalized quaternions (reflexive sfield [2, p. 81]) with involution equal to the conjugate and (2) a field $\mathfrak{K}$ which is a separable extension of degree two over a field $\mathfrak{K}_0$ with involution in $\mathfrak{K}$ equal to that automorphism of $\mathfrak{K}$ over $\mathfrak{K}_0$ different from the identity. In the latter instance, the cases $\mathfrak{K}_0=GF(3)$ or $GF(5)$ are excluded for dimension three. The only case considered by Dieudonné for index zero is the orthogonal with dimension of the vector space equal to three [2, p. 52].

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[9] See [11] for a detailed proof for groups of linear transformations.

6. **Remarks.** The proofs outlined above give a relatively uniform treatment of all of the groups considered for infinite or sufficiently high dimensions. It is worth noting the contrast with the proofs given by Dieudonné for the finite-dimensional cases. Dieudonné also makes some use of the methods introduced by Mackey as well as methods of Schreier and van der Waerden. However he relies heavily on structure theory [3] which leads to a great multiplicity of cases. This also explains his restriction to nonzero index in the orthogonal and unitary cases since the structure theory is not available here for index zero. It also explains his restriction of the coefficient domain in the unitary case. Needless to say, there is practically no structure theory available in the infinite-dimensional cases. The proofs sketched in the preceding sections rely much more heavily on Mackey's methods and represent a fairly thorough exploitation of those methods. On the other hand, these methods break down for characteristic equal to two and extension of the isomorphism theorems to infinite dimensions for characteristic two has yet to be done. Even in finite dimensions only the full linear and symplectic groups have been considered for characteristic two [2, p. 93]. Hua, in his treatment of the

questions considered here, uses still other methods based on direct calculation with matrices which are quite effective for low dimensions where the other methods run into difficulties.

The projective group associated with a group $\mathcal{G}$ of linear transformations may be defined as the group $\mathcal{G}$ modulo its center Z. If $\mathcal{G}$ is one of the classical groups, then Z reduces to scalar multiples of the identity transformation where the scalars belong to the center of the coefficient ring. When the group $\mathcal{G}$ is allowed to contain semi-linear transformations, it is natural to take Z, instead of equal to the center of $\mathcal{G}$, equal to the group of *all* scalar multiples of the identity which belong to $\mathcal{G}$. It would be desirable to extend the isomorphism theorems discussed above to the corresponding projective groups. This has been done by Dieudonné [2] for automorphisms of the linear groups in finite dimension,[10] the main result being that an automorphism of the projective group is induced by an automorphism of the associated group of linear transformations (with exceptions, of course, among the low dimensions). A study of involutions in the projective group $\mathcal{G}/Z$ leads to consideration of "projective involutions" in $\mathcal{G}$; that is, transformations T in $\mathcal{G}$ such that $T^2=\zeta I\in Z$. The problem then is to characterize within the projective group $\mathcal{G}/Z$ those elements which are determined by extremal involutions in $\mathcal{G}$. In other words, one has to distinguish, modulo Z, the extremal involutions from the other projective involutions in $\mathcal{G}$. Once such a characterization of extremal involutions is obtained, the way is clear to apply the methods of the preceding sections to the projective groups. We are, as yet, unable to deal adequately with the projective groups in infinite dimensions but hope to return to this problem at another time.

It seems probable that the isomorphism theorems can be obtained for certain coefficient domains more general than division rings. An interesting case would be that of an integral domain. As another example, consider an $\mathfrak{R}$-module $\mathfrak{Z}$ (that is, $\mathfrak{Z}$ is an additive group which admits elements of the ring $\mathfrak{R}$ as operators) and assume given an involution in $\mathfrak{R}$. In an obvious way, one can extend the notion of a self-dual vector space to this situation and so obtain generalized symplectic and unitary groups $\mathcal{L}^*(\mathfrak{Z}, \mathfrak{R})$. The generalized unitary groups obtained in this way contain as special cases groups of the form $\mathcal{L}(\mathfrak{X}, \mathfrak{D}, \mathfrak{X}^*)$. This fact suggests the possibility of an even more uniform treatment of the problems considered here.

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[10] Schrier and van der Waerden also consider the projective group.

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YALE UNIVERSITY

THE SUMMER MEETING IN MINNEAPOLIS

The fifty-sixth Summer Meeting and thirty-second Colloquium of the American Mathematical Society were held at the University of Minnesota, Minneapolis, Minnesota, Tuesday to Friday, September 4–7, 1951, in conjunction with meetings of the Mathematical Association of America, the Institute of Mathematical Statistics, the Econometric Society, and Section A of the American Association for the Advancement of Science.

Over 500 people registered for the meetings, among whom were the following 342 members of the Society:

A. A. Albert, G. E. Albert, B. A. Amirà, A. B. Ammann, E. W. Anderson, R. D. Anderson, R. L. Anderson, T. W. Anderson, H. A. Antosiewicz, K. J. Arnold, M. A. Basoco, W. F. Bauer, H. P. Beard, E. G. Begle, J. H. Bell, Theodore Bennett, A. H. Berger, R. H. Bing, H. D. Block, J. W. Bradshaw, R. W. Brink, G. S. Bruton, R. C. Buck, C. E. Burgess, L. J. Burton, L. E. Bush, J. H. Bushey, Jewell H. Bushey, W. H. Bussey, Eugenio Calabi, E. A. Cameron, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, K. H. Carlson, R. E. Carr, Evelyn Carroll-Rusk, Maria Castellani, Lamberto Cesari, Abraham Charnes, Harold Chatland, K. T. Chen, S. S. Chern, Herman Chernoff, E. W. Chittenden, F. M. Clarke, H. D. Colson, T. F. Cope, A. H. Copeland, L. M. Court, T. B. Curtz, J. F. Daly, M. E. Daniells, G. B. Dantzig, P. H. Daus, Allen Devinatz, B. K. Dickerson, C. E. Diesen, Flora Dinkines, M. H. Dipert, W. J. Dixon, C. L. Dolph, J. L. Doob, Aryeh Dvoretzky, P. S. Dwyer, J. M. Earl, W. F. Eberlein, B. J. Eisenstadt, M. E. Estill, H. P Evans, R. L. Evans, F. D. Faulkner, A. M. Feyerherm, D. T. Finkbeiner, I. C. Fischer, M. M. Flood, E. E. Floyd, J. S. Frame, Abraham Franck, H. D. Friedman, D. R. Fulkerson, W. B. Fulks, W. R. Fuller, R. E. Fullerton, Arie Gaalswyk, David Gale, H. M. Gehman, B. R. Gelbaum, Gladys Gibbens, P. W. Gilbert, S. A. Gilbert, W. M. Gilbert, R. E. Gilman, Wallace Givens, V. D. Gokhale, Michael Goldberg, D. B. Goodner, S. H. Gould, Arthur Grad, L. M. Graves, L. J. Green, L. W. Green, V. G. Grove, John Gurland, Edwin Halfar, M. E. Haller, P. C. Hammer, Frank Harary, T. E. Harris, O. G. Harrold, W. L. Hart, H. L. Harter, Charles Hatfield, Nickolas Heerema, Olaf Helmer, I. N. Herstein, M. R. Hestenes, E. L. Hill, J. L. Hodges, A. J. Hoffman, R. V. Hogg, F. E. Hohn, D. L. Holl, Carl Holtom, T. C. Holyoke, Harold Hotelling, E. M Hove, Ralph Hull, M. G. Humphreys, W. A. Hurwitz, Jack Indritz, S. L. Isaacson, W. H. Ito, E. T. Jabotinsky, L. K. Jackson, C. G. Jaeger, L. W. Johnson, B. W. Jones, R. V. Kadison, Shizuo Kakutani, W. C. Kalinowski, G. K. Kalisch, Irving Kaplansky, Leo Katz, W. H. Keen, M. E. Kellar, J. L. Kelley, L. M. Kells, D. E. Kibbey, J. C. Kiefer, W. M. Kincaid, V. L. Klee, Fulton Koehler, T. C. Koopmans, C. F. Kossack, M. S. Kramer, O. E. Lancaster, R. E. Langer, E. H. Larguier, W. G. Leavitt, J. R. Lee, Patrick Leehey, D. H Lehmer, F. C. Leone, Howard Levene, D. J. Lewis, M. T. Lewis, B. W. Lindgren, Lee Lorch, W. S. Loud, Dorothy McCoy, S. W. McCuskey, W. C. McDaniel, J. V. McKelvey, J. C. C. McKinsey, E. J. McShane, C. C. MacDuffee, G. W. Mackey, G. R. MacLane, Saunders MacLane, H. M. MacNeille, H. B. Mann, Morris Marden, E. C. Marth, Imanuel Marx, K. O. May, J. R. Mayor, L. E. Mehlenbacher, Paul Meier, B. E. Meserve, E. J. Mickle, A. N. Milgram, W. H. Mills, E. E. Moise, Deane Montgomery, J. C.

Moore, Frederick Mosteller, H. T. Muhly, Sigurd Mundhjeld, Lewis Nelson, W. J. Nemerever, E. D. Nering, John von Neumann, Jerzy Neyman, O. M. Nikodým, E. A. Nordhaus, M. J. Norris, Rufus Oldenburger, Arthur Ollivier, J. M. H. Olmsted, E. J. Olson, Alex Orden, T. G. Ostrom, E. H. Ostrow, M. P. Peisakoff, J. L. Penez, O. J. Peterson, B. J. Pettis, H. P. Pettit, P. E. Pfeiffer, C. G. Phipps, George Piranian, Everett Pitcher, J. C. Polley, George Pólya, J. E. Powell, G. B. Price, F. M. Pulliam, A. L. Putnam, Gustave Rabson, Tibor Radó, Howard Raiffa, J. F. Randolph, R. B. Rasmusen, L. T. Ratner, G. E. Raynor, M. O. Reade, W. T. Reid, Haim Reingold, H. B. Ribeiro, F. D. Rigby, L. A. Ringenberg, B. V. Ritchie, E. K. Ritter, J. H. Roberts, H. A. Robinson, Murray Rosenblatt, P. C. Rosenbloom, Arthur Rosenthal, A. E. Ross, E. H. Rothe, Herman Rubin, Walter Rudin, H. E. Salzer, Hans Samelson, James Sanders, L. R. Sario, A. C. Schaeffer, Henry Scheffé, E. V. Schenkman, Peter Scherk, E. R. Schneckenburger, R. R. Seeber, Jr., I. E. Segal, George Seifert, E. B. Shanks, L. S. Shapley, L. W. Sheridan, Seymour Sherman, Harold Shniad, S. S. Shu, Annette Sinclair, I. M. Singer, M. L. Slater, D. M. Smiley, M. F. Smiley, A. J. Smith, F. C. Smith, Ernst Snapper, J. L. Snell, W. S. Snyder, Andrew Sobczyk, E. S. Sokolnikoff, J. J. Sopka, E. H. Spanier, E. J. Specht, D. C. Spencer, Abraham Spitzbart, M. D. Springer, R. C. Staley, W. L. Stamey, M. L. Stein, H. E. Stelson, Rothwell Stephens, D. M. Stone, M. H. Stone, Irwin Stoner, E. B. Stouffer, J. V. Talacko, H. P. Thielman, G. H. M. Thomas, J. M. Thomas, J. E. Thompson, Gerhard Tintner, E. W. Titt, C. B. Tompkins, Leonard Tornheim, J. I. Tracey, E. F. Trombley, A. W. Tucker, J. W. Tukey, H. L. Turrittin, E. P. Vance, A. H. Van Tuyl, V. J. Varineau, N. H. Vaughan, Bernard Vinograde, D. F. Votaw, H. C. Wang, J. A. Ward, S. E. Warschawski, M. T. Wechsler, H. F. Weinberger, Alexander Weinstein, B. A. Welch, W. J. Wells, M. D. Wetzel, G. W. Whitehead, W. F. Whitmore, H. H. Wicke, L. A. Wolf, Jacob Wolfowitz, G. N. Wollan, Y. K. Wong, M. A. Woodbury, J. L. Yarnell, L. C. Young, J. W. T. Youngs, Arthur Zeichner, R. A. Zemlin, Antoni Zygmund.

The Colloquium Lectures, on *Topological transformation groups*, were presented by Professor Deane Montgomery of the Institute for Advanced Study on Tuesday afternoon and Wednesday, Thursday, and Friday mornings. Presiding were, in order, Professor E. W. Chittenden, Professor M. H. Stone, Professor John von Neumann, and Professor Everett Pitcher.

At a joint session of Section A and the Society at 10:30 A.M., Wednesday, September 5, Professor E. J. McShane of the University of Virginia gave an address, *Order preserving mappings of partially ordered spaces*, as a retiring Vice President of the American Association for the Advancement of Science. Professor R. W. Brink presided.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited two speakers. On Thursday, September 6, Professor R. H. Bing of the University of Wisconsin gave an address on *Partitioning continuous curves* at 2:00 P.M. Professor J. H. Roberts presided. On Friday at 2:00 P.M. Professor G. W. Whitehead of the Massachusetts Institute of Technology gave an address on *Homot-*

opy theory. Professor Hans Samelson presided.

The organized social events provided for the Society consisted of:

An informal reception on Tuesday evening at the Campus Club, Memorial Union. Members of the receiving line included: from the University of Minnesota: President James L. Morrill, Dean Theodore C. Blegen, Professor and Mrs. Raymond W. Brink, and Professor and Mrs. George C. Priester; from the College of St. Thomas: The Reverend Bernard J. Coughlin, Mrs. Lawrence W. Sheridan, and Professor L. Earle Bush.

A banquet on Wednesday evening in the main ballroom of Coffman Memorial Union. Professor W. A. Hurwitz acted as toastmaster, and there were speeches by Professor Saunders MacLane (main speaker and representative of the Mathematical Association of America), Professor John von Neumann (representing the American Mathematical Society), Professor Gerhard Tintner (representing the Econometric Society), and Professor Paul S. Dwyer (representing the Institute of Mathematics Statistics).

A piano recital on Thursday evening by Emilie Pray, concert pianist and instructor at Macalaster College in St. Paul.

Activities for the women included a visit to the Betty Crocker Kitchens of the General Mills Corporation on Tuesday afternoon, and a luncheon on Thursday at the country club house of the Automobile Club of Minneapolis.

The Council met at 8:00 P.M. on Tuesday, September 4.

The Secretary announced the election of the following sixty-one persons to ordinary membership in the Society:

Mr. Jack Morell Anderson, University of South Dakota;
Mr. Richard Jardine Arthur, Assistant, University of Illinois;
Mr. Albert Elliot Babbitt, Jr., Columbia University;
Mr. William Beck, Assistant, University of Kentucky;
Mr. Volodymyr Bohun-Chudyniv, New York, New York;
Professor Raj Chandra Bose, University of North Carolina;
Mr. Charles Bostick, Assistant, University of Illinois;
Mr. John Joseph Brady, Mathematician, Naval Ordnance Laboratory, White Oak, Maryland;
Mr. Barron Brainerd, University of Michigan;
Mr. Richard Kettel Brown, Teaching Assistant, Rutgers University;
Mr. Pablo Emilio Casas, Princeton University;
Mr. Justus Chancellor, III, Mathematician, International Business Machines Corporation, Detroit, Michigan;
Mr. George Y. Cherlin, Instructor, Rutgers University;
Mr. Joseph Charles Connell, Staff Member, Sandia Corporation, Albuquerque, New Mexico;
Mr. Stanley Frost Dice, Lecturer, University of Pittsburgh;
Mr. Donald Epstein, Teaching Assistant, Syracuse University;

Miss Mae Irene Fauth, Instructor, Chemistry Department, Pennsylvania State College;
Reverend Peter Federchuck, Instructor, St. Basil's Prep School, Stamford, Connecticut;
Mr. Walter Clemens Frey, Teaching Assistant, Rutgers University;
Mr. Henry David Friedman, Pennsylvania State College;
Mr. Seymour Ginsburg, University of Michigan;
Mr. Karl Goldberg, Brooklyn, New York;
Mr. George Gumas, Assistant Project Engineer, Sperry Gyroscope, Great Neck, Long Island, New York;
Mr. Theodore Edward Hagensee, Instructor, Chicago Jewish Academy;
Mr. Harold Joseph Hasenfus, Chief, Rocket Section, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland;
Miss Emilie Virginia Haynsworth, Fellow, University of North Carolina;
Mr. Owen Heller Hoke, University of Georgia;
Mr. Morton Roy Kenner, Mathematician, Bell Aircraft Corporation, Buffalo, New York;
Mr. Jack C. Kiefer, Research Associate, Cornell University;
Mr. Martin David Kruskal, Assistant, Institute for Mathematics and Mechanics, New York University, New York, New York;
Acting Assistant Professor Esayas George Kundert, University of Tennessee;
Mr. Hasell Thomas LaBorde, Instructor, University of North Carolina;
Mr. Christian Donald LaBudde, Mathematician, Air Weapons Research Center, Museum of Science and Industry, Chicago, Illinois;
Lieutenant Commander Patrick Leehey, Technical Officer, Mathematical Sciences Division, Office of Naval Research, Washington, D. C.;
Mr. Bernard Werner Levinger, New York, New York;
Mr. San Dao Liao, University of Chicago;
Mr. Robert Dale Lowe, Assistant, Northwestern University;
Mr. Harold V. McIntosh, Cornell University;
Professor Samuel Alexander McLeod, Lander College, Greenwood, South Carolina;
Mr. Pat Maxwell, Jr., Staff Member, Sandia Corporation, Albuquerque, New Mexico;
Mr. Rudolph Burton Merkel, Teacher, Sacramento City Unified School District, Sacramento, California;
Sister Ingonda Maria von Mezynski, Teacher, Holy Ghost College, Techny, Illinois;
Mr. Frederick David Miller, Teaching Fellow, University of Michigan;
Mr. Charles Thomas Molloy, Mathematics Section, Kellex Corporation, New York, New York;
Dr. Cathleen Synge Morawetz (Mrs.), Staff Member, Massachusetts Institute of Technology;
Mr. Sidney I. Neuwrith, Bio-Statistician, Schering Corporation, Bloomfield, New Jersey;
Mr. Robert Stephen Novosad, Instructor, Tulane University;
Mr. Martin Orr, Mathematician, Signal Corps Engineering Laboratories, Signal Corp Center and Fort Monmouth, New Jersey;
Mr. William Coe Orthwein, Aerophysics, Consolidated Vultee Aircraft Corporation, Fort Worth, Texas;
Mr. Nicholaas du Plessis, Senior Lecturer, Rhodes University, Grahamstown, South Africa;
Mr. John Joseph Randazzo, St. Louis University;

Mr. Marvin Shinbrot, Aeronautical Research Scientist, National Advisory Committee for Aeronautics, Moffet Field, California;
Miss Arienne Bernice Silverstein, Columbia University;
Mr. Louis Solomon, Harvard University;
Mr. Benjamin Ormond Van Hook, Mississippi Southern College;
Professor Orlando Eugenio Villamayer, Faculty of Engineering, University of Cuyo, San Juan, Argentina;
Mr. Milo Wesley Weaver, Instructor, University of Texas;
Mr. Edward B. West, Teaching Assistant, University of Missouri;
Assistant Professor Walter Whittier Wright, Physicist, State Engineering Experiment Station, Georgia Institute of Technology, Atlanta;
Mr. Chung-Tao Yang, Tulane University;
Mr. Joseph Zeu-tse Yao, University of Chicago;

It was reported that the following two persons had been elected as nominees of institutional members as indicated:

Cornell University: Mr. Daniel Burrill Ray.
Queens College: Mr. Alfred Gaetano Vassolotti.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Swiss Mathematical Society: Mr. Andrew Bernard Ammann, Research Associate, University of Chicago; Wiskundig Genootschap: Professor Lauwerens Kuipers, Technical Faculty, University of Indonesia;

The University of Arkansas, Fayetteville, Arkansas; The College of the Holy Cross, Worcester, Massachusetts; The University of Miami, Coral Gables, Florida; Mississippi State College, State College, Mississippi; and Montana State College, Bozeman, Montana have been elected to institutional membership.

The following appointments of representatives of the Society were reported: Professor F. C. Ogg at the inauguration of Asa Smalledge Knowles as President of the University of Toledo on May 9, 1951; Professor L. M. Blumenthal at the inauguration of Ralph L. Woodward as President of Central College on June 2, 1951; Professor E. M. Beesley at the inauguration of Malcolm A. Love as President of the University of Nevada on June 11, 1951; and Professor S. W. McCuskey at the 125th anniversary ceremonies of Western Reserve University on June 11, 1951.

The following additional appointments by the President were reported: as a committee to submit to the Policy Committee a list of nominees for the United States National Committee on Mathematics: E. G. Begle, W. T. Martin, and G. A. Hedlund. (This committee nominated Einar Hille, Marshall Stone, John von Neumann, and Norbert Wiener); as a committee to select Gibbs Lecturers for

1952 and 1953: L. V. Ahlfors (Chairman), Salomon Bochner, and Nathan Jacobson; as a committee on arrangements for the Providence meeting: F. M. Stewart (Chairman), C. R. Adams, A. A. Bennett, L. W. Cohen, H. M. Gehman, and E. H. Lee; as a committee on arrangements for the Southeast meeting: J. C. Eaves (Chairman), J. C. Currie, F. A. Lewis, E. P. Miles, W. V. Parker, W. A. Rutledge, and W. M. Whyburn; Professor K. O. Friedrichs was reappointed the Society's representative on the Advisory Board of the Applied Mechanics Reviews for a three-year term starting July 1, 1951; Professor Eric Reissner as Chairman of the Editorial Committee for the Applied Mathematics Symposium Proceedings for period July 1, 1951 to June 30, 1952; and Professor A. H. Taub reappointed a member of Editorial Committee for the Applied Mathematics Symposium Proceedings for period July 1, 1951 to June 30, 1954. (Committee now consists of: Eric Reissner (Chairman), R. V. Churchill, and A. H. Taub.)

The following dates of meetings have been approved by the Council: December 1, 1951, at California Institute of Technology; February 23, 1952, at Columbia University; April 25–26, 1952, at Columbia University; April 25–26, 1952, at the University of Chicago; May 3, 1952, at Fresno State College; June 21, 1952, at the University of Oregon; and September 2–5, 1952, at Michigan State College.

Invitations to give addresses in 1951 were announced: R. H. Bing, Summer Meeting, Minneapolis; J. C. Oxtoby for the October meeting in Washington, D.C.; E. H. Rothe for the October meeting in Norman, Oklahoma; J. L. Kelley and Herbert Robbins for the November meeting, Auburn, Alabama; and J. C. C. McKinsey for the December meeting in Pasadena.

The Council voted to accept an invitation to hold at the Carnegie Institute of Technology on June 16–17, 1952, the Fifth Symposium on Applied Mathematics.

The Executive Director reported that the postcards sent out with the programs of the April meetings showed that in many cases the programs did not reach members until as long as 18 days after mailing. Other procedures for distributing the programs are under conconsideration, but no definite plan has yet been decided on.

The headquarters of the Society will be moved from New York City to 80 Waterman Street, Providence, Rhode Island, during the week of September 17.

The bulk of the Society's library has now been moved to the University of Georgia.

The report of the Librarian was received. The Council voted to put the problem of exchanges of our periodicals in the hands of the

Editor of Mathematical Reviews and also voted to cooperate with the University of Georgia in requesting copies of dissertations to be deposited in the University of Georgia Library.

The Council voted to appoint Professor G. B. Hochschild a member of the Proceedings Editorial Committee during the absence from this country of Professor Jacobson.

The Council voted that when two or more meetings are set at dates sufficiently close so that the programs should be sent out in one mailing, the deadline for abstracts for all these meetings be set as the deadline for the earliest of these.

The Council voted to approve the following resolution, which was also approved by the Mathematical Association of America:

> The Board of Governors of the Mathematical Association of America and the Council of the American Mathematical Society respectfully call to the attention of the Congress of the United States the following considerations, which in our view indicate the extreme importance to the national welfare of having sufficient funds provided now, through the National Science Foundation, for the support of basic research and of a program of fellowships for the training of more scientists.
>
> Since the present emergency is not likely to end for quite a number of years, it is important to take a long range view of our needs for scientific research in connection with the national defense. The airplane manufacturers and the various research agencies under the Department of Defense are now unable to secure enough trained and experienced scientists and engineers to fill their requirements. This serious shortage is partly due to the diversion of students from the fields of science and engineering to the armed services and to other activities during the years 1941–45. It is also undoubtedly due in part to the fact that many young men with potential scientific ability did not have the funds to develop their talents. In order to provide an adequate supply of trained investigators for the near future, it is necessary to begin at once to provide such encouragement for the prospective scientists as is contemplated in the fellowship program of the National Science Foundation.
>
> The intimate dependence of practical developments upon basic research in science has been shown repeatedly in recent years. The universities have been the centers where basic research most naturally flourishes. The provision of funds to help support investigations in the universities is also important for our defense and general welfare in the near future. The amount suggested in the budget of the National Science Foundation is small compared with the amounts currently provided by the universities themselves, but will nevertheless be an effective stimulus in accelerating basic research.
>
> In view of the preceding, the Board of Governors of the Mathematical Association of America and the Council of the American Mathematical Society respectfully urge that the Congress appropriate adequate funds for the support of basic scientific research and of students taking scientific training, as provided in the budget of the National Science Foundation, and directs that the Secretaries of the Association and the Society forward copies of this statement to the members of the Committees on appropriation of the Senate and of the House of Representatives.

The Council voted to approve the recommendation of the Committee on Russian Translations that the preparation of these translations be continued.

The Council directed the Secretary to obtain assurance, whenever he feels it necessary, that at any scheduled meeting of the Society there will be no discrimination as to race, color, creed, or nationality.

As requested by the members present at the Annual Meeting in 1950 at Gainesville, the Council reconsidered its actions concerning the University of California. After a lengthy discussion, the Council voted to reaffirm its action of September 1, 1950. when it passed the following resolution:

> The Council of the American Mathematical Society deplores the harm done to academic freedom and scientific progress by the recent action of the Regents of the University of California in imposing arbitrary and humiliating conditions of employment on the faculty. The Council notes that this action has already resulted in a great discontent and loss of morale in the California faculty, and in the consequent desire of many distinguished faculty members to move elsewhere. The future effects of this action upon the scientific and academic work of the California faculty and upon the standing of the University will be disastrous. The Council therefore urges that the Regents reconsider their action, so as to restore academic freedom and to insure the continued high standing of the University of California.

The Council also passed the following resolution:

> Owing to the expressed reluctance of a large number of mathematicians to attend meetings of the American Mathematical Society at the University of California on account of the conditions condemned in a resolution adopted by the Council at its meeting of September 1, 1950, the Council hereby regretfully resolves that it will not consider holding any meetings of the Society at the University of California during the calendar years 1951, 1952, and 1953 unless those conditions have been alleviated in the meantime.

The Secretary was instructed to send copies of these resolutions to the editors of Science.

Abstracts of the papers read follow. Presiding officers at the sessions for contributed papers were Professors B. J. Pettis, R. E. Langer, E. H. Spanier, Tibor Radó, Dr. H. M. MacNeille, Professors J. L. Doob, Antoni Zygmund, S. E. Warschawski, O. G. Harrold, M. R. Hestenes.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 426 was presented by Professor Chatland, 459 by Dr. Kadison, 462 by Professor Gelbaum, 476 by Dr. Marx, 484 by Professor Piranian, 506 by Professor Charnes, 508 by Professor Hestenes, 518 by Dr. Hammer, 522 by Professor Sobczyk, and 533 by Professor Calabi.

Mr. Foster was introduced by Professor J. A. Ward, Professor Goffman by Professor J. W. T. Youngs, Professor Polansky by Professor S. E. Warschawski, and Dr. Yu by Professor Szolem Mandelbrojt.

ALGEBRA AND THEORY OF NUMBERS

421t. A. A. Albert: *On commutative division algebras.*

Let D be a division algebra with a unity quantity over its center F. If D is commutative and has degree two over F, then it is shown that F has characteristic two and that the subfields of D are all inseparable. An example showing the existence of such algebras is given. The result is also used in the proof of the theorem stating that every commutative power-associative finite division algebra is a field. The last and most important contribution of the paper is a construction of central division algebras of all odd dimensions n over any field F of characteristic not two such that there exists a cyclic extension of degree n over F. This result can be combined with a construction of L. E. Dickson to show that there exist finite commutative division algebras of every dimension $n>2$ over any finite field F of characteristic not two. (Received July 20, 1951).

422. A. A. Albert: *On nonassociative division algebras.*

This paper is a revision and expansion of the paper entitled *On commutative division algebras*. The main addition is a proof of the theorem stating that every finite power-associative division ring D of characteristic $p>5$ is a finite field. As in the commutative case the result holds for rings of characteristic 3 and 5 providing that it is assumed that all scalar extensions of the power-associative attached commutative algebra $D^{(+)}$ are also power-associative. (Received August 18, 1951.)

423t. Leonard Carlitz: *Diophantine approximation in fields of characteristic p.*

Let Φ denote the field consisting of the numbers $\alpha=\sum_{-\infty}^{m} c_i x^i$, $c_i \in GF(p^n)$; define $e(\alpha)=e^{2\pi i a_1/p}$ where $c_{-1}=a_1\theta^{n-1}+\cdots+a_n$, where θ defines the $GF(p^n)$. The following items are treated. 1. Kronecker's theorem for Φ. 2. Criteria for uniform distribution. 3. Estimates for "Weyl sums" $\sum e(\phi(A))$, $A\in GF[p^n, x]$, deg $A<m$, and $\phi(u)$ is a polynomial of degree less than p with coefficients in Φ. 4. Applications. 5. The sum $S_m(\alpha, \beta)=\sum e(A^2\alpha+2B\beta)$, deg $A<m$. The main tool here is the following analogue of the Hardy-Littlewood approximate functional equation: $S_m(\alpha, \beta)=\eta p^{na/2} \cdot S_{m-a}(1/\alpha, \beta/\alpha)$, where deg $\alpha=-a<0$, deg $\beta<0$, and η is a complex number of absolute value 1. 6. Upper and lower bounds for $S_m(\alpha, \beta)$. (Received July 2, 1951.)

424t. Leonard Carlitz: *Independence of arithmetic functions.*

Bellman and Shapiro (Duke Math. J. vol. 15 (1948) pp. 229–235) proved the algebraic independence of a certain set of arithmetic functions; a short direct proof was given by L. I. Wade (ibid. p. 237). In the present note the ordinary product of arithmetic functions is replaced by the Dirichlet problem. It is then proved that the functions $I_0, \cdots, I_r$ are algebraically independent, where $I_k(m)=m^k$; indeed if $\Phi(u_0, \cdots, u_r)$ denotes a power series which converges for $|u_i|<1+\epsilon$, $\epsilon>0$, then $\Phi(I_0, \cdots, I_r)=0$ implies the identical vanishing of Φ. If $Q_k(m)=m^k$ for m quadratfrei and 0 otherwise, then similar results hold for $I_0, \cdots, I_r, Q_0, \cdots, Q_s$. (Received July 2, 1951.)

425t. Leonard Carlitz: *Some applications of a theorem of Chevalley.*

The well known theorem of Chevalley on systems of equations with coefficients in a finite field (Abhandlungen aus dem Mathematischen Seminar der Hansischen

Universität vol. 11 (1936) pp. 73–75) is applied (1) to show that certain systems of equations with polynomial coefficients have nontrivial solutions. For example if $f(u_1, \cdots, u_s)$ is a polynomial of degree not greater than k and with coefficients in $GF[q, x]$, $f(0, \cdots, 0)=0$, then the equation $f(U_1, \cdots, U_s)=0$ with $U_i \in GF[q, x]$ always has a nontrivial solution provided $s \geqq k^2+1$; in general this value of s cannot be diminished. (2) Chevalley's theorem is applied to certain problems of approximation in the field Φ consisting of the numbers $\sum_{-\infty}^{m} c_i x^i$, $c_i \in GF(q)$. For example, if $\alpha \in \Phi$, $m \geqq 1$, $k \geqq 1$, there exist polynomials A, B, $A \neq 0$, deg $A \leqq km$, such that deg $(A^k\alpha - B) < -m$. (Received July 2, 1951.)

426. Harold Chatland and H. B. Mann: *On polynomials reducible modulo every prime ideal.*

In a forthcoming paper H. B. Mann has shown that an irreducible polynomial over an algebraic number field cannot have a root in the ground field modulo every prime ideal. In this paper necessary and sufficient conditions are given that a polynomial factors modulo every prime ideal. It is shown that the polynomial, $f(x)$, of degree n, with coefficients in an algebraic number field, factors modulo every prime ideal if and only if the Galois group of $f(x)$ does not contain a cyclic permutation of degree n. More precisely, if g is the order of the group of $f(x)$ and $m = m(n_1, n_2, \cdots, n_r; f_1, f_2, \cdots, f_r)$ is the number of permutations consisting of n_1 cycles of degree f_1, n_2 cycles of degree f_2, and so forth, and if S is the set of primes modulo which $f(x)$ decomposes into n_1 irreducible factors of degree f_1, n_2 of degree f_2, and so forth, then $\sum_{\mathfrak{p} \subset S} 1/(N(\mathfrak{p}))^s = (m/g) \log (1/(s-1)) + O_{s \to 1} (s-1)$. It follows that an irreducible polynomial of prime degree over an algebraic number field cannot factor modulo every prime ideal. The sufficiency of the condition that the polynomial factor modulo every prime ideal holds also when the ground field is any field. It has been shown by H. B. Mann that the condition is not necessary in the general case. (Received July 25, 1951.)

427*t*. Eckford Cohen: *Arithmetic functions of polynomials.*

Let R be a fixed polynomial of degree r in $D = GF[p^n, x]$. A simple set of arithmetic functions denoted by $\epsilon_Z(A) = \epsilon(ZA, R)$ are defined in terms of pth roots of unity. It is shown that any arithmetic function defined for polynomials in D of degree less than r can be represented uniquely by a sum of these ϵ-functions. Applications of this result are made to the number of representations of polynomials by linear and bilinear sums. The methods of this paper have analogues in the rational case. (Received July 23, 1951.)

428*t*. Harvey Cohn: *Periodic algorithm for cubic forms.*

A reduction theorem of three-dimensional lattices is proved using as a starting point Minkowski's critical rectangular parallelepiped. The theorem states, in effect, that any three noncollinear points (ξ_i, η_i), where $i = 1, 2, 3$, can be transformed by an integral unimodular transformation $\xi_i' = (a\xi_i + b\eta_i + c)/\lambda$, $\eta_i' = (a'\xi_i + b'\eta_i + c')/\lambda$ where $\lambda = a''\xi_i + b''\eta_i + c''$, det $(ab'c'') = 1$, into points (ξ_i', η_i') lying one in each of the regions R_1: $\{\xi' + \eta' + 1 \leqq 0, \xi' \leqq 0, \eta' \leqq 0\}$; R_2: $\{\xi' + \eta' + 1 \geqq 0, \xi' \leqq 0, \eta' \geqq 0\}$; R_3: $\{\xi' + \eta' + 1 \geqq 0, \xi' \geqq 0, \eta' \leqq 0\}$. In the preliminary report of this result (Bull. Amer. Math. Soc. Abstract 56-3-220) the theorem was promised in an equivalent form applicable to "continued fraction" constructions for three-dimensional lattices. (Received July 5, 1951.)

429*t*. Jesse Douglas: *On the basis theorem for finite abelian groups.*

Supplementing the proof given by the author in Proc. Nat. Acad. Sci. U.S.A vol. 37 (1951) pp. 359–362 (cf. Bull. Amer. Math. Soc. Abstract 57-4-274), this paper supplies two additional simple proofs of the "fundamental theorem of abelian groups." The detailed account will appear in later issues of Proc. Nat. Acad. Sci. U. S. A. vol. 37 (Aug., Sept., 1951). (Received July 11, 1951.)

430*t*. A. E. Foster: *A generalized harmonic conjugate for commutative algebras*. Preliminary report.

This paper extends the work of Wagner (Duke Math. J. vol. 15 (1948) p. 455). With every set of partial differential equations of the second order in the form $\partial^2u/\partial x_r\partial x_j=\sum_{i=1}^{n} c_{rji}\ \partial^2u/\partial x_1\partial x_i$ $(r=2, \cdots, n;\ j=r, \cdots, n)$ there can be associated a unique commutative distributive algebra over the field of the c's such that the set will be the generalized Laplace equations if the c_{rji} are such that the algebra is associative and of the Frobenius type. It is shown that if the algebra is of this type, it is always possible to determine the conjugates of a function satisfying the above set of equations. (Received July 23, 1951.)

431. J. S. Frame: *The classes and representations of the groups of* 27 *lines and* 28 *bitangents*.

The finite groups G of order 51840 and H of order 2,903,040, related to the exceptional Lie algebras E_6 and E_7, contain simple subgroups of index 2 denoted by G_0 and H_0. G is the group of automorphisms of the 27 lines on a general cubic surface, which is a subgroup of index 28 in the group H_0 of the automorphisms of the 28 bitangents to a plane quartic curve of genus three. In this paper matrices are given for a set of 36 conjugate hyperplane reflections S_i of which six generate a six-dimensional orthogonal representation of G, and it is shown that if T_k and U_k are respectively products of two or three permutable S_i's, then every element of G can be written either as a U_jU_k or as a U_jT_k. The 25 classes and irreducible characters of G are then determined. Classes are described by a symbol $1^\alpha 2^\beta 3^\gamma \cdots$ in which some of the integers α, β, γ may be negative. A complete set of six basic invariants for G are found. Then the 30 classes and irreducible characters of H_0 are found using induced representations and modular theory. Finally all large subgroups of H_0 are determined from the character table. (Received July 11, 1951.)

432*t*. Leonard Gillman: *On intervals of ordered sets*.

The following statements are considered. $P(\aleph_\alpha)$: $\beta<\alpha$ implies $2^{\aleph_\beta}<\aleph_\alpha$; $P'(\aleph_\alpha)$: $\beta<\omega_{cf(\alpha)}$ implies $2^{\aleph_\beta}<\aleph_\alpha$; P: $\aleph_\alpha$ singular implies $P(\aleph_\alpha)$; $Q(\aleph_\delta, \aleph_\alpha)$: every ordered set of power $\aleph_\delta$ has a family of $\aleph_\alpha$ mutually disjoint intervals; $Q(\aleph_\alpha)$: $Q(\aleph_\alpha, \aleph_\alpha)$; Q: $\aleph_\alpha$ singular implies $Q(\aleph_\alpha)$. Obviously $Q(\aleph_0)$ is true, while both $Q(2^{\aleph_0})$ and $Q(\aleph_1)$ are false. Results: (1) if $\aleph_\alpha$ is singular then $P(\aleph_\alpha)$ implies $Q(\aleph_\alpha)$; (2) for every α, $Q(\aleph_\alpha)$ implies $P'(\aleph_\alpha)$. It follows that: (3) $Q(\aleph_\alpha)$ fails for every regular $\aleph_\alpha$ which is not strongly inaccessible; (4) $Q(2^{\aleph_\alpha})$ fails for every α; (5) if ω_α is regular and $\aleph_{\omega_\alpha}$ is singular, then $P(\aleph_{\omega_\alpha})$ and $Q(\aleph_{\omega_\alpha})$ are equivalent; (6) P and Q are equivalent. If $\aleph_\alpha$ is strongly inaccessible then $Q(\aleph_{\alpha+1}, \aleph_\alpha)$ holds, but the question of $Q(\aleph_\alpha)$ itself is not settled in this case (for $\alpha>0$). (Received June 15, 1951.)

433. Wallace Givens: *Polarities and signature in continuous geometry*.

Let $\mathfrak{R}$ be an irreducible regular ring of order not less than 3, Z its center (necessarily a field) assumed not of characteristic two, and $L=\overline{R}_{\mathfrak{R}}$ the lattice of principal right ideals in $\mathfrak{R}$. [Notations agree with those of von Neumann (cf. his five papers in Proc. Nat. Acad. Sci. U.S.A. 1936–1937).] Assume $x\rightarrow x'$ an antiisomorphism of $\mathfrak{R}$ so $(x)_r\rightarrow(x'a)_l^r\equiv(x)_r^*$ is a dual endomorphism (correlation) in L. Excluding rank $a=1/n$ in the discrete case of order n, a necessary and sufficient condition that $(x)_r^{**}\supseteqq(x)_r$ is $a'=\alpha a$, $\alpha\in Z$. For such a in $\mathfrak{R}$ the correlations are called polarities in agreement with the classical theory of linear complexes, quadrics and antiquadrics. As in the geometry of the quadric cone, one can reduce considerations to the case in which a^{-1} exists by introducing a complement of the vertex $(a)_l^r$. Theorems proved include: (1) dim $[(x)_r\cap(x)_r^*]=\dim(x)_r-\text{rank}(x'ax)$ so $(x)_r$ is totally isotropic if and only if $x'ax=0$; (2) if $(e)_r$ is totally isotropic, there exists a totally isotropic $(f)_r$ such that dim $(e)_r=\dim(f)_r$ and $(e)_r\cap(f)_r=(0)_r$; (3) a totally isotropic subspace of maximum dimension δ exists; and, (4) $a=e'af+f'ae+g'ag$ where $e+f+g=1$, e, f and g are suitable orthogonal idempotents, $(e)_r$ and $(f)_r$ are totally isotropic of dimension δ, and $g'ag$ is definite (that is, for $x\in g\mathfrak{R}g$, $x'ax=0\rightarrow x=0$). (Received July 23, 1951.)

434*t*. Franklin Haimo: *Scripts and their centers.*

Scripts, as defined in Rosenbloom, *The elements of mathematical logic*, New York, 1950, cannot have centers with more than one element unless such scripts have only one prime. If this prime is not in the center, then the latter set is linearly ordered by divisibility. A script is the set sum of four of its sub-scripts. These components are defined in terms of the prime pre- and post-factors of their elements. Two by two, they intersect in the common intersection of all four, the sub-script generated by the finite powers of the primes of the script. (Received July 16, 1951.)

435. Nickolas Heerema: *An algebra determined by a binary cubic form.*

Consider the linear associative algebra $A=A(a, b, c, d)$ generated over a field F by R and S where $ax^3+bx^2y+cxy^2+dy^3\equiv(Rx+Sy)^3$. As in the quadratic case, leading to Clifford algebras, R and S are assumed to commute with the elements of the coefficient field F, which is taken to have more than two elements. The approach is to select, by an economical process, a collection B of monomials which can be shown to contain a basis for A. Using representations by infinite matrices it is then possible to show that B is actually a basis. Some of the principal results are: (1) $A(a, b, c, d)$ is infinite-dimensional for any choice of a, b, c, and d, and an extensive class of bases for A is identified. (2) Every properly homomorphic image of A is finite-dimensional. Hence A is not a direct sum of nonzero algebras over F. (3) If $f(x, y)$ has nonproportional linear factors, the center of A is isomorphic to $F(u, v)/\{u^3+v^2+k^2v\}$ for $k\neq 0$ in F, where $F(u, v)$ designates the polynomial domain in two commuting indeterminants u and v. (Received July 23, 1951.)

436*t*. Nathan Jacobson and C. E. Rickart: *Homomorphisms of Jordan rings of self-adjoint elements.*

Let $\mathfrak{A}$ be an associative ring with an involution $a\rightarrow a^*$ and let $\mathfrak{H}$ denote the set of self-adjoint elements $h=h^*$. Then $\mathfrak{H}$ is a special Jordan ring. In the present paper the investigation of Jordan homomorphisms of rings begun in a previous paper [Trans. Amer. Math. Soc. vol. 69 (1950) pp. 479–502] is continued by studying Jordan homomorphisms of rings of type $\mathfrak{H}$. The principal result is that if $\mathfrak{A}$ is a general matrix ring

$\mathfrak{S}_n$, $n \geqq 3$, with an involution such that $e_{ii}^* = e_{ii}$ $(i=1, \cdots, n)$ and every element of $\mathfrak{H}$ is of the form $a+a^*$, then any Jordan homomorphism of $\mathfrak{H}$ can be extended to an associative homomorphism of $\mathfrak{A}$. This result can be extended to locally matrix rings and in this form is applicable to involutorial simple rings with minimal one-sided ideals. Jordan isomorphisms of the Jordan ring of self-adjoint elements of an involutorial primitive ring with minimal one-sided ideals onto a second Jordan ring of the same type are also studied. (Received July 9, 1951.)

437. Irving Kaplansky: *Algebras of type I.*

In a previous paper (Ann. of Math., March, 1951) the author studied AW^*-algebras, an abstract analogue of W^*-algebras (weakly closed algebras of operators on Hilbert space). In this paper a structure theory is given for AW^*-algebras of type I. The main results are as follows. 1. Two such algebras, which are homogeneous in the appropriate sense, are isomorphic if and only if their centers are isomorphic. 2. An AW^*-algebra of type I is W^* if and only if its center is W^*. 3. A *-automorphism leaving the center elementwise fixed is inner by a unitary element. (Received July 13, 1951.)

438. H. B. Mann: *On identically solvable congruences.*

Let $F(x)$ be a polynomial with coefficients in a field Ω. The congruence $F(x) \equiv 0(p)$ is called identically solvable in Ω if it has a solution in Ω for every prime ideal p of Ω. The following theorem is proved: Let Ω be an algebraic number field and let S be the set of all prime ideals p for which the congruence $F(x) \equiv 0(p)$ has a solution in Ω. Let n be the order of the Galois group G of $F(x)$ and m the number of elements of G that leave at least one root of $F(x)$ fixed. Then $\lim_{s \to 1+} \sum_{p \in S} N(p)^{-s} / -\log (s-1) = m/n$. Moreover the congruence $F(x) \equiv 0(p)$ is identically solvable in Ω if and only if $m=n$. This theorem is obtained as a consequence of a theorem of Frobenius (Sitzber. Ber. Akad. (1896) pp. 689–705). In particular the congruence of the theorem is not identically solvable if $F(x)$ is irreducible or if $F(x) = x^k - a$, where a is not a kth power. The theorem need not hold if Ω is not an algebraic number field even if Ω has infinitely many discreet non-Archimedean valuations. (Received July 30, 1951.)

439*t*. L. J. Mordell: *On cubic equations $z^2 = f(x, y)$ with an infinity of integer solutions.*

It is shown that the equation $z^2 = p^2 + \lambda x + ax^2 + bxy + cy^2 + Ax^3 + Bx^2y + Cxy^2 + Dy^3$, where the coefficients are integers and where $p \neq 0$, $c = 2p$, and $p^2(b^2 - 4ac) + \lambda^2 c$ is positive and not a square, has an infinity of integer solutions. (Received July 16, 1951.)

440*t*. Leo Moser: *On small quadratic nonresidues.*

Some unpublished results of A. Brauer and some results of L. Rédei (Nieuw Archief voor Wiskunde vol. 23 (1950) pp. 150–162) on the distribution of small quadratic nonresidues are extended to smaller intervals. It is proved that for every $\epsilon > 0$ there exists an $N(\epsilon)$ such that for $p > N$, p prime, $p \equiv 1$ (mod 4), the interval $[1, (p/2)^{1/2}]$ contains at least $(c-\epsilon)(p/2)^{1/2}$ nonresidues. $c = 1/2 - (1/2 - 3/\pi^2)^{1/2}$. For all primes $p > N$, the interval $[1, p^{1/2}]$ is shown to contain at least $(c-\epsilon)p^{1/2}$ nonresidues. Explicit results of this type, free of ϵ, are also obtained. (Received July 23, 1951.)

441*t*. T. G. Ostrom: *Some consequences of Hall's multiplier theorem.*

A set of integers $a_0, a_1, \cdots, a_n$ is called a difference set mod N if the set $\{a_i - a_j\}$ contains each nonzero residue exactly once. A multiplier is a number q such that the set $\{qa_i\}$, $i=0, 1, \cdots, n$, is the same as the set $\{a_j+s\}$, $j=0, 1, \cdots, n$, for some s. Let N_1 be a prime factor of N. We shall say that N_1 is of type I if there is some multiplier q such that the exponent to which q belongs mod N is greater than the exponent to which it belongs mod N_1. We shall say that N_1 is of type II if every multiplier q belongs to the same exponent mod N as it does mod N_1. This paper obtains the following consequences from a theorem of Hall: 1. Unless every factor of N is of type II, the existence of a difference set mod N necessitates the existence of a difference set mod one or more of its factors. It follows that a difference set with $n=x^r$, $(r, 3)=1$, cannot exist unless there is a difference set with $n=x$. 2. Unless every factor of N is of type I, it is necessary that $n\equiv p^r$ mod N, where p is a prime factor of n and $3r$ divides n or $n+1$. 3. If every prime factor of N is of type II, the number of multipliers divides n or $n+1$. (Received July 9, 1951.)

442*t*. Peter Scherk: *On complexes in abelian groups*. Preliminary report.

Given an additive abelian group G of order $g\leqq\infty$. Let $A=\{a_1, \cdots, a_r\}$, $B=\{b_0, b_1, \cdots, b_s\}$, $C=\{c_1, \cdots, c_t\}$ denote finite nonempty complexes in G. Then $A+B=\{a_i+b_k\}$, and $C-B$ is the set of all elements $d\subset G$ such that $d+B\subset C$. Theorem 1: Let $r+s<g$, $A+B=C$. Suppose that for each m the equation $(r+m)(b_k-b_0)=0$ has not more than m solutions $b_k\subset B$ ($k>0$; $m=0, 1, \cdots, s-1$). Then $t\geqq r+s$. Theorem 2: Let $s<t$, $A=C-B$. Suppose that for each m the equation $(t-m)(b_k-b_0)=0$ has not more than m solutions $b_k\subset B$ ($k>0$; $m=0, 1, \cdots, s-1$). Then $r\leqq t-s$. Theorem 1 generalizes a theorem by Cauchy and Davenport on cyclic groups of prime order. Its proof follows closely Davenport's pattern. The two theorems and their proofs are connected by a duality theorem essentially due to Hadwiger. (Received July 20, 1951.)

443. Peter Scherk: *On sets of integers*.

Let $n>0$ be fixed. Definitions: I=set of all non-negative integers not greater than n; $A=\{a\}$, $B=\{b\}$, $\cdots$ =subsets of I; $A+B=\{a+b\}\cap I$; $A-B$=largest C such that $C+B\subset A$; $\tilde{A}$=complement in I of $\{n-a\}$; $A(x)$=number of all the $a\leqq x$. Khintchine's inversion formula can readily be generalized as follows: $A-B=(\tilde{A}+B)^{\sim}$, $A+B=(\tilde{A}-b)^{\sim}$. By means of these formulas the following result and its dual are established: Let $0\leqq r$, $0\leqq s$; $r+s<n$; $r\subset A$, $s\subset B$, $n\not\subset C=A+B$. Then there exists an m such that $C(n)-C(n-m)\geqq A(r+m)-A(r)+B(s+m)-B(s)$. The number m satisfies the following requirements: (1) $n-m\subset C$. (2) There is no decomposition $r+s+m=a+b$ where $r\leqq a\leqq r+m$, $s\leqq b\leqq s+m$; in particular $r+m\not\subset A$, $s+m\not\subset B$. (3) $m=n-r-s$ or $0<m\leqq(n-r-s)/2$. (Received July 16, 1951.)

444*t*. M. F. Smiley: *The lemma of Zassenhaus for half-loops*.

If H, K, L are subgroups of a group G, K is normal in G, and L is normal in H, then LK is normal in HK and $LK\cap H$ is normal in H and the identity mapping of G induces an isomorphism of the corresponding quotient groups. This isomorphism theorem is extended to *half-loops* (that is, multiplicative systems with a right unit and unique right solvability), and yields the usual Schreier theorem. Ore's generalization of Remak's theorem applies to half-loops and the isomorphism of the factors in two direct decompositions into indecomposable factors are induced by a single automorphism, as usual. (Received July 3, 1951.)

445. Leonard Tornheim: *Asymmetric minima of indefinite binary quadratic forms and asymmetric approximation by rationals.*

Let $f=ax^2+bxy+cy^2$ have determinant $b^2-4ac=1$. Let p be the greatest lower bound of the non-negative values of f for integral x, y and q the least upper bound of the negative values. For k a positive integer let $M^{-1/2}=\min(kp, -q)$. Markoff and others have found properties of the set of values of M when $k=1$. Segre has determined the smallest value of M for each value of k. This result is extended by finding the next smallest value and showing that it as well as all values close enough to it are condensation points. Analogous results hold for asymmetric approximation to a real number θ by rational numbers x/y. The problem is the determination of the largest value of N such that an infinite number of rationals x/y exist for which $-k/y^2N \leqq x/y-\theta \leqq 1/y^2N$. Proofs depend on the properties of continued fractions. (Received July 23, 1951.)

446. Bernard Vinograde: *The matrix equations $AX=XA'$ and $B^2=I$.*

Let A be nonsingular and nonderogatory with coefficients in any field of zero characteristic. Let $B=\sum_1^n x_iA^{i-1}$. In seeking a rational solution of $B^2=I$ one may proceed by replacing A by its companion matrix M and equating the coefficients of simplest structure. If one symmetrizes the quadratic relations which are thus obtained, a system of symmetric matrices $\{S_i\}$ of Hankel type result with the properties: (1) $MS_i=S_iM'$ and (2) the columns of S_i are the ith columns of the powers of M. Hence the transforms of the S_i's form a basis of the rational symmetric solutions of $AX=XA'$. The comparison of the two matrix equations is accomplished by (1) using the Kronecker product $M\times M^{-1}$ to display the structure of the S_i's and (2) comparing differentials. (Received July 23, 1951.)

ANALYSIS

447*t*. Lipman Bers: *New treatment of boundary value problems for minimal surfaces with singularities at infinity.* Preliminary report.

In a recent paper (Trans. Amer. Math. Soc. vol. 70 (1951) pp. 465–491) the author established an existence theorem for a class of boundary value problems for minimal surfaces with singularities at infinity. The boundary curve P was assumed to deviate but little from a convex curve. In the present paper this condition of "essential convexity" is removed. The new proof is based on an a priori estimate resulting from the theory of quasiconformal transformations. (Received July 23, 1951.)

448*t*. Lipman Bers and Shmuel Agmon: *The expansion theorem for pseudo-analytic functions.* Preliminary report.

Theorem. Let $w(z)$ be regular pseudo-analytic for $|z-z_0|<R$ (cf. L. Bers, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 130–136; vol. 37 (1951) pp. 42–47). Then $w(z)$ can be expanded in a series of global formal powers which converges for $|z-z_0|<R$. This strengthens the result announced in the notes quoted above. A similar theorem holds for formal Laurent series. (Received July 23, 1951.)

449*t*. D. G. Bourgin: *Some multiplicative functionals.*

Suppose G and G' refer to an Abelian compact group and to its character group respectively and let $L_2(G)$ be a ring with multiplication interpreted as convolution. The general form of a norm continuous multiplicative functional M on $L_2(G)$ to the complex field is found in terms of the Fourier transform of x. The determining set is a finite subset of G'. These and similar conclusions depend on the finiteness of determining sets of continuous multiplicative functionals on such spaces as for instance $L_p(S)$, $p \geqq 1$, and so on, where S is discrete. Other cases where the determining set is finite include $C(S)$, S compact, where weak replaces norm continuity for M. These results may be viewed as the special one-dimensional cases of a representation theory for the multiplicative semi-group parts of certain Banach algebras. (Received July 20, 1951.)

450t. A. P. Calderón and Antoni Zygmund: *Singular integrals in the theory of the potential.*

If P, Q are points of E^n, the length of the vector $P-Q$ will be written $|P-Q|$ Let $\Omega(P)$ be any function defined on the surface Σ of the n-dimensional unit sphere with center O, satisfying condition Lip α $(\alpha>0)$, with $\int_\Sigma \Omega(P)dP=0$. Let $K(P-Q) =|P-Q|^{-n}\Omega[(P-Q)|P-Q|^{-1}]$, and let $K_\lambda(P-Q)=K(P-Q)$ for $|P-Q|\geqq 1/\lambda$, $K_\lambda(P-Q)=0$ otherwise. Let $\tilde{f}_\lambda(P)=\int_{E^n}K_\lambda(P-Q)f(Q)dQ$. Then, for $\lambda\rightarrow\infty$, (1) If $f\in L^p$, $1<p<\infty$, the function $\tilde{f}_\lambda(P)$ tends to a limit $\tilde{f}(P)$, in the metric L^p, almost everywhere, and majorized by a function from L^p. (2) If $|f|(1+\log^+|f|)$ is integrable, $\tilde{f}_\lambda(P)\rightarrow\tilde{f}(P)$ in the metric L^1 over every set of finite measure. (3) If $f(P)dP$ is replaced by $\mu(dP)$, where μ is a completely additive set function with $\int_{E^n}|\mu(dP)|<\infty$, $\tilde{f}_\lambda(P)$ tends to a finite limit almost everywhere. These results have applications to the problems of the differentiability of the potentials. Take, for example, $n=2$. (4) Let $f(s, t)$ vanish near the point at infinity and let $|f|\log^+|f|\in L$. Let $u(x, y) =\int_{E^2}f(s, t)\log(1/r)dsdt$. Then $u(x, y)=\int_{-\infty}^{x}\int_{-\infty}^{y}\tilde{f}(s, t)dsdt$, for some $\tilde{f}\in L$. The partial derivatives u_x, u_y are absolutely continuous on almost every line parallel to either axis. The derivatives u_{xx}, u_{xy}, u_{yy} exist almost everywhere and are given by the classical expressions. Moreover, u has p.p. a second differential. (5) Under the assumptions of (4), the Newtonian potential $U(x, y)=\iint_{E^2}f(s, t)r^{-1}dsdt$ is absolutely continuous on almost every line parallel to either axis. If $f\in L^q$, $q>2$, then U has p.p. a first differential. (Received July 16, 1951.)

451. R. E. Carr: *Pattern integration with Riemann double integrals.*

The proper Riemann double integral $\iint_R f(x, y)dA$ is defined to be the $\lim_{n\rightarrow\infty}\sum_{k=1}^{n} f(\xi_k^{(n)}, \eta_k^{(n)})A_k^{(n)}$, where $A_k^{(n)}$ is the kth subregion when R is divided into n parts, and $(\xi_k^{(n)}, \eta_k^{(n)})$ is any point in $A_k^{(n)}$. (It is assumed that as $n\rightarrow\infty$ the maximum diameter of each sub-region approaches zero.) The pattern integral is defined to be the corresponding limit, providing it exists, when the summation is restricted to a prescribed subset P of the set $N=\{k\}_1^n$. The case is considered in which R is the unit square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, and in which the subdivision is made by the horizontal lines $x=i/m$, $i=1, 2, \cdots, m-1$, and the vertical lines $y=j/n$, $j=1, 2, \cdots, n-1$. The principal theorem shows that, if $\{\beta_{jk}\}$ is any prescribed double sequence of complex numbers where $|\beta_{jk}|<M$ for all j and k, and if $\lim_{m,n\rightarrow\infty}(1/mn)\sum_{j,k=1}^{m,n}\beta_{jk}=\beta$, then, from the existence of $\iint_R f(x, y)dA$, $\lim_{m,n\rightarrow\infty}(1/mn)\sum_{j,k=1}^{m,n}\beta_{jk}f(\xi_j^{(m,n)}, \eta_k^{(m,n)}) =\beta\iint_R f(x, y)dA$, where $(j-1)/m\leqq\xi_j^{(m,n)}\leqq j/m$, $(k-1)/n\leqq\eta_k^{(m,n)}\leqq k/n$. The extension of this theorem to multiple integrals of higher order is immediate. (Received May 21, 1951.)

452. Lamberto Cesari: *Some connections between Lebesgue area and measure theory.*

Let S be any Fréchet surface in E_3 of finite Lebesgue area $L(S)$, and $T: x = x(w)$, $w \in Q$, any representation of S on a simple closed Jordan region Q. Let $\mathfrak{G}$ be the collection of all maximal continua g of constancy for T on Q (proper continua and single points); $\mathfrak{M}$, $\mathfrak{O}$, $\mathfrak{B}$ respectively the collections of all subsets of Q, of all subsets of Q open in Q, of all Borel subsets of Q; $\mathfrak{M}_0$, $\mathfrak{O}_0$, $\mathfrak{B}_0$ the subcollections of $\mathfrak{M}$, $\mathfrak{O}$, $\mathfrak{B}$ of those sets which are sums of continua $g \in \mathfrak{G}$. Let $G(O, T)$ be the Geöcze area of the part of S defined by T on any set $O \in \mathfrak{O}$. Utilizing methods previously introduced [Memorie Acad. Italia vol. 13 (1943) pp. 1323–1481] it is proved that *for any sequence O_i of sets $O_i \in \mathfrak{O}_0$ we have $G(\sum O_i, T) \leqq \sum G(O_i, T)$*. By making use also of the general theory of set functions [H. Hahn and A. Rosenthal, *Set functions*, 1948] it is shown that *the function $G(O, T)$, $O \in \mathfrak{O}_0$, can be extended to a function $\phi(M, T)$ defined for all $M \in \mathfrak{M}_0$, coinciding with $G(O, T)$ for all $O \in \mathfrak{O}_0$, and ϕ is a regular ordinary measure function, is a content function, is totally additive in $\mathfrak{B}_0$, and is continuous.* Further results are obtained for restricted classes of representations. (Received July 16, 1951.)

453t. V. F. Cowling: *On analytic continuation by quasi-Hausdorff methods.*

In this paper regions in the complex plane in which a Taylor series is summed to its "proper" value by means of quasi-Hausdorff methods are determined (see G. H. Hardy, *Divergent series*, Oxford University Press, 1949, pp. 277–288). The method is similar to that employed by R. P. Agnew in his determination of regions in the complex plane in which a Taylor series is summed by Hausdorff methods. The lengthy nature of the results prevents their being stated here even in the special case of the geometric series. (Received July 23, 1951.)

454t. Albert Edrei: *On entire and meromorphic functions which have only real zeros and real ones.*

In this paper, the author proves that, if $f(z)$ is an entire function, and if the roots of each of the two equations $f(z) = a$, $f(z) = b$ $(a \neq b)$ are all real, then the order of $f(z)$ is finite and does not exceed one. The statement becomes false if the word entire is replaced by meromorphic. However, there exist analogous theorems for meromorphic functions which are stated and proved in the paper. As an application of the simplest of the theorems, the author proves a conjecture of Schoenberg on the generating function of a totally positive sequence. (Received August 2, 1951.)

455. B. J. Eisenstadt: *The component of the identity of the space of continuous functions into the circle.*

Let $R_{2q}(X)$ denote the component of the identity of the space of continuous functions from a topological space X into the reals mod $2q$. $R_{2q}(X)$ can be made, in a natural way, into a metric, abelian group. In this paper, necessary and sufficient conditions are put on a metric, abelian group so that it be equivalent to $R_{2q}(X)$ for some q and for some compact, connected space X. A pseudo multiplication by real scalars can be introduced on $R_{2q}(X)$ and the points of X define pseudo linear functionals from $R_{2q}(X)$ to the reals mod $2q$. Thus, in the characterization, conditions are put on the abstract group to ensure the existence of a pseudo multiplication and of sufficiently many pseudo linear functionals. The looked for space X is then obtained as a subset of the set of all pseudo linear functionals. In picking out the points of X, certain Banach spaces associated with the group are investigated, a new linear functional is discussed, and a new characterization of the Banach space of real, continuous functions is given. (Received August 1, 1951.)

456*t*. R. S. Finn: *Isolated singularities of nonlinear partial differential equations.*

Equations of the form (1) $(\sigma\phi_x)_x+(\tau\phi_y)_y=0$ are considered. Let $\sigma=f(\phi_x, \phi_y)$, $\tau=g(\phi_x, \phi_y)$, and let (1) be of elliptic type. Let $\phi(x, y)$ be a solution single-valued in a neighborhood of an isolated singular point q. Then if near q, $(\sigma^2+\tau^2)^{1/2}|\nabla\phi|=o(1/r)$ where r is the distance to q, the singularity at q is removable. Previous results of Bers on the minimal surface equation (Ann. of Math. vol. 53 (1951) pp. 364–386) and of the author (Bull. Amer. Math. Soc. Abstract 57-3-212) appear as special cases. If $\sigma=\tau=f(\phi_x^2+\phi_y^2)$, every solution with n-valued gradient at q may be represented in the form $\phi(x, y)=c$ arc tan $y/x+\chi(x, y)$, where $\chi(x, y)$ is n-valued and bounded at q. Also considered are nonlinear equations of the form $\sum_1^r(\sigma_\nu\phi_{x_\nu})_{x_\nu}=0$. If near an isolated singular point q, $(\sum_1^n \sigma_\nu^2)^{1/2}|\nabla\phi|=(1/r^{n-1})$, $\phi(x, y)$ is bounded at q. The proofs depend on repeated application of several forms of Green's identity, and on an integration process defined over level surfaces of a function. (Received July 23, 1951.)

457*t*. Evelyn Frank: *On certain determinantal equations.*

The nature of the roots of certain determinantal equations is derived by the use of continued fractions. Also, conditions on determinantal equations are found which give the number of roots with positive and negative real parts. The determinantal equation $|A-zI|=0$ has all roots pure imaginary or zero if the elements a_{jk} and $-a_{kj}$ are conjugate complex constants and the a_{jj} are pure imaginary or zero, $j, k=1, 2, \cdots, n$. (Received May 21, 1951.)

458*t*. M. S. Friberg: *A method for the effective determination of conformal maps.*

A method similar to that of Theodorsen is derived for conformal mapping of nearly circular regions. Let C: $\rho=\rho(\theta)$, $0\leqq\theta\leqq 2\pi$, ($\rho''(\theta)$ continuous) be a closed Jordan curve and $\beta(\theta)$ the angle between the radius vector and the external normal at the point $w=\rho(\theta)e^{i\theta}$. Suppose $w=f(z)$ maps $|z|<1$ conformally onto the interior of C such that $f(0)=0, f'(0)>0$, and let $\theta(\phi)=\arg f(e^{i\varphi})$. Then $\theta(\phi)$ and $\theta'(\phi)$ satisfy the integro-differential equation (*) $\log\theta'(\phi)=\log\cos\beta[\theta(\phi)]+(1/2\pi)\int_0^\pi\{\beta[\theta(\phi+t)]-\beta[\theta(\phi-t)]\}\cot(t/2)dt\equiv T[\theta(\phi)]$. Under suitable assumptions regarding $\beta(\theta)$ the author solves (*) by successive approximations. If $\theta_0(\phi)=\phi$, $\theta_n(\phi)=T[\theta_{n-1}(\phi)]$, explicit estimates are obtained for $|\theta_n-\theta|$ and $|\theta_n'-\theta'|$, which show the convergence to be *geometric*. The actual *numerical* solution of (*) is obtained by replacing (*) by an analogous vector equation (in p-dimensional space). This vector equation is solved by iteration and the difference between the nth iterate and $\theta_n'(\phi)$ is estimated. The numerical procedure is similar to the one employed by A. Ostrowski in connection with the Theodorsen-Garrick method (National Bureau of Standards, 1949). (Received July 25, 1951.)

459. Bent Fuglede and R. V. Kadison: *Determinant theory in finite factors.*

A determinant is defined in factors of type II_1 and is shown to have the usual desirable properties on the regular operators. Actually the "determinant" which is defined reduces, in the n-dimensional case, to the absolute value of the nth root of the actual determinant. With the aid of the theory developed, it is proved that the trace of an arbitrary operator in a factor of type II_1 lies in the convex hull of the spectrum of the operator. (Received May 2, 1951.)

460*t*. P. R. Garabedian: *Orthogonal harmonic polynomials.*

It is shown that the harmonic polynomials $\cos h\phi\int_0^\pi P_n'(z+i\rho\cos t)\cos ht\,dt$ $\cdot\sin h\phi\int_0^\pi P_n'(z+i\rho\cos t)\cos ht\,dt$ are orthogonal over all the prolate spheroids $z^2\mathrm{ch}^{-2}\alpha+\rho^2\mathrm{sh}^{-2}\alpha<1$ in the sense of the norm $\iiint f^2\rho\,d\rho\,d\phi\,dz$ for harmonic functions f. Here ρ, ϕ, z are cylindrical coordinates, and P_n is the Legendre polynomial of degree n. It follows that the same harmonic polynomials are orthogonal over the surface of each of these spheroids with respect to the weight function $|1-(z+i\rho)^2|^{1/2}$. A similar complete orthogonal system of harmonic polynomials is found for the oblate spheroids $z^2\mathrm{sh}^{-2}\alpha+\rho^2\,\mathrm{ch}^{-2}\alpha<1$, and analogous results are obtained with the Dirichlet integral as norm in both the prolate and the oblate cases. The kernels for the complete orthonormal systems thus defined are set up, and in terms of them expressions are given for the Green's functions of the Laplace and biharmonic equations in a spheroid. The results can be extended to the case of a shell region between two confocal spheroids. (Received July 13, 1951.)

461*t*. P. R. Garabedian and D. C. Spencer. *Complex boundary value problems.*

In a cell M of the $2k$-dimensional Euclidean space of the variables x_j, y_j, differential forms ϕ are discussed which are pure in the sense that they can be expressed in terms of the dz_j only, $z_j=x_j+iy_j$. For these pure forms, complex operators d and δ are introduced which are an analogue of Hodge's operators d and δ in k-dimensional space. A boundary value problem for the system of partial differential equations $\delta d\phi=0$ in M is posed, and the corresponding existence theorem is developed by a method which combines the Fredholm integral equation and orthogonal projection. The Green's and Neumann's forms G and N for the system $\delta d\phi=0$ are defined, and the kernel form K for forms satisfying the generalized Cauchy-Riemann equations $d\beta=\delta\beta=0$ is introduced. The fundamental identity $K=dG+\delta N$ is proved for forms of each order p, $1\leq p\leq k$. When $p=k$ the kernel form K reduces essentially to the Bergman kernel function in k complex variables. Thus the present theory yields a generalization of the methods of classical potential theory which applies, in particular, to the study of analytic functions of several complex variables. (Received July 13, 1951.)

462. B. R. Gelbaum and G. K. Kalisch: *Measure in semigroups.*

This investigation concerns conditions implying that a semigroup S with a cancellation law (no identity or commutativity, unless so stated) with bounded invariant measure is a group. (1) If the measure in S satisfies a weakened version of "shearing in $S\times S$ is measurability preserving" (see P. R. Halmos, *Measure theory*, New York, 1950) then S is a group. A counterexample shows that the shearing condition may not be omitted entirely. (2) If S is locally compact with an invariant, bounded Borel measure, then S is a group. (3) In this case, if S is commutative or satisfies the above shearing condition, its topology may be weakened so that S becomes a separated compact topological group whose Haar measure coincides with the original measure.

(4) In the abelian case, if S has no topology relevant to the measure and no shearing condition is assumed, the measure on S can be extended to its quotient group with boundedness and invariance preserved. Here the hypothesis of boundedness may be replaced by the condition: if xF is measurable ($x \in S$, $F \subset S$), then so is F. Then the measure is again invariantly extensible to the quotient group. (Received January 11, 1951.)

463*t*. Casper Goffman: *A generalization of the Riemann integral.*

Every property $\mathfrak{P}$ of a set relative to an interval, which is such that (a) if $(S, I)\mathfrak{P}$ (the set S has the property $\mathfrak{P}$ relative to the interval I), and $T \subset S$, then $(T, I)\mathfrak{P}$, and (b) if $(S, I)\mathfrak{P}$, then $(CS, I)\mathfrak{P}$ (the complement CS of S does not have the property $\mathfrak{P}$ relative to I), yields upper and lower integrals—the upper and lower $\mathfrak{P}$-integrals—for every bounded real function defined on the interval $[0, 1]$. In particular, if $(S, I)\mathfrak{P}$ means that $S \cap I$ is empty, the $\mathfrak{P}$-integral is the Riemann integral, and if $(S, I)\mathfrak{P}$ means that $S \cap I$ may be covered by a sequence of intervals the sum of whose lengths is less than 1/2 the length of I, then the $\mathfrak{P}$-integral is the Lebesgue integral. It is shown that not all $\mathfrak{P}$-integrals are additive, but a condition is found which assures additivity. (Received July 12, 1951.)

464*t*. A. W. Goodman: *Inaccessible boundary points.*

By the use of infinite products an $f(z)$ is constructed which is analytic in $|z| < 1$ and maps that region conformally onto a region with inaccessible boundary points. (Received July 23, 1951.)

465*t*. P. R. Halmos: *Commutators of operators.*

(All italic capital letters in what follows denote bounded linear transformations of an infinite-dimensional complex Hilbert space into itself.) If $C = PQ - QP$, then C is called a *commutator*; if $C = P^*P - PP^*$, then C is called a *self-commutator*. Wintner has asked whether or not it is true that if C is a commutator, then the infimum of $|(Cx, x)|$, extended over all unit vectors x, is equal to zero. The main purpose of this note is to prove that the answer is no. The positive results from which this negative answer is easily deduced are the following ones. Theorem 1: if A is Hermitian, then A is the sum of two self-commutators Theorem 2: if A is Hermitian, then there exists an Hermitian B such that $A + iB$ is a commutator. As easy consequences of Theorems 1 and 2 one obtains the assertions (i) every (not necessarily Hermitian) A is the sum of four commutators, and (ii) every additive functional (trace), that vanishes on all commutators, vanishes identically. (Received May 31, 1951.)

466*t*. Einar Hille: *A note on Cauchy's problem.*

A Cauchy problem is formulated for the functional equation $y'(t) = U[y(t)]$ or, more generally, $y^{(n)}(t) = U^n[y(t)]$, where U is a linear operator on a B-space Y to itself. A solution defined for $t > 0$ such that $\limsup_{t \to \infty} t^{-1} \log \|y(t)\| < \infty$ is of normal type. Such solutions are uniquely determined by their initial values provided U is closed, the resolvent $R(\lambda; U)$ exists for $\lambda > \lambda_0$, and $\lambda \|R(\lambda; U)\|$ stays bounded when $\lambda \to \infty$. If U generates a semi-group $\{T(t)\}$ of linear bounded operators, then $T(t)[y_0]$ gives the only solution of normal type when $y_0 \in D[U]$, the domain of U. Conversely, if U satisfies the uniqueness conditions and if the equation has a solution $y(t) = y(t; y_0)$ for $y_0 \in D[U]$ and $\|y(t; y_0)\| \leqq Me^{\alpha t}\|y_0\|$, then U generates a semi-group $\{T(t)\}$ and $y(t; y_0) = T(t)[y_0]$. Applications are given to Cauchy's problem for linear partial differential equations. (Received August 8, 1951.)

467t. Einar Hille: *Behavior of solutions of linear second order differential equations.*

A study is made of the equation $w''=\lambda F(x)w$ where $F(x)$ is positive and continuous for $0\leqq x<\infty$, $xF(x)\notin L(0, \infty)$, and $\lambda=\mu+\nu i$ is a complex parameter with $\nu\neq 0$ if $\mu<0$ It is shown that certain fundamental solutions describe spirals in the complex plane as $x\rightarrow\infty$ and properties of monotony and convexity of absolute value and argument are determined as well as integrability properties over $(0, \infty)$. The main problem is the determination of sufficient conditions for $\mu\leqq 0$ under which the subdominant tends to zero when $x\rightarrow\infty$ and $F(x)$ times this solution is in $L(0, \infty)$. Extremal properties for the rate of growth of solutions of this class of differential equations are indicated. (Received June 22, 1951.)

468t. Einar Hille: *On the generation of semi-groups and the theory of conjugate functions.*

A new proof, based on elementary identities satisfied by the resolvent, is given of the theorem according to which a linear closed operator U of domain dense in the complex B-space X such that $\lambda R(\lambda; U)$ is a contraction operator for $\lambda>0$, is the infinitesimal generator of a semi-group $T(\xi)$ given by the strong limit of $[(n/\xi)R(n/\xi; U)]^n$. In an application to conjugate functions it is shown that the derivative of the conjugate satisfies the indicated conditions if $X=L_p(-\infty, \infty)$, $1<p<\infty$, and is the infinitesimal generator of the semi-group of Poisson transforms for the upper half-plane. Only for the case $p=2$ had the author previously published a proof based on the spectral theorem. (Received June 22, 1951.)

469. W. H. Ito: *On conjugate functions.*

The paper is concerned with an extension of the classical theorem of M. Riesz on the pth means of conjugate functions in the unit circle (Math. Zeit. vol. 27 (1927)), to conjugate functions harmonic in more general regions. 1. Let C be a simple, closed, Jordan curve with a continuously turning tangent, containing $z=0$ in its interior, $I(C)$, and let $F(z)=u(z)+iv(z)$ be analytic in $I(C)$, satisfy $v(0)=0$, and be continuous on $I(C)+C$. Then $[\int_C|v|^p ds]^{1/p}\leqq A_p(C)[\int_C|u|^p ds]^{1/p}$, $1<p<\infty$, where $A_p(C)$ depends on only p and C. 2. If, instead of the continuously turning tangent condition, we substitute the hypothesis that C be convex, the above inequality also holds in the range $2\leqq p<\infty$. We can relax the assumption of continuity of $F(z)$ in favor of a less restrictive hypothesis in both theorems 1 and 2. A lemma on conjugate functions on the unit circle due to A. Zygmund (Fund. Math. vol. 13 (1929) Lemma α) is extended to the case of the boundary values of functions conjugate within a convex contour. (Received July 18, 1951.)

470t. Meyer Jerison: *On algebras associated with a compact group.* Preliminary report.

If G is a compact group and R is a Banach algebra, then the set of continuous mappings of G into R, with multiplication defined by convolution and the norm defined as the supremum of the norms of the images under the mapping, is a Banach algebra which will be denoted by $C(G, R)$. It is clear that without some restriction on G or R, neither G nor R will be determined by the algebra $C(G, R)$. However, the following theorem can be proved: If $C(G, R)$ is (isometric and) isomorphic to $C(G', R')$, G and G' are compact abelian groups, and the only idempotents of R and R' are their respective identities, then G is isomorphic to G' and R is isomorphic to R'. This result is obtained by proving first that if $R=R'$ is the field of complex numbers, then G and G' are isomorphic. In the latter theorem, it is not necessary to assume that the groups are abelian. (Received July 11, 1951.)

471. J. L. Kelley: *Banach spaces with the extension property.*

A Banach space B has the *extension property* if: if F is a bounded linear function on a subspace of a Banach space C, the values of F being in B, then F has an extension F' to all of C such that $\|F\|=\|F'\|$. *Theorem.* A Banach space with the extension property is equivalent to the space of continuous real-valued functions on an extremally disconnected compact Hausdorff space. The converse, as well as a weakened form of the above theorem, has been established by L. Nachbin (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 28–46) and by D. B. Goodner (ibid. vol. 69 (1950) pp. 89–108). (Received July 23, 1951.)

472. V. L. Klee: *Convex sets in linear spaces.* II.

This is a sequel to an earlier paper bearing the same title (Duke Math. J. vol. 18 (1951) pp. 443–466). It is demonstrated that for a convex subset X of an arbitrary linear system L, the basic questions concerning polygonal connectedness of $L\sim X$ have the same answers as in the case of a two-dimensional L. Some characterizations of hyperplanes are given. In answer to a question of Erdös, it is proved that no separable Banach space can be covered by fewer than c hyperplanes. Finally, it is proved that every non-reflexive separable Banach space contains a pair of disjoint bounded closed convex sets which cannot be separated by a hyperplane. This extends a result of Dieudonné and, when combined with a theorem of Tukey, provides a new characterization of reflexivity. (Received July 20, 1951.)

473*t*. A. J. Lohwater: *A uniqueness theorem for a class of harmonic functions.*

Let $u(r, \theta)$ be harmonic in $|z|<1$, $z=re^{i\theta}$, and let the integral $\int_0^{2\pi}|u(r, \theta)|d\theta$ be bounded independently of r. It is proved that, if $u^*(\theta)=\lim_{r\to 1} u(r, \theta)=0$ for almost all θ and if $u^*(\theta)$ is infinite only for $\theta=\theta_k$, where $0<\theta_k<\theta_{k+1}<2\pi$ and $\lim_{k\to\infty} \theta_k=\theta^*<2\pi$, then there exist constants c^* and c_k, $c_k\neq 0$ $(k=1, 2, \cdots)$ with $\sum|c_k|<\infty$, such that $u(r, \theta)\equiv c^*K(r, \theta-\theta^*)+\sum_{k=1}^{\infty} c_kK(r, \theta-\theta_k)$, where $K(r, \theta-\alpha)$ is the Poisson kernel $(1-r^2)/[1+r^2-2r\cos(\theta-\alpha)]$. If $u^*(\theta)=0$ for almost all θ, and if $u^*(\theta)$, wherever else it may exist, is finite, then $u(r, \theta)$ is identically zero in $|z|<1$. (Received July 18, 1951.)

474*t*. A. J. Lohwater: *On the Schwarz reflection principle.*

Let $w=f(z)$ be meromorphic in $|z|<1$ and let $|f(re^{i\theta})|$, $z=re^{i\theta}$, have radial limit 1 for almost all θ belonging to an arc A $[0\leqq a_1<\theta<a_2\leqq 2\pi]$ of $|z|=1$. It is shown that if P is a singular point of $f(z)$ on A, then, given an arbitrary point $e^{i\lambda}$ of $|w|=1$, either $e^{i\lambda}$ is in the range $R(P)$ of $f(z)$ at P, or else there exists a Jordan arc L in $|z|<1$ terminating at a point ζ of $|z|=1$ arbitrarily close to P, such that as $z\to\zeta$ along L, $f(z)\to e^{i\lambda}$ (that is, $e^{i\lambda}$ is an asymptotic value of $f(z)$ "near P"). If, furthermore, $f(z)$ has a finite number of zeros and poles in a neighborhood of the singular point P, the union of $R(P)$ and the set of asymptotic values of $f(z)$ "near P" is precisely one of the sets $|w|\leqq 1$, $|w|\geqq 1$, or the extended plane. These theorems extend results of Carathéodory [Comment. Math. Helv. vol. 19 (1947) p. 266], Nevanlinna [Annales Academiae Scientiarum Fennicae (A) vol. 32 (1929) no. 7, p. 28], and Seidel [Trans.

Amer. Math. Soc. vol. 36 (1934) p. 208]. (Received July 18, 1951.)

475. G. R. MacLane: *Approximation by the derivatives of an entire function.*

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n/n!$ in $|z|<R$. Then the family $\{f^{(m)}(z)\}$ is normal in some neighborhood of $z=0$, and all its limit functions are holomorphic there (that is, none is the constant ∞) if and only if $|a_n|\leqq A$ for some constant A and all n. If this condition is satisfied, then $f(z)$ is entire and the family of derivatives is normal in the z-plane. If there are exactly p distinct limit functions, then $f(z)=g(z)+\sum b_n z^n/n!$, where $b_n\rightarrow 0$ (as $n\rightarrow\infty$) and $g(z)=\sum_{n=0}^{p-1} A_n \exp(w^n z)$, with $w=\exp(2\pi i/p)$ and g.c.d. $\{p, 1 \operatorname{sgn} |A_1|, 2 \operatorname{sgn} |A_2|, \cdots, (p-1) \operatorname{sgn} |A_{p-1}|\}=1$. Now $|a_n|\leqq A$ implies $|f(z)|=O(\exp|z|)$, but the reverse implication is false. A result in the opposite direction is: there exists an entire function $F(z)$ such that (1) $|F(z)|=O(\exp(1+\epsilon)|z|)$, for any $\epsilon>0$, and (2) if D is any simply-connected domain of the z-plane and $q(z)$ is any function holomorphic in D, then there exists a subsequence, n_k, of the positive integers such that $F^{(n_k)}(z)\rightarrow q(z)$ uniformly in any closed subset of D. (Received July 20, 1951.)

476. Imanuel Marx and George Piranian: *On the characterization of Lipschitz functions.*

Let $f(t)$ be a continuous function of t. The function $\alpha[f;t]=\liminf_{h\to 0}\log|f(t+h)-f(t)|/\log|h|$ may be called the Lipschitz function of $f(t)$. For any constant α, $0<\alpha<1$, W. S. Loud [Proceedings of the American Mathematical Society vol. 2 (1951)] has constructed a function $f(t)$ which has α as Lipschitz function in a sense somewhat stricter than that of the statement above. The authors show that if $\alpha(t)$ is a Lipschitz function, it is the limit inferior of a sequence of continuous functions of t. Conversely, if $0\leqq\alpha(t)\leqq 1$ and $\alpha(t)$ is the limit inferior of a sequence of continuous functions, there exists a function $f(t)$ such that $\alpha[f;t]=\alpha(t)$. (Received July 20, 1951.)

477*t*. T. S. Motzkin and J. L. Walsh: *On the derivative of a polynomial and the Chebyshev polynomial.*

If $\mu(z)$ is positive on the point set E: $(z_1, z_2, \cdots, z_n)$, the problem of determining the polynomial $T_{n-1}(z)\equiv z^{n-1}+A_1 z^{n-2}+\cdots+A_{n-1}$ which minimizes $\max[\mu(z)|T_{n-1}(z)|, z \text{ on } E]$ is essentially equivalent to the problem of determining the polynomial $t_{n-2}(z)$ which for given $f(z)$ minimizes $\max[\mu(z)|f(z)-t_{n-2}(z)|, z \text{ on } E]$. Let $f_1(z)\equiv B_0 z^{n-1}+B_1 z^{n-2}+\cdots+B_{n-1}$ be the polynomial of degree $n-1$ which coincides with $f(z)$ on E. Suppose $B_0\neq 0$. *The value of* $\arg\{[f(z)-t_{n-2}(z)]/B_0\}=\arg[T_{n-1}(z)]$ *in each point of E is independent of* $\mu(z)$ *and* $f(z)$. *The origin lies in the convex hull of the points* $[f(z_i)-t_{n-2}(z_i)]$. *All zeros of the polynomial* $f_1(z)-t_{n-2}(z)$ *lie in the convex hull of E.* Previous related work is due to Fejér, Fekete, and von Neumann. (Received July 23, 1951.)

478*t*. D. J. Newman: *A new proof of a conjecture of Pólya.*

Pólya has a conjectured that an entire function of zero type cannot be bounded over the integers unless it is a constant. A proof of this statement is given in Paley and Wiener, *Fourier transforms in the complex domain.* This proof, however, is somewhat involved and uses rather deep theorems on Fourier transforms. In this note the author gives a simpler proof of the conjecture of Pólya without any recourse to

Fourier transforms. The author uses Wigert's theorem which states that if $f(z)$ is an entire function of zero type then the function $g(z)=\sum_{n=0}^{\infty} f(n)z^n$, convergent for $|z|<1$, is an entire function of $1/(1-z)$; he also uses the fact that $\sum_{n=-\infty}^{-1} f(n)z^n$ represents the analytic continuation of $g(z)$ to $|z|>1$. These two facts imply that $g(z)(1-|z|)$ is bounded over the entire complex z-plane, if $f(n)$ is to be bounded. It is then shown that, since the only singularity of $g(z)$ is at $z=1$, and since $g(z)(1-|z|)$ is bounded, it follows that $g(z)=a+c/(1-z)$. From this it follows, by comparing coefficients, that $f(n)=c$, or that $f(z)-c$ is an entire function of zero type which vanishes at the integers. Such a function, however, must be identically zero, and the theorem is proved. (Received June 1, 1951.)

479t. D. J. Newman: *On series of the form $\sum \pm x^n/n!$.*

The author discusses the effect of introducing ± signs into various series. A typical question which is answered is, "Can ± signs be introduced into the series $\sum x^n/n!$ such that the resulting function tends to 0 as $x\to\pm\infty$?" It is shown in fact that the only way in which $\sum \pm x^n/n!$ can tend to zero at $+\infty$ is for the signs to be alternating and this answers the previous question in the negative. The general tendency of the results seems to indicate that $\sum \pm x^n/n!$ is large at $+\infty$ for most choices of the sequence of ± signs, but it is shown however that almost all of the series involved are $o(e^x)$ as $x\to+\infty$. The major result of the paper is the following: The only time $\sum \pm x^n/n!$ will be $O(e^{(1-\epsilon)x})$ is when the ± signs involved are periodic after a while, that is after a finite number of exceptional signs, also the period must be not greater than $2\pi/\cos^{-1}(1-\epsilon)$. Series directly related to the exponential such as $\sum x^{an}/(an)!$ are also considered, it is shown that for $a>2$ it can never be made into a bounded function over the positive axis by an introduction of ± signs. (Received June 1, 1951.)

480t. H. W. Oliver: *An existence theorem for nth Peano differentials.*

A real-valued function of two variables $f(x, y)$ is said to have an nth Peano differential at (x, y), $n=1, 2, \cdots$, if there exist numbers $f_{pq}(x, y)$, $1\leqq p+q\leqq n$, such that (1) $f(x+h, y+k)=f(x, y)+\sum_{1\leqq p+q\leqq n} h^pk^q/p!q!f_{pq}(x, y)+o(\rho^n)$, $\rho\to 0$, where $\rho=(h^2+k^2)^{1/2}$. The following theorem is proved: if f is measurable in a domain and if, for (x, y) in the measurable set E, $f(x+h, y+k)=f(x, y)+\sum_{1\leqq p+q\leqq n-1} h^pk^q/p!q!f_{pq}(x, y)+O(\rho^n)$, as $\rho\to 0$, then pp. in E, there exist numbers $f_{pq}(x, y)$, $p+q=n$, such that (1) holds. This is an extension, on the one hand, of a result of Rademacher and of Stepanoff for the first differentials; and, on the other, of a result due to Denjoy and to Marcinkiewicz and Zygmund for the nth Peano derivatives of a function of one variable. (Received July 20, 1951.)

481t. H. W. Oliver: *Borel derivative and exact Peano derivatives.*

A real-valued function $f(x)$ has an nth Peano derivative $f_n(x)$ at x if there exist numbers $f_k(x)$ such that $f(x+h)=f(x)+\sum_{k=1}^{n} h^k/k!f_k(x)+o(h^n)$, $h\to 0$; it has a (Borel) B_n-derivative $B_nf'(x)$ if $B_nf'(x)=\lim_{h\to 0} 1/h\int_0^h 1/t_1\int_0^{t_1}\cdots 1/t_{n-1}\int_0^{t_{n-1}}\{(f(x+t_n)-f(x))/t_n\}dt_n\cdots dt_1$. An exact nth Peano derivative, e.n.P.d., is one which exists at every point of an interval. It is shown that (f_n denotes an e.n.P.d.): (a) f_n has the property of Darboux; (b) mean-value theorems analogous to the ordinary mean-value theorem hold; (c) if f_n is bounded either above or below, it is the ordinary nth derivative at every point; (d) if $f_n\equiv 0$, f is a polynomial of degree $n-1$; (e) there is an everywhere dense, open set where f_n is the ordinary nth derivative; (f) the set $\{x;\ \alpha<f_n(x)<\beta\}$ is, for all $\alpha<\beta$, either void or of positive measure; (g) a monotone function has an ordinary first derivative wherever it has a B_n-derivative;

(h) under proper restrictions on f, for example, continuity, the $(n+1)$th Peano derivative of the nth integral of f exists if and only if $B_nf'(x)$ exists. (Received July 20, 1951.)

482t. B. J. Pettis: *On a vector space construction by Hausdorff.*

In 1932 Hausdorff constructed in any Banach space a second category linear subspace that is not a Banach space. His construction method can be used for a variety of *Gegenbeispiel* purposes. For example, in n-dimensional real vector space there exists an additive proper subgroup that is closed under multiplication by rational scalars, has vacuous interior, and is second category, everywhere dense, non-Baire, and non-measurable, and has both it and its complement having for measurable subsets only sets of measure zero and for Baire subsets only sets of the first category. Another application shows that in any second category linear topological space there exists a continuum of pairwise disjoint everywhere dense linear manifolds. These examples include the characteristics of examples due to Hamel, Kodaira, Halperin, Tukey, and Klee. They also show that in some recent theorems in topological groups certain hypotheses can not be dropped. (Received July 12, 1951.)

483. P. E. Pfeiffer: *Equivalence of measures on infinite product spaces.*

The measures considered are general probability measures on infinite product spaces. Necessary and sufficient conditions for equivalence are developed in terms of certain quasi-equivalence relations on suitable subclasses of the measurable sets. The class F (known to be a sigma-algebra of sets) is defined to be the class of those sets which are either $n+1, n+2, \cdots$-cylinder sets for each positive integer n or the empty set (for notation, see Halmos: *Measure theory*). A relation between certain conditional probabilities plus equivalence on F is shown to be necessary and sufficient for equivalence. Examples are given to exhibit the roles of these conditions. Comparison is made with (1) a theorem by Y. Kawada (Math. Japonicae vol. 1 pp. 170–177, Theorem 7) and (2) a theorem of S. Kakutani (Ann. of Math. vol. 49, p. 218). With the aid of Theorem 3 of Kawada's paper, new proofs of (2) and a modified form of (1) are given. Methods of the paper involve use of standard theorems of measure theory, Kawada's "projection" of measures, properties of the class F, and properties of conditional probabilities. (Received July 9, 1951).

484. Deborah Piranian and George Piranian: *Uniformly accessible Jordan curves with large sets of relative harmonic measure zero.*

Lohwater and Seidel [Duke Math. J. vol. 15 (1948) pp. 137–143] have constructed a Jordan region R whose boundary passes through a collinear set E of positive Lebesgue measure in such a way that E has harmonic measure zero relative to R. Also, Lohwater and G Piranian [Michigan Mathematical Contributions vol. 1 (1951) pp. 1–4] have exhibited a Jordan region R whose boundary passes through a set which has positive two-dimensional Lebesgue measure and has harmonic measure zero relative to R. The present authors simplify the previous constructions in such a way that the additional condition indicated in the title is satisfied. The proofs no longer depend on Lavrentieff's theorem on finite accessibility [Rec. Math. (Mat. Sbornik) vol. 36 (1929) pp. 112–115], but are based on the principle of Montel and Lavrentieff [C R. Acad. Sci. Paris vol. 184 (1927) pp. 1407–1409]. (Received July 12, 1951.)

485*t*. Everett Pitcher: *The projection property of quadratic functions.*

The following abstract approach to characteristic roots is developed. Let (A) $B(\eta, \phi, \lambda)$ define a real-valued, symmetric additive, homogeneous function of $\eta, \phi \in Y$, which is differentiable in λ for $0 \leqq \lambda < b$. Set $I(\eta, \lambda) = B(\eta, \eta, \lambda)$. Assume (B), (C), (D) of Bull. Amer. Math. Soc. Abstract 55-1-24, where the *index property* (that the index of $I(\cdot, \lambda)$ be finite and equal to the number of characteristic roots less than λ) and the *minimum property* are defined. The *projection property* is the property that with any interval $0 \leqq \lambda \leqq e < b$ there is a family of projections π_λ of Y onto a subspace Z of finite dimension such that (i) if $I(\eta, \lambda) < 0$ then $I(\pi_\lambda \eta, \lambda) < 0$ and (ii) if $\pi_\lambda \zeta = 0$ then $B(\pi_\lambda \eta, \zeta, \lambda) = 0$. *The projection property is equivalent to the index and minimum properties.* For quadratic functions of the calculus of variations, projections are readily defined by use of broken extremals in the manner which has been developed by Morse (see his exposition in *The calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications, vol. 18). (Received July 23, 1951.)

486. F. J. Polansky: *On the conformal mapping of variable regions.*

Let D and E denote two bounded simply-connected domains which contain the origin in the complex w plane. Let $f(z)$ and $g(z)$ be the functions which map D and E conformally onto the domain $|z| < 1$ in the complex z plane in such a way that the origins correspond and $f'(0)$ and $g'(0)$ are positive. A "distance" e between the boundaries is defined, and we prove that, uniformly in $0 < r < 1$, $\int_{C_r} |f(t) - g(t)| \, |dt| \leqq K(A_1^{1/2} + A_2^{1/2}) e^{1/8}$, $\int_{C_r} |f'(t) - g'(t)| \, |dt| \leqq \phi(e, \delta)$, where C_r is the circle $|t| = r < 1$. Here K is an absolute constant, A_1 and A_2 are the areas of the regions. For the second inequality the boundaries are presumed to be rectifiable Jordan curves of length l_1 and l_2 and $\delta = |l_1 - l_2|$. The function ϕ depends on e, δ and certain other parameters which are directly characterized by geometric properties of the curves. Furthermore $\phi \rightarrow 0$ as $e, \delta \rightarrow 0$. (Received July 23, 1951.)

487. Gustave Rabson: *Fejer's theorem on $SU(2)$.*

The characters $\chi_n(x)$ of the special unitary group $SU(2)$ of two by two matrices are known explicitly. In terms of these the Dirichlet kernels, $D_n(x) = \sum_1^n k\chi_k(x)$, may be defined, and the Fejer p-kernels may be constructed from these using (C, p) summation. It may then be shown that (1) $\int |D_n(x)| dx$ is of order n; (2) $\int |F_n^1(x)| dx$ is of order $\log n$; (3) $\int |F_n^2(x)| dx$ is bounded; and (4) $F_n^3(x)$ is positive (this is an immediate consequence of a result due to Fejer in J. London Math. Soc. vol. 8 (1933) pp. 53–62). From (1) and (2) it follows from a result due to Lebesgue that there are continuous functions whose Fourier series diverge even in the sense of $(C, 1)$. From (3) it follows, as in the classical proof of Fejer's theorem, that the Fourier series of a continuous function will converge in the sense of $(C, 2)$. It may also be shown that the Fourier series of any function in L^2 will converge in the sense of $(C, 3)$ almost everywhere. From this and from an unpublished result due to J. Korevaar it follows that if $n_{k+1}/n_k > \lambda > 1$ then the sequence of n_kth partial sums of the Fourier series of a function in L^2 will converge to it almost everywhere. This result is due to Kolmogorov in the classical case. (Received July 23, 1951.)

488*t*. L. T. Ratner: *A note on semi-linear spaces.*

In the present note we begin by considering abstract semi-linear spaces (which differ from linear spaces in omission of one of the distributive laws: $(a+b)x \neq ax + bx$); later we introduce a topology or norm. Various hyperspaces of semi-linear spaces are themselves semi-linear spaces, and it is these hyperspaces which are the subject of primary interest in this study. Problems arising in connection with the study of multi-valued transformations in linear and semi-linear spaces are treated by hyperspace methods. Results are obtained on a "minimum hypothesis" basis. (Received July 26, 1951.)

489. M. O. Reade: *On certain conformal mappings of the exterior of the unit circle.* Preliminary report.

Let $w=f(z)=z+a_1/z+a_2/z^2+\cdots$ be regular and schlicht for $\infty>|z|>1$, and continuous for $\infty>|z|\geqq 1$; let the image of $|z|=1$ be a Jordan curve C. The principle results are the following ones. (i) If C is convex, then the image of each circle $|z|=r>1$ is convex, and (ii) if C is star-shaped, with respect to $w=0$, then the image of each circle $|z|=r>1$ is star-shaped, with respect to $w=0$. Integral representations of each type of mapping function are discussed. The derivations follow Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, Leipzig, Teubner, 1913. (Received July 23, 1951.)

490. P. C. Rosenbloom: *The distribution of values of entire functions.*

Let $\eta(w)$ be any entire function which satisfies a differential equation of the form $P\eta'+Q\eta=1$, where P and Q are polynomials in w. Then if $f(z)$ is any entire function, we can obtain very precise relations between the distribution of the zeros of $\eta(f(z))$ and the growth of $f(z)$. For example, if η is a polynomial with no repeated roots, then we obtain a very simple proof of Nevanlinna's second fundamental theorem. If $\eta(w)=\exp(w/2\pi i)-1$, then we obtain very precise results on the distribution of the points where the function $f(z)$ takes on integral values. (Received June 22, 1951.)

491. Walter Rudin: *Inversion of second order generalized differential operators.*

Let $Ly=y''+py'+qy$, where p and q are continuous on $[a, b]$. Suppose that the system $Ly=0$, $y(a)=y(b)=0$, is incompatible, and let $K(x, t)$ be the corresponding Green's function. The operator L is generalized as follows. If F is defined in (a, b), and $a<x<b$, let $y_h(t)$ be such that $Ly_h=0$, $y_h(x+h)=F(x+h)$, $y_h(x-h)=F(x-h)$. Put $\Delta_hF(x)=y_n(x)-F(x)$, and $\Lambda F(x)=\lim_{h\to 0}2\Delta_hF(x)/h^2$. $\Lambda^*F(x)$, $\Lambda_*F(x)$ are defined likewise, with lim sup and lim inf in place of lim. If $F''(x)$ exists, then $\Lambda F(x)=LF(x)$. The following theorem generalizes a result well known if $F\in C''$. Suppose (1) F is continuous and bounded on (a, b); (2) $\Lambda^*F(x)>-\infty$, $\Lambda_*F(x)<+\infty$ on (a, b), except possibly on countable sets E_1 and E_2; (3) $\limsup_{h\to 0}\Delta_hF(x)/h\geqq 0$ on E_1, $\liminf_{h\to 0}\Delta_hF(x)\leqq 0$ on E_2; (4) there exists a measurable function g such that $g(x)\leqq\Lambda^*F(x)$ and $\int_a^b(x-a)(b-x)|g(x)|dx<\infty$. Then $\Lambda F(x)$ exists p.p., and $F(x)=-\int_a^bK(x, t)\Lambda F(t)dt+y(x)$ for all x on (a, b), where $Ly=0$, and $y(a)=F(a+)$, $y(b)=F(b-)$. (The existence of $F(a+)$, $F(b-)$ is part of the conclusion.) The theorem can be applied to uniqueness problems for Sturm-Liouville series. (Received July 18, 1951.)

492. L. R. Sario: *Principal functions on Riemann surfaces.*

Let R be an arbitrary Riemann surface, closed or open, of finite or infinite genus. Consider a subregion D of R, relatively bounded by k. A functional L will be called a *normal linear operator* in D if it associates to every harmonic function v on k a harmonic function Lv in D satisfying the following conditions: (1) $Lv=v$ on k, (2) $\int_kd\overline{L}v=0$, (3) $L1\equiv 1$, (4) $Lv\geqq 0$ if $v\geqq 0$ on k, (5) $L(c_1v_1+c_2v_2)=c_1Lv_1+c_2Lv_2$. Consider on R

the analytic relative boundary l of a compact or noncompact subregion of R. In $S=R-l$ let s be a single-valued real function such that both branches permit a harmonic continuation across l. *If* $\int_{l^{+}+l^{-}}\bar{d}s=0$, *then there exists on* R *a single-valued real function* p, *harmonic on* l *and such that* $p-s=L(p-s)$ *in* S. *The function* $p=p(z; s, L)$ *is unique up to an additive constant and will be called the principal function for given* s, L *on* R. *If* $s\neq Ls$, *then* p *is not constant.* By selecting different initial functions s and different operators L the corresponding principal function $p(z; s, L)$ furnishes the solution to different mapping, existence, uniqueness, and boundary value problems (Received July 17, 1951.)

493*t*. J. T. Schwartz: *Elliptic differential equations.* I. *Compact manifolds.*

If t is a generic covariant tensor of order p on a real analytic manifold M, then a notion of ellipticity for linear systems of differential equations of the form $L(t)=t'$ can be defined. If the coefficient functions occurring in L are analytic, the equation $L(T)=0$ has meaning for an arbitrary current of covariant order p. It is shown that a current satisfying such an equation is an analytic function. If L is self-adjoint in the formal sense, and M is compact, it is shown that L is essentially self-adjoint as an operator in Hilbert space. The inverse $(L-i)^{-1}$ is compact. The proof involves a generalization to elliptic systems of F. John's recent construction of a fundamental solution. (Received August 21, 1951.)

494*t*. J. T. Schwartz: *Elliptic differential equations.* II. *Noncompact manifolds.*

In the case where the real analytic manifold M is not compact, the space of solutions of an elliptic system $L(t)=0$ is discussed from the point of view of compactness and reproducing-kernel properties. If L is a formally positive elliptic operator, then its boundary-value theory on a noncompact manifold is discussed, use being made of the theory of semi-bounded operators. This discussion applies to arbitrary boundary conditions which are related in a suitable formal way to the positivity of L. (Received August 21, 1951.)

495. Seymour Sherman: *On a conjecture of Gelfand and Neumark.*

In their abstract characterization of uniformly closed subalgebras of bounded operators on hilbert space as Banach * algebras R satisfying six axioms, Gelfand and Neumark [*On the imbedding of normed rings into the ring of operators in hilbert space*, Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp. 197–213] have conjectured that axiom 6, $x^{*}x+e$ possesses a two-sided inverse element for each $x\in R$, is redundant. If the self-adjoint elements in R are ordered by letting $x\geqq 0$ mean $x=\sum_{k=1}^{n} z_{k}^{*}z_{k}$, $z_{k}\in R$, then it follows that, in the presence of the first five axioms, axiom 6 is equivalent to the *monotonicity of the norm* in the sense that for each $x, y\in R$, $0\leqq x\leqq y$ implies $\|x\|\leqq\|y\|$. Thus the suspected axiom, couched in algebraic terms, is replaced by an equivalent axiom phrased in the concepts of partially-ordered, normed, linear spaces It is also shown that if the norm of a C^{*} algebra is monotone with respect to a partial order whose cone of non-negative elements contains the set of positive semi-definite operators of the algebra, then this cone is just the set of positive semi-definite operators. (Received February 13, 1951)

496. I. M. Singer: *Maximal abelian algebras of finite factors.*

Let M and M' be a finite factor and its commutator, respectively, for which $Tr_M(I) = Tr_{M'}(I)$. Let $I_{M,M'}$ be the anti-isomorphism of M onto M' determined by $Tg = I_{M,M'}(T) \cdot g$, $T \in M$, and depending upon a fixed choice of a uniformly distributed vector g [Murray and von Neumann, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 208–248]. For any maximal abelian subalgebra A of M, the multiplicity of the weakly closed algebra $\bar{A}$ generated by A and $I_{M,M'}(A)$ is an invariant of A under the group of automorphisms of M. Any multiplicity can occur if M is approximately finite. If A is isomorphic to the continuous functions on a compact Hausdorff space X, then the uniform closure of the algebra generated by A and $I_{M,M'}(A)$ is naturally isomorphic to the continuous functions on $X \times X$ while the complete Boolean algebra of projections in $\bar{A}$ is naturally isomorphic to a complete Boolean algebra of subsets $X \times X$. If $\bar{A}_1$ and $\bar{A}_2$ have multiplicity one, then a necessary and sufficient condition that an isomorphism Φ carrying A_1 onto A_2 can be extended to an automorphism of M is that Φ can be extended to an isomorphism of $\bar{A}_1$ onto $\bar{A}_2$. (Received July 23, 1951.)

497. M. L. Slater: *Some generalizations of a lemma of Cesari.*

In a paper entitled *On the absolute extremals for the integrals on parametric surfaces* (Bull. Amer. Math. Soc Abstract 57-1-26) Cesari used the following lemma: Let $f(t)$ be continuous and nonincreasing in $0 \leqq t \leqq a$, greater than 0 in $0 \leqq t < a$, and satisfy $K[f(t)]^2 \geqq \int_t^a f(t)dt$. Then $a \leqq 2Kf(0)$. The author gives the following generalizations: Let $K_{1,\phi}$ $(0, a)$ be the set of non-negative functions $f(t) \in L(0, a)$ which are >0 a.e. in $[0, a]$ and satisfy $\phi(f(t)) \geqq \int_t^a f(t)dt$ throughout the closed interval $[0, a]$. $K_{1,\phi}$ is nonvoid and there exists $f_* \in K_{1,\phi}$ such that $f_*(t) \leqq f(t)$ for all $f \in K_{1,\phi}$ and all $t \in [0, a]$ if $\phi(x)$ $(x \geqq 0)$ has the following properties: (1) $\phi(0) = 0$ and $\phi(x)$ is strictly increasing, (2) $\phi \in C'$, (3) $\lim_{x \to \infty} \phi(x)/x = \infty$, (4) $\phi'(x)/x \in L(0, b)$ for any $b > 0$. $f_*(t)$ will satisfy $\phi(f_*(t)) = \int_t^a f_*(t)dt$. If in particular $\phi(x) = Kx^\alpha$ $(\alpha > 1)$ then $f_*(t) = ((\alpha-1)/\alpha K)^{1/(\alpha-1)}(a-t)^{1/(\alpha-1)}$. Again, let $K_{2,\phi}$ $(0, a; 0, b)$ be the set of non-negative functions $f(t, u) \in L(0, a; 0, b)$ which are greater than 0 a.e. in $[0, a; 0, b]$ and satisfy $\phi(f(t, u)) \geqq \int_t^a \int_u^b f(t, u)dtdu$ throughout the closed rectangle $[0, a; 0, b]$. $K_{2,\phi}$ is nonvoid and there exists $f_* \in K_{2,\phi}$ such that $f_*(t, u) \leqq f(t, u)$ for all $f \in K_{2,\phi}$ and all $(t, u) \in [0, a; 0, b]$ if $\phi(x)$ satisfies (1), (2), and (3) above and $\phi'(x)$ is in one of the classes $O(x^\alpha)$ $(\alpha > 0)$, $O((\log 1/x)^{-\beta})$ $(\beta > 2)$, $O((\log 1/x)^{-2}(\log \log 1/x)^{-\gamma})$ $(\gamma > 2)$, $\cdots$, as $x \to 0$. $f_*(t, u)$ will satisfy $\phi(f_*(t, u)) = \int_t^a \int_u^b f_*(t, u)\,dtdu$. If in particular $\phi(x) = Kx^\alpha$ $(\alpha > 1)$, then $f_*(t, u) = (1/K)^{1/(\alpha-1)}((\alpha-1)/\alpha)^{2/(\alpha-1)}((a-t)(b-u))^{1/(\alpha-1)}$. (Received July 20, 1951.)

498. J. E. Thompson: *Some studies of conformal mapping on the boundary.*

This paper is concerned with the conformal mapping of infinite strip regions onto parallel strips. "Normal strip regions" are defined and the Ahlfors-Dufresnoy-Ferrand inequalities are modified and applied to obtain quite general distortion theorems concerning the mapping functions of such strips. A special type of normal strip region, called a quasi L strip, is introduced and more precise results are proved concerning its mapping function, generalizing some results of S. E. Warschawski. The paper concludes with a discussion of conditions under which the mapping function of Jordan regions satisfies a uniform Lip α condition on the boundary. Direct methods based on estimating the Dirichlet integral are used to obtain fairly general results of this type. More precise results are obtained by using the above-mentioned strip methods. For example, if $f(z)$ is the function which maps $|z| < 1$ conformally onto a Jordan region and if $0<\alpha\leqq 1$, conditions are given under which $f(z)$ is uniformly Lip $(\alpha-\epsilon)$ on $|z|=1$. Also for a certain class of curves a necessary and sufficient condition that $f(z)$ be uniformly Lip α on $|z|=1$ is given. (Received July 19. 1951.)

499*t*. Leonard Tornheim: *Normed fields over the complex field.*

A normed field F over the complex field C is a field containing C and having a norm satisfying $\|\alpha\|>0$ if $\alpha\neq 0$, $\|\alpha+\beta\|\leqq\|\alpha\|+\|\beta\|$, $\|\alpha\beta\|\leqq\|\alpha\|\,\|\beta\|$, and $\|c\alpha\|=|c|\,\|\alpha\|$ for any α, β in F and c in C. Then $F=C$, a result closely related to the Gelfand-Mazur theorem. A proof is presented based upon these simple facts. If $\alpha-c$ has an inverse for every complex c, then $\|1/(\alpha-c)\|$ attains a maximum M for c on a bounded closed nonvacuous set C_0 in the complex plane. Also for such an α, the sum S_n of $1/n(\alpha-c_0-ur)$, where u ranges over all nth roots of unity and $r<\|1/(\alpha-c_0)\|^{-1}$, has limit $1/(\alpha-c_0)$, whereas for c_0 a boundary point of C_0 and r properly chosen, $\|S_n\|$ is bounded away from M. This contradiction implies that $\alpha-c$ has no inverse for some c and, since F is a field, $\alpha-c=0$; hence $\alpha=c$. (Received July 30, 1951.)

500*t*. J. L. Walsh and D. M. Young: *On the accuracy of the numerical solution of the Dirichlet problem by finite differences.*

A solution of the difference equation analogue of Laplace's equation is discrete harmonic (d.h.). For d.h. functions the authors use analogues of harmonic measure and the Schwarz alternating process, as well as the explicit solution of the difference equation for the rectangle, to estimate errors for a d.h. function as an approximate solution of the Dirichlet problem for rectangular and other regions. The error is expressed in terms of either the derivatives or the modulus of continuity $\omega(\delta)$ on the boundary of the solution $u(x, y)$ of the Dirichlet problem. For example for the unit square the absolute value of the error in the closed region is bounded by the maximum of A and B where $A=2\omega[2^{1/2}(h^{2/7}+h)]+8Mh^{2/7}$ and $B=145\ Mh^{2/7}+24\omega(2h)+192\ Mh$ and where $2M$ is the oscillation of $u(x, y)$ on the boundary and h is the mesh size. (Received August 16, 1951.)

501*t*. H. C. Wang: *A theorem on fixed points under a group of homeomorphisms.*

In a recent paper, A. D. Wallace proved that if a cyclic group Z of homeomorphisms of a Peano continuum leaves fixed an end point, then Z must leave fixed another point, and in a later paper [Fund. Math. vol. 36], he asked the question whether this still holds if Z is assumed to be compact instead of cyclic. This note aims at answering his question by proving the following theorem: *Let E be an arcwise connected normal space, and G a group of homeomorphisms of E leaving fixed an end point p. G has no other fixed point if and only if, for each neighbourhood N of p, $G(N)=E$.* As a direct consequence of this theorem, it follows that the answer to the above question of Wallace is affirmative. (Received July 25, 1951.)

502*t*. J. G. Wendel: *Left centralizers and isomorphisms of group algebras.* Preliminary report.

A bounded linear operator A on a (real or complex) group algebra $L(G)$ is called a *left centralizer* in case it commutes with all left multiplications: $A(xy)=x(Ay)$, x, $y\in L(G)$. Using the Riesz-Kakutani theorem it is shown that to each such A there corresponds a unique regular (real or complex) bounded measure μ such that $Ax=x*\mu$, where $(x*\mu)(g)=\int x(gh^{-1})d\mu(h)$. Every right translation on $L(G)$ is an *isometric* left centralizer; it is shown that this property characterizes the right translations, up to scalar factors of unit modulus, and that the strong closure of the manifold spanned by the right translations in the space of all bounded operators on $L(G)$ coincides with the set of left centralizers. Now let T be an isomorphism of $L(G)$ upon a second group algebra $L(\Gamma)$, $\|T\| \leqq 1$. It is proved that G and Γ are bicontinuously isomorphic, and that T is induced by a point mapping followed by multiplication by a character. This generalizes a previous theorem (Pacific Journal of Mathematics vol. 1 (1951)) wherein the same conclusions were obtained for isometric T; but in the apparently more general case T turns out to be isometric after all. (Received July 18, 1951.)

503*t*. Bertram Yood: *Properties of linear transformations preserved under addition of a completely continuous transformation.*

In this paper a study is made of various properties of bounded linear transformations on a Banach space which are possessed by transformations of the form $I+U$ where I is the identity and U is completely continuous. These are the principal properties given by Riesz, Acta Math. vol. 41 (1918) and Schauder, Studia Math. vol. 2 (1930) as well as certain related properties. The classes of transformations possessing these properties are characterized. (Received July 20, 1951.)

504*t*. C. Y. Yu: *On some functions holomorphic in an infinite region.*

S. Mandelbrojt indicated the following proposition: If a function is holomorphic and bounded in a half-strip of the z-plane containing the half-axis ox as a part of its central line and if this function and a certain infinite sequence of its derivatives vanish at the origin, then it is identically zero. In the present paper, one considers a function $F(z)$ holomorphic in a region Δ of the z-plane defined by $x \geqq d$, $|y| \leqq g(x)$, where $-\infty < d < 0$ and where $g(x)$ is a certain positive continuous function tending to zero with $1/x$. In this case, if in Δ, $F(z)$ tends to zero rapidly enough and uniformly with respect to y as x tends to infinity, and if $F(z)$ and a certain infinite sequence of its derivatives vanish at the origin, then $F(z)$ is identically zero. In order to establish this proposition, one proves at first a lemma on some entire functions of zero order by means of a theorem of G. Valiron (Annales de la Faculté des Sciences de l'Universite de Toulouse vol. 27 (1913) pp. 117–257, §32) and then applies a result of Mandelbrojt on generalized quasi-analyticity (*Séries adhérentes, régularisations descuites, applications*, Paris, Gauthier-Villars, in press; for a particular case, see Ann. École Norm. vol. 63 (1946) p. 372). (Received May 7, 1951.)

APPLIED MATHEMATICS

505. W. F. Bauer: *Modified Sturm-Liouville problems.*

The eigenvalue problem $y''+(\lambda+p)y=0$, $a_1y(0)+a_2y'(0)+a_3y''(0)=0$, $b_1y(1)+b_2y'(1)+b_3y''(1)=0$ has the following orthogonality relation which can be conveniently expressed as an inner product: $[y_i, y_j] \equiv \int_0^1 y_iy_jdx-(a_3/a_2)y_i(0)y_j(0)+(b_3/b_2)y_i(1)y_j(1)=0$. Under the conditions $a_2a_3 \leqq 0$ and $b_2b_3 \geqq 0$ the eigenvalues are shown to be real and simple. With the aid of the inner product notation and the Green's function it is shown that under further conditions on the coefficients a_i and b_i and on $p(x)$ the problem has an infinite number of real, positive eigenvalues and a minimal property is established for them. An expansion theorem is then obtained for functions of class C^2 which satisfy the boundary conditions by establishing the

Parseval's relation and the uniform convergence of the series. The eigenvalue problem arises in the study of the transient temperatures of a solid which is radiating heat from its faces while one of its ends is in contact with a well-stirred liquid whose container is in itself radiating heat into the surrounding medium. Properties of the eigenvalues and an expansion theorem can also be obtained by the application of the Laplace transform to the boundary value problem. (Received July 16, 1951.)

506. Abraham Charnes and H. J. Greenberg: *Plastic collapse and linear programming.* Preliminary report.

The critical value of the loading parameter at which plastic collapse occurs in structures or continua for certain types of loading has been characterized by means of a maximum and a minimum principle by Greenberg, Prager, and Drucker. They proved also that maximizing (resp. minimizing) solutions automatically satisfy compatibility, hence correspond to collapse states. It is shown here, for structures, that the maximum problem can be reduced in various ways to an equivalent form—maximization of a linear functional subject to linear inequalities. Application of linear programming techniques yields: (1) a new minimum principle for each reduction to a programming problem, (2) a criterion distinguishing collapse states from noncollapse states, (3) systematic algebraic procedures for computation of collapse states. (Received July 23, 1951.)

507. C. L. Dolph: *On the transmission and reflection problem for electro-magnetic waves.*

If S is a closed surface separating two homogeneous media with different dielectric, conductivity, and permeability constants upon which an electro-magnetic wave is incident from the exterior, the total electric field is shown to be a system of three linear Fredholm-Stieltjes integral equations. This representation is obtained by means of the fundamental tensor solution of the vector wave equation (cf. for example, Schwinger and Levine, Comm. Pure and App. Math. Dec. 1950) and the technique is a generalization of that of Sternberg (Compositio Mathematica vol. 3 (1936)). If the permeabilities of the two media agree, the device of Hilbert (*Gründzuge einer allgemeinen Theorie der linear Integralgleichungen,* Teubner, 1912, p. 206) can be used to reduce the system to a single Fredholm integral equation of the second kind. Thus this problem is conceptually no more difficult than the corresponding problem for the scalar wave equation. (Received July 12, 1951.)

508. G. E. Forsythe, M. R. Hestenes, and J. B. Rosser: *Iterative methods for solving linear equations.*

Several iterative methods are given for solving the equations $Ax=b$, where A is a given matrix and b is a vector. These methods appear to be particularly adapted to high speed computing machines. They have the property that if there were no round-off error the solution would be obtained in at most n steps where n is the rank of A. In the singular case the least square solution is obtained. At each iteration the problem is started anew. Accordingly there is no accumulation of errors. In the hermitian case the method is based on the following result. If A, $B>0$ are hermitian matrices which commute, then the system $b, Ab, \cdots, A^nb$ may be replaced by a set of B-orthogonal vectors by the algorithm $z_0=b$, $z_1=z_0-a_0Az_0$, $z_{i+1}=b_iz_i-a_iAz_i+c_{i-1}z_{i-1}$. (Received July 23, 1951.)

509*t*. G. E. Forsythe and T. S. Motzkin: *An extension of Gauss's transformation for improving the condition of systems of linear equations.*

Let the real n-by-n matrix A, symmetric or not, have a non-negative semi-definite quadratic form of rank $n-d$ $(0 \leqq d < n)$. Let s be any n-rowed column-vector. Let $B=B(s)$ be the $(n+1)$-by-$(n+1)$ matrix formed by bordering A with an $(n+1)$th column As, an $(n+1)$th row s^TA, and an element $b_{n+1,n+1}=s^TAs$. (T denotes transposition.) Let $\{\lambda_i\}$, $\{\mu_i\}$ be the eigenvalues of A, B, respectively. In a lemma it is proved that, for all s, $0=\mu_0=\lambda_0=\mu_1=\lambda_1= \cdots =\lambda_d=\mu_d<\lambda_{d+1}\leqq\mu_{d+1}\leqq\lambda_{d+2}\leqq\mu_{d+2} \leqq \cdots \leqq\lambda_n\leqq\mu_n<\infty$, and conversely, given any $\{\mu_i\}$ separated by the $\{\lambda_i\}$ as above, that one can find an s for which the $\{\mu_i\}$ are the eigenvalues of B. A familiar condition number $P(A)$ is extended to these singular matrices by the definition $P(A) =\lambda_n/\lambda_{d+1}$. It is related to the speed of convergence of certain iterative processes for finding one solution of consistent systems of linear equations $Ax=b$. By the lemma we find that, by suitably choosing s, one can make $P(B)$ assume any value in the interval $\lambda_n/\lambda_{d+2}\leqq P(B)<\infty$, and no other value. Gauss (Letter to Gerling, 26 December 1823, *Werke*, vol. 9, pp. 278–281) proposed the special value $s=(1, 1, \cdots, 1)^T$ for speeding the relaxation solution of certain normal equations arising in surveying. (Received July 20, 1951.)

510*t*. Abolghassem Ghaffari: *On one-dimensional Brownian motion.*

In a problem of one-dimensional Brownian motion it is required to solve the functional equation (1) $f(x, s; y, t)=\int_V f(x, s; z, u)f(z, u; y, t)dz$, $s<u<t$, where f is the displacement-distribution function. By applying Fréchet's method (Proc. London Math. Soc. vol. 39 (1935) pp. 515–540) one is led to take for the solution of (1) the series (2) $f(x, s; y, t)=\sum_{n=0}^{\infty} A_n(x, s)B_n(y, t)$, where A_n and B_n form a complete biorthonormal set of functions over V. Using this method and taking for the region V the interval $(-\infty, +\infty)$ one finds that (3) $f(x, s; y, t)=\sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)\theta^n(s, t)$, where the functions $\phi_n(x)=H_n(x) \exp(-x^2)/(2^n n!\pi^{1/2})^{-1/2}$ form a complete orthonormal set of functions over $(-\infty, +\infty)$; $\theta(s, t)=a(s)/a(t)$ where $a(s)$ is a positive increasing function $\neq 0$, and $H_n(x)$ is the nth Hermite polynomial. It is shown that, for s, t fixed such that $s<t$ and x, y varying arbitrarily, the series (3) is absolutely and uniformly convergent and also that the solution (3) is square summable over the whole plane. It is further shown that the solution (3) satisfies the equation $f_{xx}+[2a(s)/a'(s)]f_s+(1-x^2)f=0$. It is proved that for $s=t$ and $x\neq y$, $f(x, s; y, s)=0$ and also for $s=t$ and $x=y$, $f(x, s; x, s)=\infty$. The author has shown that the solution (3) is not among those obtained by A. Kolmogoroff (Math. Ann. vol. 104 (1931) pp. 415–458). (Received July 11, 1951.)

511*t*. P. G. Hodge, Jr.: *The Brownian motion of coupled systems in neutral equilibrium.*

It is well known that colloidal particles suspended in a liquid will engage in random movements, known as *Brownian motion*. More recently, Barnes and Silverman [Review of Modern Physics vol. 6 (1934) pp. 162–192] considered the random deflections of measuring instruments from a position of stable equilibrium, due to the random impacts of the surrounding fluid molecules. In the present paper, the Brownian motion of systems in neutral equilibrium is considered. The kinetic energy of the system is assumed to be expressible in generalized coördinates x_i, in the form $K =(1/2)\sum a_{ij}\dot{x}_i\dot{x}_j+b_ix_i+c$, where a_{ij}, b_i, and c may be functions of the coordinates

x_i, but not of the velocities $\dot{x}_i$. If the functions a_{ij}, b_i, and c are subject to certain restrictions, and if the generalized coordinates x_i are initially orthogonal, the mean-square deviations can be computed as functions of time, even in the case of neutral equilibrium. The method used is a generalization of that of Infeld [*On the theory of Brownian motion*, Applied Mathematics Series, no. 4, University of Toronto Press, Toronto, 1940]. The effect of certain approximations which were not discussed by Infeld is here considered in some detail. (Received June 15, 1951.)

512*t*. Andrew Sobczyk: *Resistances of spheres, and of packages of spheres.*

The resistance R between contacts at ends of a diameter of a conducting sphere, corresponding to a central plane angle 2α, is shown to satisfy the inequality $(1/\alpha) \leqq (2\pi aR/K) \leqq (1.442/\alpha)$, where a is the radius, K the resistivity of the material of the sphere. This result is obtained by use of the device of inserting appropriate insulating and conducting sheets, of integration with spherical shells as elements, of the known solutions of Laplace's equation for the point and hemispherical contact cases, and of methods in Smythe, *Static and dynamic electricity*. It is pointed out that there is an error of approximation in a recent paper by A. Gemant on resistance of soils. With view to obtaining a theory of the conductivity of a bar of powdered material, the resistances of packages of spheres, for the various possible packings of spheres in space described in Hilbert and Cohn-Vossen, *Anschauliche Geometrie*, are determined as functions of the contact angle α. (Received July 26, 1951.)

513. M. L. Stein: *Gradient methods in the solution of systems of linear equations.*

Let $Ax=b$ be a system of linear equations. Let a star (*) denote "conjugate transpose." Let H be a positive hermitian matrix. The minimum of $f(x) = (b-Ax)^*H(b-Ax)$ yields a solution of $Ax=b$ should one exist and otherwise a best fit of b by Ax in the metric H. The following algorithm for minimizing $f(x)$ has been tested experimentally. Let x_i be known, $\eta_i = A^*H(b-Ax_i)$, $\gamma_i = \eta_i^*\eta_i/\eta_i^*A^*HA\eta_i$. Then $x_{i+1} = x_i + \beta_i\gamma_i\eta_i$ where β_i is a scalar such that $1/\beta_i + 1/\bar{\beta}_i \geqq 1+\delta$, $|\beta_i| \geqq \delta$, $0<\delta<1$. For $\beta_i \equiv 1$ this is the optimum gradient method. Tests show that $\beta_i \equiv \beta < 1$ gives faster convergence than $\beta_i \equiv \beta \geqq 1$. In sharp contrast to methods using $\beta \geqq 1$ those using $\beta < 1$ show an instability which yields large accelerations in convergence. This self-acceleration property is significant for the use of the method on computors having limited storage capacity. Byproducts of the method are estimates of bounds for eigenvalues of A. Tests in which the problem $Ax=b$ was transformed into an equivalent eigenvalue problem are also discussed. Modifications of the Hestenes-Karush optimum-α method (M. R. Hestenes and W. Karush, *A method of gradients for the calculation of the characteristic roots and vectors of a real symmetric matrix*, to appear in NBS Journal of Research) are compared. (Received July 20, 1951.)

514. H. F. Weinberger: *An error estimate for the method of Weinstein.*

The method of Weinstein (Mémorial des Sciences Mathématiques, no. 88, 1937) gives upper bounds for the eigenvalues λ_i of the projection of a completely continuous positive definite operator L into a linear subspace 2 of the Hilbert space $\mathfrak{H}$ in which L operates. The eigenvalues κ_i and eigenfunctions u_i of L in $\mathfrak{H}$ are assumed to be known. These upper bounds are the corresponding eigenvalues of the projection of L

into $\mathcal{H}\ominus\{p_1,\cdots,p_n\}$ where $p_1,\cdots,p_n$ are any vectors (constraints) in $\mathcal{H}\ominus\mathfrak{Q}$. It is shown that a uniform error estimate which can be made arbitrarily small is obtained by taking p_i as the projection into $\mathcal{H}\ominus\mathfrak{Q}$ of the eigenvector u_i. A fundamental inequality due to Aronszajn (Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) p. 476, Corollary I′) shows that the desired eigenvalues λ_i are related to the eigenvalues κ_i^n in $\mathcal{H}\ominus\{p_1,\cdots,p_n\}$ by the inequality $\kappa_i^n\geqq\lambda_i\geqq\kappa_i^n-\kappa_{n+1}$. Because L is completely continuous, the error estimate $\kappa_{n+1}\rightarrow 0$ as $n\rightarrow\infty$. Since both upper and lower bounds are obtained, it is no longer necessary to use the Rayleigh-Ritz method or its generalizations. (Received June 25, 1951.)

515. Alexander Weinstein: *On cracks in shafts under torsion and seven-dimensional electrostatics.*

Let x and ρ be the coordinates in the meridian plane of a shaft of revolution subjected to torsion. The problem of a flat circular crack perpendicular to the x-axis is solved by an explicit determination of two functions ϕ and ψ satisfying the equations $\rho^3\phi_x=\psi_\rho$ and $\rho^3\phi_\rho=-\psi_x$. By putting $\psi=y^4\Phi(x,\rho)$, the problem is reduced to the determination of the electrostatic potential Φ of a disc in a space of seven dimensions in terms of ellipsoidal coordinates. The functions ϕ, ψ, and Φ can also be expressed in terms of Bessel functions by integrals of the Sonine-Schafheitlein type. (Received May 14, 1951.)

516*t*. L. A. Zadeh: *On the theory of filtration of signals.*

Let U be the signal space (generally a Hilbert space). A network N is called an ideal filter if it is idempotent, that is, if it is equivalent to a tandem combination of two networks each of which is identical with N. Thus, operation on the signals with an ideal filter corresponds to the projection of U on some linear manifold $\mathfrak{M}_P$ along a complementary manifold $\mathfrak{M}_S$. $\mathfrak{M}_P$ and $\mathfrak{M}_S$ constitute the "pass-band" and "stop-band" of the ideal filter. Two ideal filters N and N' are called complementary if the pass-band of N is the stop-band of N' and vice-versa. In general, an ideal filter is not physically realizable inasmuch as its impulsive response $W(t,\xi)$ does not vanish for $t<\xi$. However, in most cases one can approximate to $W(t-\beta,\xi)$, where β is a sufficiently large constant, by a physical filter. Two (or more) sets of signals $\{u_1(t)\}$ and $\{u_2(t)\}$ are called disjoint if the manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ spanned by $\{u_1(t)\}$ and $\{u_2(t)\}$, respectively, are disjoint. Such signals can be transmitted simultaneously (multiplexed) and separated at the receiving end by two complementary ideal filters N_1 and N_1' of which N_1 has $\mathfrak{M}_1$ for the pass-band and $\mathfrak{M}_2$ for the stop-band. (Received July 18, 1951.)

Geometry

517*t*. D. O. Ellis: *Notes on abstract distance geometry.* III. *On self-congruences of metroids.*

This note deals with the relation between certain algebraic properties of a metroid and the self-congruences of the metroid. For the terminology see David Ellis, *Geometry in abstract distance spaces*, Publicationes Mathematicae Debrecen vol. 2 (1951) no. 1. The principal results are: 1. *A metroid with left cancellation* (*right cancellation*) *and a left zeroid* (*right zeroid*) *is metrically irreducible*; 2. *In a metroid with skew cancellation* ($x^*a=a^*y$ implies $x=y$) *every self-congruence is involutory*; 3. *If a metroid has a simply transitive group of motions every element of the form* x^*x *is idempotent*; 4. *In an associative metroid with unit the following are pairwise equivalent*: (a) *the group of motions is simply transitive*, (b) *the metroid has the property of free mobility*, (c) *all* *elements are nilpotent*, (d) *the metroid is (isomorphic to) a subgroup of the additive group of a Boolean ring*. (Received July 18, 1951.)

518. P. C. Hammer and Andrew Sobczyk: *Planar line families. II. Covering properties of outwardly simple line families.*

A family of lines in the plane is said to be outwardly simple if it simply covers all points in a region exterior to some circle, including points at infinity. The existence of instantaneous centers of rotation of such a family F is fully discussed. The accumulated positive area swept out in a bounded region by lines in the family is defined and various inequalities noted. Letting S_k be the set of points in the plane each of which lies on exactly k lines in an outwardly simple family F of lines, then we prove that the planar measure, $m(S_k)$, is zero if k is even or infinite. (Received July 6, 1951.)

519*t*. P. C. Hammer and Andrew Sobczyk: *Planar line families. III. Outwardly n-fold families.* Preliminary report.

A family of lines will be said to *cover the plane n-fold* if every point in the finite plane and every point on the line at infinity lies on precisely n lines. The line at infinity will not be included as a member of the line family. The following results are obtained. (1) No family of lines covers the plane simply (1-fold). (2) There exists a family of lines which covers the plane n-fold for every $n \geqq 2$—assuming the continuum hypothesis and the well ordering axiom. (3) There exists no family of lines which covers the plane continuously n-fold—for $n=1, 2, \cdots$. (4). For every $n \geqq 1$ there exist families of lines covering continuously n-fold the exterior of any circle. (Points at infinity are assumed to be outside any bounded region.) Such a line family is called an *outwardly n-fold family of lines*. (Received July 6, 1951.)

520*t*. W. R. Hutcherson. *Fifth order neighborhood of an involution of period thirteen.*

An imperfect point (W. R. Hutcherson, *Point non parfait et courbes invariables*, Bulletin de la Société Royale des Sciences de Liège no. 11 (1950) pp. 485–489; *A cyclic involution of period eleven*, Canadian Journal of Mathematics vol. 3 (1951) no. 2) on a certain surface is found to possess perfect points in the neighborhood of orders three and four for I_{11}, whereas for I_{13} it is discovered to be five and three respectively for the two invariant directions. (Received June 27, 1951.)

521*t*. I. J. Schoenberg: *A generalization of the classical isoperimetric inequality to closed curves in higher even-dimensional spaces.*

Let Γ: $x_i = x_i(t)$ $(i=1, \cdots, 2n)$ be a closed curve in E_{2n} where $x_i(t)$ are continuous functions of period 2π. We restrict ourselves to the class Ω of curves Γ which have the following property: There is no hyperplane in E_{2n} which is crossed more than $2n$ times by Γ. Curves Γ with this property are called *convex in E_{2n}*; for $n=1$ we obtain the ordinary plane convex curves. Convex curves in E_{2n} are known to be rectifiable. Let $\Gamma \in \Omega$ and let L be its length. Let $K=K(\Gamma)$ denote the least convex extension of Γ and let $V=V(K)$ be the $2n$-dimensional volume of $K(\Gamma)$. Then the following inequality holds: $L^{2n} \geqq (2\pi n)^n n!(2n)!V$, where we have the equality sign if and only if Γ is obtained by a similitude, rotation and perhaps a reflexion from the curve Γ_0 defined by the equations $x_1 = \cos t$, $x_2 = \sin t$, $x_3 = (1/2) \cos 2t$, $x_4 = (1/2) \sin 2t, \cdots,$ $x_{2n-1} = (1/n) \cos nt$, $x_{2n} = (1/n) \sin nt$. These are the curves which maximize $V(K)$ for a fixed value of their length L within the class of convex curves. (Received July 20, 1951.)

522. Andrew Sobczyk and P. C. Hammer: *Symmetrical types of convex regions.*

If a bounded convex region C has involutory symmetry in each of a set of complementary linear subspaces $L_1, \cdots, L_k$ of n-dimensional linear space L, then C is symmetric with respect to the origin (that is, it may be taken as the unit sphere for a Minkowski or Banach norm on L). Possible regions which have specified sections $C_1, \cdots, C_k$ in $L_1, \cdots, L_k$ are discussed. Convex regions are classified according to kinds of symmetry which they may possess; rotational and discrete symmetry are special cases. The notion of *convex transformation* is defined and studied. Classes of convex transformations, including affine transformations as subclass, are introduced; the existence and properties of completely unsymmetrical convex regions (group of symmetries in a corresponding class consists of only the identity) are investigated. (Received July 6, 1951.)

LOGIC AND FOUNDATIONS

523*t*. S. C. Kleene: *Permutability of inferences in Gentzen's calculi LK and LJ.*

After establishing the possible interchanges of logical inferences in Gentzen's calculi *LK* and *LJ*, the following theorem is proved. In a proof in *LK* or *LJ* without cut, one may assign the logical symbol occurrences L in the endsequent to classes $C_1, \cdots, C_k$, and then order the inferences belonging to those symbol occurrences (call them also L) so that those by symbols of C_1 are uppermost, those by C_2 next, and so on, provided the classification meets two restrictions: (1) If $L_1 \in C_i$, $L_2 \in C_j$, and L_2 contains L_1 within its scope, then $i \leqq j$. (2) If $L_1 \in C_i$, $L_2 \in C_j$, and L_1/L_2 is one of the following exceptional pairs of inferences, then $i \leqq j$. For *LK*: $(\forall\rightarrow$ or $\rightarrow\exists)/(\rightarrow\forall$ or $\exists\rightarrow)$. For *LJ*: $\forall\rightarrow/\rightarrow\forall$, $(\forall\rightarrow$ or $\rightarrow\exists)/\exists\rightarrow$, $(\supset\rightarrow$ or $\neg\rightarrow)/(\rightarrow\supset$ or $\rightarrow\neg)$, $(\supset\rightarrow$, $\rightarrow\vee$, $\neg\rightarrow$ or $\rightarrow\exists)/\vee\rightarrow$. Counterexamples are given to the theorem with (1), or any one of the exceptions under (2), omitted. (Received June 11, 1951.)

524*t*. J. C. C. McKinsey, A. C. Sugar, and Patrick Suppes: *A set of axioms for classical particle mechanics.*

A system of axioms is given for classical particle mechanics, based on five primitive notions: (1) P is the set of *particles*; (2) T is a set of real numbers measuring elapsed *times*; (3) for $p \in P$, $m(p)$ is a positive real number, measuring the *mass* of p; (4) for $p \in P$ and $t \in T$, $s(p, t)$ is a vector giving the *position* of p at time t; (5) for $p \in P$, $t \in T$, and n a positive integer, $f(p, t, n)$ is a vector giving the nth *force* acting on p at time t. The axioms, which are eight in number, include a form of Newton's Second Law, and otherwise consist chiefly of so-called "closure axioms," and axioms giving continuity properties. Theorems are established which provide grounds for considering this an adequate axiomatization of the usual particle mechanics; in particular, it is proved that every model can be embedded in a model which in addition satisfies Newton's third law (and the first law, as would be expected from less formal developments, is of course a theorem). It is also shown that the axioms are independent; and—some received opinions to the contrary notwithstanding—that m, s, and f are not definable in terms of the other notions, even if Newton's third law is added as an axiom. (Received July 9, 1951.)

STATISTICS AND PROBABILITY

525*t*. Abolghassem Ghaffari: *On a functional equation arising in the the theory of chain probabilities.*

The author considers one-dimensional Chapman's functional equation (C) $f(x, s; y, t) = \int_V f(x, s; z, u)f(z, u; y, t)dz$, $s<u<t$, where f denotes the probability density which is subjected to specific conditions such as: $f(x, s; y, t) \geqq 0$; $\int_V f(x, s; y, t)dy = 1$; $f(x, s; y, s) = 0$; $f(x, s; x, s) = \infty$. Taking for the region V the interval $(-\infty, +\infty)$ and trying for the solution of (C) a preassigned exponential expression $f(x, s; y, t) = A(s, t) \exp [-(a(s, t)x^2+2b(s, t)xy+c(s, t)y^2)]$, one finds that (1) $f(x, s; y, t) = g^{1/2}(t)[\pi(H(t)-H(s))]^{-1/2} \exp (xg^{1/2}(s)-yg^{1/2}(t))^2(H(t)-H(s))^{-1}$, where $g(s)$ is an arbitrarily chosen positive continuous function and $H(s)$ is an increasing function. The solution (1), satisfying the equation (C) and the supplementary conditions indicated above, is then the most general such solution and moreover contains the solution of L. Bachelier as a special case. (Received July 11, 1951.)

526*t*. H. P. Mulholland: *On the distribution of a convex even function of several independent rounding-off errors.*

The following theorem, in which $\phi(w)$ may be any continuous convex even function of the real vector $w = (w_1, w_2, \cdots, w_n)$ with $\phi(0) = 0$, generalizes Z. W. Birnbaum's results (*On random variables with comparable peakedness*, Ann. Math. Statist. vol. 19 (1948) pp. 76–81) for the case $\phi(w) \equiv |w_1+w_2+ \cdots +w_n|$. Let u_i $(i=1, 2, \cdots, n)$ be independent random variables with unimodal continuous distributions symmetrical about 0 and cumulative distribution functions (c.d.f.'s) $P_i(x)$, and similarly for v_i with c.d.f.'s $Q_i(x)$: let the distribution of u_i be more peaked about 0 than that of v_i, that is, $P_i(x) \geqq Q_i(x)$ $(x \geqq 0; i=1, 2, \cdots, n)$. Then the distribution of $\phi(u)$ is more peaked about 0 than that of $\phi(v)$. The theorem is applicable when u_i is a (composite) rounding-off error, that is, $P_i(x)$ is the c.d.f. of the l_i-fold convolution of the rectangular distribution on $(-1/2, 1/2)$, and $Q_i(x)$ is the normal c.d.f. with mean 0 and variance σ^2, if we take $\sigma \geqq \rho_l(l/12)^{1/2}$ where $l = \max l_i$, $\rho_1 = (6/\pi)^{1/2}$, and $1<\rho_l<\rho_2<1.0707$ $(l>2)$. This result is simpler and stronger than H. H. Goldstine and J. von Neumann's fundamental lemma (*Numerical inverting of matrices of high order*. II, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 188–202). (Received July 2, 1951.)

527. Murray Rosenblatt: *On a question of J. L. Doob.*

Let $X(t)$ be a time-homogeneous Markov process with a denumerable number of states. J. L. Doob has asked whether there are Markov chains $X(t)$ whose sample functions have discontinuities that are not well-ordered with probability one (Trans. Amer. Math. Soc. vol. 58 (1945) pp. 455–473). An example of such a Markov chain is given. The transition probabilities of the given process satisfy the "second system of differential equations" (forward equations) but do not satisfy the "first system of differential equations" (backward equations) (A. Kolmogorov, Math. Ann. vol. 104 (1931) pp. 415–458). (Received July 16, 1951.)

528. J. L. Snell: *Applications of martingale system theorems.*

A sequence of random variables $\{x_n\}$ with $E\{|x_n|\} < \infty$ is called a martingale if $E\{x_n | x_1 \cdots x_{n-1}\} = x_{n-1}$ with probability one for all n, a semimartingale if $E\{x_n | x_1 \cdots x_{n-1}\} \geqq x_{n-1}$ with probability one for all n. If $\{x_n\}$ is a martingale (semimartingale) the interpretation of x_n as a gambler's fortune playing a fair (favorable) game has suggested theorems, called system theorems. Such theorems have been proven by Doob and Halmos. In this paper these system theorems are generalized and their interpretation in terms of von Neuman's theory of games is given. The theorems are then applied to the study of convergence properties of the random variables of a martingale and semimartingale. It is proven that lim $x_n(\omega)$ exists and is finite for almost all ω in the set $\{\omega | \inf_k \sup_{n \geqq k} E\{|x_n| | x_1 \cdots x_2\} < \infty\}$ if $\{x_n\}$ is a semimartingale. Finally, the system theorems are applied to the problem of the existence of Bayes solutions in the theory of statistical decision functions. Results of Arrow Blackwell and Girshick are obtained from martingale theory. (Received June 29, 1951.)

TOPOLOGY

529*t*. Richard Arens: *Extension of mappings.*

Let A be a closed subset of a normal space X. Let f be a continuous mapping into a Banach space L. Then f can not always be continuously extended to X (even if L is a Hilbert space), although it always can when $f(A)$ is separable. Two separate necessary and sufficient conditions that f can be extended to X are (1) every neighborhood finite covering of A has a neighborhood finite refinement which can be extended neighborhood finitely to X, (2) every pseudo-metric on A has a refinement which can be extended to all of X. If an extension of f is possible it may be so chosen that the new values lie in the convex hull of the closure of $f(A)$, but when moreover A is a G_δ (or compact) the new values can be forced to lie in the convex hull of $f(A)$. However, in the absence of the G_δ condition it is sometimes necessary for mappings into the plane (but never, for mappings into the line) to have extended values outside of the convex hull of $f(A)$. (Received July 27, 1951.)

530*t*. R. H. Bing: *Convexifying continuous curves so as to obtain many unique segments.*

It is shown that each continuous curve M has a dense subset R and a convex metric $D(x, y)$ such that if p and q are two points of R there is only one arc in M from p to q whose length is $D(p, q)$. The method used is to get a decreasing sequence of core partitionings of M and define the convex metric $D(x, y)$ in terms of the number of elements of the various partitionings it takes to link x and y. (Received July 23, 1951.)

531*t*. C. E. Burgess: *Concerning continua and their complementary domains in the plane.*

A continuum K in the plane is said to be compactly connected if every pair of its points is contained in a compact subcontinuum of K. The following theorems are proved: (1) If the continuum M is the boundary of the sum of a finite number of its complementary domains and there is a sequence of mutually exclusive compactly connected continua in M converging to M, then either M is indecomposable or there is only one pair of indecomposable proper subcontinua of M whose sum is M. (2) If M is a continuum containing no domain and some domain intersects M and only a finite number of complementary domains of M, and there is a sequence of mutually exclusive continuous curves in M converging to M, then either M is indecomposable or it is the sum of two of its proper subcontinua of which one is indecomposable. (Received July 11, 1951.)

532. C. E. Burgess: *Concerning continua in the plane.* Preliminary report.

The following theorems are proved: (1) In order that for each three domains R_1, R_2, and R_3 intersecting the compact continuum M there should exist three complementary domains of M each intersecting each of the domains R_1, R_2, and R_3, it is necessary and sufficient that either M be the boundary of each of three of its complementary domains or there exist a sequence of distinct complementary domains of M converging to M. (2) If M is a compact decomposable continuum and for each two domains R_1 and R_2 intersecting M there exist three complementary domains of M each intersecting each of the domains R_1 and R_2, then there is a collection G consisting of less than five indecomposable continua such that M is the sum of the continua of G and such that, furthermore, if X and Y are two continua of G, then $X+Y$ is a continuum which is irreducible between some two points, and either there is a complementary domain of M having $X+Y$ as its boundary or there is a sequence α of distinct complementary domains of M converging to $X+Y$ such that each domain of α is a complementary domain of $X+Y$. (Received July 11, 1951.)

533. Eugenio Calabi and Beno Eckmann: *A class of simply connected, compact complex manifolds which are not algebraic.*

It is demonstrated that the product of any two odd-dimensional spheres $S^{2p+1} \times S^{2q+1} = M$ can be given a complex analytic structure. The second Betti number of M vanishes (except in the trivial case $p=q=0$), so that M does not admit any Kähler metric, hence M can not be an algebraic manifold (theorem of Lefschetz-Hodge). In the special case where $p=0$ a complex structure on $S^1 \times S^{2q+1}$ was first exhibited by Hopf (*Courant anniversary volume*, 1948, p. 168) and the variety is uniformizable, while in the general case p, $q \geqq 1$, the manifold M, being simply connected, can not be uniformized. The manifold M can be considered as a complex analytic fibre space over the product of two complex projective spaces of dimension p, q respectively, the fibre being a 2-torus with any specified conformal structure. (Received July 23, 1951.)

534. K. T. Chen: *Isotopy invariants of links.*

This is an improved result of a previous paper [Bull. Amer. Math. Soc. vol. 57 (1951) p. 143] of the author. By a δ-approximation L' of an arbitrary link L in the euclidean 3-space E, we mean a polygonal link L' in E such that there is a homeomorphism $\phi: L \rightarrow L'$ with $|\phi(x)-x| < \delta$ for any $x \in L$. It is proved that, for each $d \geqq 1$, there is a $\delta > 0$ such that any two δ-approximations L' and L'' of L have $G'/G'_d \cong G''/G''_d$, where G'_d and G''_d are the dth lower central commutator subgroups of the fundamental groups $G' = \pi(E-L')$ and $G'' = \pi(E-L'')$ respectively. Set $H_d = G'/G'_d$. Thus, for each d, H_d is well defined for L up to isomorphism. Furthermore, this group H_d is invariant under any isotopy of the link L, that is, a continuous family of mappings $h_t: L \rightarrow E$, $0 \leqq t \leqq 1$, each of which maps L topologically into E with $h_0(x) = x$, $x \in L$. If L is polygonal, then $H_d \cong G/G_d$, where $G = \pi(E-L)$. Through the groups H_d, one finds some numerical invariants which distinguish the members of a certain infinite series of links, each of which consists of only two curves with vanishing linking number. (Received July 9, 1951.)

535. Mary E. Estill: *Concerning a problem of Souslin.*

In order that there exist a linear nonseparable space not containing uncountably many mutually exclusive domains it is necessary and sufficient that there exist a locally connected nonseparable space not containing uncountably many mutually exclusive domains which satisfies the first three parts of R. L. Moore's axiom 1 such that, if A and B are two points, there exists a countable point set separating A and B. It is also a necessary and sufficient condition for the existance of a locally connected, nonseparable space not containing uncountably many mutually exclusive domains which satisfies the first three parts of R. L. Moore's axiom 1 such that, if A and B are two points, there exists a separable point set separating A and B. (Received July 23, 1951.)

536*t*. E. E. Floyd: *On periodic maps and the Euler characteristics of associated spaces.*

Let p be a fixed prime. If A is a locally compact Hausdorff space, denote by $H_i(A)$ the Čech homology group of compact i-cycles of A modulo cycles bounding on compact subsets of A, with the integers mod p as coefficients. Denote by dH_i the minimum number of generators of $H_i(A)$, and by $\chi(A)$ the sum $\sum(-1)^i dH_i$, in case these numbers are defined and finite. Let now T be a periodic map of period p on a locally compact finite-dimensional Hausdorff space X. Denote by L the fixed point set of T, and by Y the orbit decomposition space of T. Then if $\chi(X)$ exists, we prove that $\chi(X)+(p-1)\chi(L)=p\chi(Y)$, where all terms are well-defined. Various consequences are obtained. An immediate one is that $\chi(L)=\chi(X)$ mod p. (Received July 23, 1951.)

537. E. E. Floyd: *On related periodic maps.*

Consider two periodic maps S and T of period p on a compact n-manifold X. The problem studied is as follows. Suppose S and T are related in some simple topological fashion. What, then, are the implied relationships between the fixed point sets of S and T? Between their orbit decompositions spaces? For example, if S and T are homotopic then, when properly interpreted and p is prime, the Euler characteristics of the fixed point sets are equal. Results of another type concern the case in which S is a fixed map of period q^a, q prime. Then there exists $\epsilon>0$ such that if T is of the same period as S, and $\rho(S, T)<\epsilon$, then their fixed point sets have isomorphic mod q homology groups. (Received July 18, 1951.)

538*t*. V. G. Gorciu: *Continuous-convergence in a non-metric uniform space.* Preliminary report.

A Hausdorff, nonmetric, uniform space is defined by assigning to each point an indexed set of neighborhoods. The fundamental set of indices $\Lambda(\lambda, \mu, \cdots; >)$ is directed, noncountable and does not possess a last element. *Definition.* The directed set of functions $\{f_\lambda(x)\}$ is said to be continuous-convergent at a point $a\in\overline{A}$ if for every directed set of points $\{x_\lambda\}\rightarrow a$, the directed set $\{f_\lambda(x_\lambda)\}$ converges; $x_\lambda\in A$. If $\Lambda_1(\lambda_1, \mu_1, \cdots; >)$ and $\Lambda_2[\lambda_2, \mu_2, \cdots; >]$ are two sets order-isomorphic with Λ, a new directed set $\Lambda_3[\Lambda_1\cup\lambda_2; >]$ is defined by introducing following partial ordering: $\lambda_2>\lambda_1$, $\mu_2>\mu_1, \cdots; \lambda>\mu\leftrightarrow\lambda_1>\mu_1$, $\lambda_2>\mu_2$, $\lambda_1>\mu_2$, $\lambda_2>\mu_1$. In this new set, both Λ_1 and Λ_2 are cofinal subsets. By using this new set, it is easily proved that, in the case of continuous convergence, the limit of $\{f_\lambda(x_\lambda)\}$ does not depend on the directed set $\{x_\lambda\}\rightarrow a$ that is chosen. A necessary and sufficient condition that $\{f_\lambda(x_\lambda)\}$ be continuous-convergent, at a point $a\in\overline{A}$, is established. (Received July 18, 1951.)

539. O. G. Harrold: *Almost locally polyhedral spheres.*

Recently examples of topological spheres in space have been given illustrating the fact that a set need be only slightly pathological for the classical Schoenflies relation to fail. In particular, if a sphere is sufficiently twisted in the vicinity of two points, the fundamental group of a complementary domain may be nontrivial. A supplementary result is derived. Set $K \subset R^n$, Euclidean space, is called locally polyhedral at $p \in R^n$ if and only if for some subdivision of R^n into convex cells: (1) only a finite number of cells meet any spherical domain; (2) there is a neighborhood U of p in R^n such that $U \cap K$ is a union of cells of the subdivision or their lower dimensional faces. If K contains no points where this property fails, call K locally polyhedral. If a topological sphere in R^3 is locally polyhedral save for one (two) point(s), then both complementary domains are (at least one complementary domain is) simply connected. (Received July 23, 1951.)

540*t*. S. T. Hu: *Cohomology theory in topological groups.*

For a given topological group Q and a given topological abelian group G, three kinds of cohomology groups of Q over G are defined in this paper, namely, the pth cohomology group $H^p(Q, G)$, the pth cohomology group $H^p(Q, G)$ with empty supports, and the reduced cohomology group $H^p(Q, G)$ for any integer $p \geqq 0$. They form in a natural way an exact sequence. Among the various applications, the following decomposition theorem has been proved. If E is a compact connected topological group with a center C which is a Lie group, then E is locally isomorphic with the direct product $C \times Q$ of the center C and the quotient group $Q = E/C$. If E is a compact Lie group, then this is a classical result of E. Cartan. (Received July 23, 1951.)

541*t*. S. T. Hu: *On imbedding a local group in a topological group.*

For any given local group L, there is constructed in this paper a topological group Q_L and a local homomorphism h_L of L into Q_L such that, for every local homomorphism f of L into a topological group G, there exist a homomorphism F of Q_L into G and a neighborhood U of the identity element in L such that $f(x) = Fh_L(x)$ for all $x \in U$. Then it is proved that L is locally isomorphic with a local subgroup of some topological group if and only if there exists a neighborhood V of the identity element in L such that, for any given element $v \in V$, there is a local homomorphism f of L into a topological group G such that $f(xy) = f(x)f(y)$ for all $x, y \in V$ and that $f(v)$ is different from the identity element of G. (Received August 20, 1951.)

542*t*. J. R. Jackson: *A note on function spaces.*

Let X and Y be T-spaces, $\{X_\alpha, Y_\alpha\}$ be a collection of pairs of subsets $X_\alpha \subset X$ and $Y_\alpha \subset Y$, and denote by $Y^X\{X_\alpha, Y_\alpha\}$ the compact-open topologized space of continuous functions on X into Y which carry each X_α into the corresponding Y_α. Suppose $X_0 = \bigcap X_\alpha$ and $Y_0 = \bigcap Y_\alpha$ are nonempty. A topological imbedding of Y_0 in $Y^X\{X_\alpha, Y_\alpha\}$ is obtained by identifying each point $y_0 \in Y$ with the constantly y_0-valued function. If $x_0 \in X_0$, then $\rho: Y^X\{X_\alpha, Y_\alpha\} \rightarrow Y_0$, defined by $\rho(f) = f(x_0)$, is a retraction: *Y_0 may be imbedded as a retract of $Y^X\{X_\alpha, Y_\alpha\}$*. It follows that if property P of topological spaces is preserved by retractions, then a necessary condition that $Y^X\{X_\alpha, Y_\alpha\}$ have property P is that Y_0 have property P. Another consequence is that the homotopy group $\Pi_m(Y^X\{X_\alpha, Y_\alpha\}, y_0)$ is a split extension of some subgroup by $\Pi_m(Y_0, y_0)$. [For definitions and theorems used above, see S. T. Hu, *Homotopy theory*, vol. 1, Department of Mathematics, Tulane University, 1950.] (Received July 2, 1951.)

543t. Shizuo Kakutani: *Continuity of proper functions on minimal dynamical systems.*

Let $x'=\phi(x)$ be a homeomorphism of a compact metric space X onto itself. The pair (X, ϕ) is called a *dynamical system.* (X, ϕ) is called *minimal* if for any $x\in X$ the orbit $O(x)=\{\phi^n(x)\,|\,n=0, \pm 1, \pm 2, \cdots\}$ is dense in X. (X, ϕ) is called *equicontinuous* if the iterations $x'=\phi^n(x)$ of $x'=\phi(x)$ for $n=0, \pm 1, \pm 2, \cdots$ are equicontinuous on X. Given two dynamical systems (X, ϕ), (Y, ψ), a mapping $y=\sigma(x)$ of X into Y is called a *homomorphism* if $\psi(\sigma(x))=\sigma(\phi(x))$ for all $x\in X$. It is proved that, for any Baire homomorphism of a minimal dynamical system into an equicontinuous dynamical system, there exists a continuous homomorphism which differs from it only on a set of first category. From this follows that, for any Baire proper function on a minimal dynamical system, there exists a continuous proper function which differs from it only on a set of first category. This result makes it possible to develop a purely topological ergodic theory of minimal dynamical systems without using the notion of invariant measures or invariant integrals. For example, the notions of minimal dynamical systems with a pure point spectrum and minimal dynamical systems without point spectrum (that is, topologically weakly mixing dynamical systems) can be introduced and their properties can be discussed. (Received July 23, 1951.)

544t. W. S. Massey and G. W. Whitehead: *The Eilenberg-MacLane groups and homotopy groups of spheres.*

It is known that for any fixed value of p, the homotopy groups $\pi_{n+p}(S^n)$ are isomorphic for all sufficiently large values of n (H. Freudenthal, Compositio Math. vol. 5 (1937) pp. 299–314). Identify these isomorphic groups and call the resulting group G_p. If Π and G are any abelian groups, then $A_q(\Pi, G)$ will denote the q-dimensional homology group of the abstract complex, $A(\Pi)$ (see Eilenberg and MacLane, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 657–663) with coefficients in the group G. With this notation, the authors' main result, relating the groups $A_q(\Pi, G)$ and homotopy groups of spheres, may be stated as follows. For any abelian group Π, there exists a Leray-Koszul sequence $(E^n(\Pi), d_n)$, $n=2, 3, 4, \cdots$, of abelian groups with differential operators having the following four properties: (1) $E^n(\Pi)$ has a bigraded structure defined by a doubly-indexed family of subgroups, $E^n_{p,q}(\Pi)$. (2) The differential operator d_n is homogeneous of degree $(-n, n-1)$. (3) The group $E^2_{p,q}(\Pi)$ is naturally isomorphic to $A_p(\Pi, G_q)$. (4) If $E^\infty(\Pi)$ denotes the limit group of the sequence, then $E^\infty_{1,0}(\Pi)=\Pi$, and $E^\infty_{p,q}(\Pi)=0$ for all other pairs (p, q). For the definition of a Leray-Koszul sequence, see H. Cartan, C. R. Acad Sci. Paris vol. 226 (1948) pp. 148 and 303; also, J. Leray, Journal de Mathématiques vol. 29 (1950) pp. 1–139. (Received July 3, 1951.)

545t. W. S. Massey and G. W. Whitehead: *The $(n+3)$-dimensional homotopy group of the n-sphere.*

Let Π be an infinite cyclic group. By making use of the calculation by Eilenberg and MacLane of the groups $A_1(\Pi), \cdots, A_5(\Pi)$ (see Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) pp. 657–663), and of the Leray-Koszul sequence associated with the group Π (see the preceding abstract), it is possible to prove that the homotopy groups $\pi_{n+3}(S^n)$ are all of order 12 or all of order 24 if $n>4$. It readily follows from the Freudenthal suspension theorems (Compositio Math. vol. 5 (1937) pp. 299–314) and the fact that the suspension $\pi_6(S^3)\rightarrow\pi_7(S^4)$ is an isomorphism into (see

Hurewicz and Steenrod, Proc. Nat. Acad. Sci. U.S.A. vol. 27 (1941) pp. 60–64) that if $\pi_8(S^5)$ is a group of order m, then $\pi_6(S^3)$ is a group of order $m/2$; hence $\pi_6(S^3)$ must be of order 6 or 12. However, from the existence of an element in $\pi_6(S^3)$ of nonzero generalized Hopf invariant (see Blakers and Massey, Ann. of Math. vol. 53 (1951) p. 202), and from the fact that the suspension $\pi_6(S^3)\rightarrow\pi_7(S^4)$ is an isomorphism into (an unpublished result discovered independently by G. W. Whitehead and P. Hilton), one concludes that $\pi_6(S^3)$ cannot be a group of order 6. Therefore, $\pi_6(S^3)$ is a group of order 12, and $\pi_{n+3}(S^n)$ is a group of order 24 if $n>4$. The structure of these groups is not determined; however, $\pi_6(S^3)$ is cyclic or not according as the map $(x, y)\rightarrow x^6yx^{-6}y^{-1}$ of $S^3\times S^3$ into S^3 is essential or not. (Received July 3, 1951.)

546t. C. E. Miller: *The topology of rotation groups.*

A boundary formula is obtained for the cell complex of J. H. C. Whitehead (Proc. London Math. Soc. (1944) p. 243) on the Stiefel manifold $V(n, r)$ of r-frames in euclidean n-space. From this the homology and cohomology groups of $V(n, r)$ are computed. The cup products with rational and with modulo 2 coefficients are computed, by using appropriate mappings and spaces to reduce the problem to known or simple cases. The Steenrod squaring homomorphisms are computed similarly, and these are then used to show the non-existence of cross sections to certain of the bundles $V(n, r)\rightarrow S^{n-1}$. This gives a simple proof of the Steenrod-Whitehead Theorem (Proc. Nat. Acad. Sci. U.S.A. (1951) p. 58) asserting that 2^k vector fields do not exist on S^{n-1}, where 2^k is the largest power of 2 dividing n. Similar techniques are used to analyze the factor space $R(2n)/U(n)$, where $R(m)$ $(U(m))$ is the rotation (unitary) group in m real (complex) variables. $R(n)$ is homeomorphic to $V(n, n-1)$ and is a special case of $V(n, r)$. The Pontrjagin product is also computed for $R(n)$ and it is shown that the Pontrjagin and cohomology rings are not isomorphic when modulo 2 coefficients are used. (Received June 18, 1951.)

547. E. E. Moise: *Affine structures in 3-manifolds.* I. *Polyhedral approximations of solids.*

Theorem 1. Let S be a set in Euclidean 3-space which is homeomorphic to the Cartesian product of a compact connected 2-manifold and a closed linear interval. Then there is a polyhedral surface, homeomorphic to the given 2-manifold, which separates the two components of the boundary of S from each other. The conclusion of this theorem can be strengthened by requiring the polyhedral surface to satisfy a rather weak regularity condition. The methods of proof are for the most part elementary. Forthcoming in Ann. of Math. (Received July 18, 1951.)

548t. E. E. Moise: *Affine structures in 3-manifolds.* II. *Positional properties of 2-spheres.*

Theorem 1. Given polyhedral 2-spheres S and S', lying in the interior of a cube P in Euclidean 3-space E^3, there is a piecewise linear homeomorphism f of E^3 onto itself which is the identity outside P and maps S upon S'. This theorem evidently implies a strengthened form of a well known theorem of J. W. Alexander (Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 6–8) to the effect that every polyhedral 2-sphere in E^3 is the boundary of a topological 3-cell. Theorem 1 has already been given, essentially, by W. Graeub (Sitzungsberichte der Heidelberger Akademie der Wissenschaften (1950) pp. 205–272). If $K\subset E^3$ is homeomorphic to a (finite) polyhedron, then K is *semi-locally tamely imbedded* if there is an open neighborhood U of K, and a homeomorphism f throwing U into E^3, such that $f(K)$ is a polyhedron. *Theorem* 2. The complement of a semi-locally tamely imbedded 2-sphere in E^3 is homeomorphic to the complement of the unit sphere $x^2+y^2+z^2=1$. Forthcoming in Ann. of Math. (Received July 18, 1951.)

549. O. M. Nikodým: *Two general criteria for continuity of linear functionals.*

Let L be an abstract real linear space on which an open-set-topology is given, satisfying the following conditions: (1) Any translation of an open set is open; (2) if $E\neq\emptyset$ is open and $I\neq 0$, then $I\cdot E$ is open; (3) if E is an open set containing the zero vector $0^{\rightarrow}$, then there exists an open set E_1 such that $0^{\rightarrow}\in E_1$, $E_1\subseteq E$, and if $x\neq 0^{\rightarrow}$, $x\in E_1$, then the closed segment $\langle 0^{\rightarrow}, x\rangle\subseteq E_1$. No separation axiom is supposed. L is more general than a linear topological space. For linear functionals $f(x)$ on L the following is proved: $f(x)$ is continuous if and only if there exists an open set $A\neq\emptyset$ and a real number μ such that $f(x)$ does not admit, on any point of A, the value μ. If the condition (3) is dropped, the theorem is not true, but in this case $f(x)$ is continuous if and only if there exists an open set $A\neq\emptyset$ on which $f(x)$ is bounded from above. These criteria contain, as particular cases, all criteria known until now. A generalization of the theorem by J. P. LaSalle on the existence of nonvanishing linear continuous functionals is given. (Received July 11, 1951.)

550*t*. M. J. Norris: *Pointwise periodic mappings of simple curves.*

It is shown that every pointwise periodic mapping of the real numbers other than the identity is a strictly decreasing involution. The nature of these involutions is determined in terms of the strictly increasing functions with the positive real numbers as domain and range. Finally the same problem is studied for pointwise periodic mappings of simple curves. (Received July 23, 1951.)

551*t*. M. J. Norris: *Semi-regularity and strict extension.*

The notions of semi-regularity and strict extension as defined by M. H. Stone (*Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. vol. 41 (1937) no. 3) are considered in this paper as pointwise properties of spaces. Stone has shown that a semi-regular space is a strict extension of every dense subset. In this paper some auxiliary conditions on the space are found which, with the condition that the space is a strict extension of every dense subset at the point p, assure semi-regularity at p. The most natural of these conditions is local semi-regularity at p. An example is given to show that a space can be a strict extension of every dense subset at p without being semi-regular at p. The question as to whether or not a space can be a strict extension of every dense subset at every point without being semi-regular is not answered. (Received July 23, 1951.)

552. Everett Pitcher: *Suspension of primary components of homotopy groups of spheres.*

The Freudenthal theorems on suspension may be extended in the following form. Let $w\geqq 3$ denote a prime and $C(G, w)$ the set of elements in the Abelian group G with order a power of w. Suppose $n\geqq 3$. Let E_w: $C(\pi_r(S^n), w)\rightarrow C(\pi_{r+1}(S^{n+1}), w)$ denote the syspension homomorphism with restricted domain. *Theorem* 1: *E_w is an isomorphism onto if $r\leqq r_0=\min(3n-3, 2n+2w-5)$ and a homomorphism onto if $r=r_0+1$. Theorem* 2: *If $w\leqq n-1$ and n is even then $\pi_{2n+2w-4}(S^n)$ contains an element of* *order w which is not a suspension of any element.* Theorems 1 and 2 depend on results of Serre (C. R. Acad. Sci. Paris vol. 232, pp. 142–144, Theorems 3, 4) and on the Freudenthal theorems in the strong form of G. W. Whitehead (Ann. of Math. vol. 51, pp. 192–237, (3.49)) and are proved through use of the model introduced by the writer (Bull. Amer. Math. Soc. Abstract 56-1-38). Theorem 1 improves on the remark of Serre (op. cit. p. 143). Theorem 2 shows that r_0 in Theorem 1 cannot be increased. (Received July 23, 1951.)

553. G. H. M. Thomas: *Simultaneous partitionings of two sets.* II.

The concept of simultaneous partitionings was introduced by the author in a previous paper (Bull. Amer. Math. Soc. Abstract 57-4-365). The following additional result is proved in the present paper. Theorem: Suppose M_1 is a compact partitionable set in a metric space, and M_2 is a closed partitionable subset. Then there exists a sequence $\{G_i\}$ of simultaneous partitionings of M_1 and M_2 such that all the elements of G_i have diameters less than $1/i$; G_{i+1} is a refinement of G_i, and if U_{ij}, $j=1, 2, 3, \cdots, m$, is any collection of elements of G_i then the interior of $\sum_{j=1}^{m} \overline{U_{ij}}$ and the interior of $\sum_{j=1}^{m} \overline{U_{ij}M_2}$ are each uniformly locally connected. Since a set in a metric space is partitionable if and only if it has property S, and every continuous curve has property S, the theorem holds in case M_1 and M_2 are continuous curves. (Received July 20, 1951.)

554t. W. R. Utz: *A note on unrestricted regular transformations.*

The action of unrestricted regular transformations (cf. W. A. Wilson, *On certain types of continuous transformations of metric spaces*, Amer. J. Math. vol. 57 (1935) pp. 62–68) on metric spaces is considered. A typical theorem is that if M is a bounded metric space and $T(M)=M$ is unrestricted regular and has equicontinuous powers, then T is an isometry. (Received July 2, 1951.)

J. W. T. YOUNGS,
Associate Secretary

BOOK REVIEWS

Contributions to the theory of games. Ed. by H. W. Kuhn and A. W. Tucker. (Annals of Mathematics Studies, no. 24.) Princeton University Press, 1950. 16+201 pp.

The modern theory of games had its inception in von Neumann's paper in the Mathematische Annalen for 1928, and was developed and popularized in the book by von Neumann and Morgenstern. Its great popularity is due to its mathematical interest and to two other reasons. One of these is the connection between the theory of games and the theory of statistical decision functions initiated independently by Wald in 1939. The other is the unhappy state of the world in these last years, and the abundance of wars, hot and cold. To the mathematician confronted with a problem in strategy it is very appealing to have recourse to a theory and an established procedure like seeking a minimax solution. Much of what is now called (military) operations analysis seems to be simply an application of the theory of games.

The present study comprises a number of diverse papers on the theory of games, with a comprehensive introduction by the editors. In this introduction the editors give a brief description of each paper and conclude with a list of questions for which, in their opinion, it is very important to have answers forthcoming. Both the summary and list of questions are very clearly and adequately put, and can serve the busy reader as a guide to further reading in this volume and elsewhere. The excellent introduction and the general theme furnish the bonds of unity for the papers in the volume, many of which were originally submitted to the Annals of Mathematics and transferred here by common consent. Since the individual papers will be reviewed in Mathematical Reviews, it seems best to this reviewer to content himself here with a rather general description of the contents of the volume.

The first paper, by Weyl, is a reprint of his 1935 paper on the equivalence of the two definitions of the convex closure of a finite set of points in Euclidean space. The second paper, also by Weyl, gives a proof of the Main Theorem (of von Neumann) when all elements of the payoff matrix lie in an ordered field. (The Main Theorem is the one which asserts that a finite game is determined when mixed strategies are admitted.)

Continuing Part I, on finite games, Shapley and Snow give a characterization of the optimal (mixed) strategies in terms of submatrices of the payoff matrix. In principle this characterization gives an algorithm for computing the optimal solutions, but in practice the computational effort required will be too great. Brown and von Neumann give a finite system of first order differential equations in one independent variable t, with the property that any limit point, as $t \rightarrow \infty$, of a solution, is an optimal strategy. At least one limit point exists and it is said that the procedure of solving the differential equations can be mechanized with relative ease.

The procedure of Brown and von Neumann makes use of the fact that the solution of any finite game can be made to depend upon the solution of a symmetric game, that is, one with a skew-symmetric payoff matrix. In another paper (no. 7) Gale, Kuhn, and Tucker give a superior method of "symmetrizing" a game, and a characterization of the optimal solutions of a symmetric game.

Bohnenblust, Karlin, and Shapley in one paper, and Gale and Sherman in another, give geometric characterizations of the totality of solutions of a zero-sum two-person game. There is some overlap between the two papers.

Kuhn (paper no. 9) and Nash and Shapley (paper no. 10) analyze simplified versions of poker.

In paper no. 8 Gale, Kuhn, and Tucker study what happens to games under certain classes of transformations of their matrices. Such transformations are of great importance in actual computational problems.

J. C. C. McKinsey discusses the question of strategic equivalence of n-person ($n>2$) games and proves some results on this subject. Roughly speaking, two games are strategically equivalent if the bargaining powers of the players and their tendencies to form coalitions are the same.

A paper (no. 12) by Karlin initiates the second part of four papers on infinite games. The principal result seems to be a slightly stronger version of Theorem 2.23 of Wald's book *Statistical decision functions*. (The two authors did their work independently and approximately at the same time.) The reviewer is informed that the second part of the paper contains the result that if one admits strategies which are only finitely additive then every zero-sum two-person game is determined. The reviewer cannot vouch for this, however, as he gave up the task of reading this paper before the end. This paper, which seems to contain interesting results, is written in a singularly inept and chaotic manner.

Paper no. 13 by Bohnenblust and Karlin generalizes to Banach spaces theorems of Ville and Kakutani which were useful tools in proving the Main Theorem for finite games.

The volume concludes with two papers: Paper no. 14, by Dresher, Karlin, and Shapley, is concerned principally with zero-sum two-person games with the kernel $K(x, y) = \sum_{i,j} a_{ij}r_i(x)s_j(y)$, where the functions $r_i(x)$ and $s_j(y)$ are continuous, and x and y, the pure strategies of the two players, are points on the unit interval. Paper no. 15 by Bohnenblust, Karlin, and Shapley, discusses zero-sum two-person games with payoff function $M(x, y)$, which is such that, for every x in an arbitrary set A, $M(x, y)$ is a continuous convex function of y. The y-player is shown to have an optimal pure strategy (y is a point in a compact, convex n-dimensional region of a finite Euclidean space). The main result of the paper is that the x-player has an optimal mixed strategy which assigns probability one to a set which contains at most $(n+1)$ points x.

This volume has the merit of bringing together between two covers a large number of interesting papers on game theory which would otherwise be inconveniently scattered over many periodical issues. Such enterprises could profitably become a feature also in other branches of mathematics. The editors deserve much credit for a painstaking job, and for their lucid and stimulating introduction.

J. WOLFOWITZ

Hydrodynamics, a study in logic, fact, and similitude. By Garrett Birkhoff. Princeton University Press, 1950. 14+186 pp. $3.50.

The material of this book formed the contents of a series of lectures at the University of Cincinnati in 1947, and this perhaps accounts for the unusual and unconventional choice of material for a book with the title *Hydrodynamics.* There are five chapters in the book, most of which have only a loose connection with the others.

The reviewer found it difficult to understand for what class of readers the first chapter was written. For readers who are acquainted with hydrodynamics the majority of the cases cited as paradoxes belong either in the category of mistakes long since rectified, or in the category of discrepancies between theory and experiment the reasons for which are also well understood. On the other hand, the uninitiated would be very likely to get wrong ideas about some of the important and useful achievements in hydrodynamics from reading this chapter. In the case of air foil theory, for example, the author treats only the negative aspects of the theory. It has always seemed to the reviewer that the Kutta-Joukowsky theory of airfoils is one of the most beautiful and striking accomplishments in applied mathematics. The fact that the introduction of a sharp trailing edge makes possible a physical argument, based on consideration of the effect of viscosity, that leads to a purely mathematical assumption regarding the behavior of an analytic function which in its turn makes the solution to the flow problem unique and also at the same time furnishes a value for the lift force, represents a real triumph of mathematical ingenuity. The fact that the theory does not solve all of the problems concerned with airfoils should not be surprising; no physical theory is every final nor without limits within which it is valid. These aspects of the Kutta-Joukowsky theory are not mentioned by the author.

At the end of the first chapter some general observations regarding the philosophy and correct attitude toward applied mathematics are made and some questions are raised. Most workers in the field would agree quite well with the author's observations, though they are perhaps better informed in some cases than the author would seem to imply. For example, they know quite well why it is that a small change in viscosity may have radical consequences while a small change in compressibility has little effect. The reason for this is clear from the mathematical formulation: the small coefficients involving viscosity occur in terms containing derivatives of the highest order in the system of differential equations, and thus developments in the neighborhood of zero viscosity involve boundary layer effects because of the loss of order of the differential equations in the limit. The terms involving compressibility occur, on the other hand, in a quite innocuous fashion.

The second chapter of the book is concerned with problems involving liquids with free boundaries and a study of jets, wakes, cavities behind solids moving through liquids, the theoretical explanation of the action of shaped charges (in which subject the author himself has made a basic contribution), and other like topics. The mathematical theory fundamental for these problems is in the main the classical Kirchhoff-Helmholtz theory, and it is given a brief but readable presentation. The recent literature on this subject is referred to very extensively in this chapter, including much material that is not well known. This chapter of the book should be very useful.

The third chapter is concerned with modeling and dimensional analysis. The reviewer found it interesting reading. The discussion of modeling, which is carried out on the basis of concrete examples, was particularly well done.

The fourth and fifth chapters, entitled *Group theory and fluid mechanics*, and *Virtual mass and groups* respectively, are, as the titles indicate, concerned with various roles which the concept of a group can play in fluid mechanics. For example, in chapter four the connection of transformation groups expressing various symmetries with the integration theory of partial differential equations is discussed.

J. J. STOKER

Introduction à la géométrie non euclidienne par la méthode élémentaire. By G. Verriest. Paris, Gauthier-Villars, 1951. 8+193 pp. 1000 fr.

This exposition begins with the foundations of Euclidean geometry: Hilbert's axioms of incidence, order, congruence, and parallelism. Setting aside the axiom of parallelism, the author develops a number of theorems belonging to both Euclidean and hyperbolic geometry. He takes care to avoid any appeal to intuition; for example, he proves in detail that the sum of two supplementary angles is equal to the sum of two right angles. This part of the work, making no assumption of continuity, culminates in the "second theorem of Legendre": If, in at least one triangle, the angle-sum is equal to two right angles, the same holds for every triangle. In Chapter IV, Dedekind's axiom of continuity is used to prove the axiom of Archimedes and the "first theorem of Legendre": The angle-sum of a triangle cannot exceed two right angles. The author mentions that this was anticipated by Saccheri, and that it would not hold without continuity [Dehn, Math. Ann. vol. 53 (1900) pp. 436–439]. Chapter V establishes the equivalence, in the presence of the other axioms, of three possible formulations of the Euclidean axiom: Euclid's own Postulate V, the unique parallel through a given point to a given line, and the angle-sum of a triangle being equal to two right angles. It follows that all three are contradicted in hyperbolic geometry.

The next three chapters are on Hilbert's treatment of area and its connection with angle-sum. Chapter IX deals with simple properties of parallels and ultraparallels in the hyperbolic plane. On p. 142 the author omits, as being "assez compliquée," the proof of the existence of a common parallel line to two given rays [Hilbert, *Grundlagen der Geometrie*, Leipzig, 1913, p. 151; Coxeter, *Non-Euclidean geometry*, Toronto, 1947, p. 205]. He excuses the omission by declaring that this result will not be required later; but he actually uses it on p. 150 and again on p. 156, each time assuring the reader that it will not be needed any more! There is a very readable account of circles, horocycles, and hypercycles, with a short paragraph on the extension to three dimensions.

Elliptic geometry is relegated to a single chapter at the end. The statement that every two coplanar lines intersect is shown to imply that each line of the plane has one or two absolute poles. The two alternative hypotheses are carefully examined, with appropriate modifications of the axioms of incidence and order. There is a neat treatment of Clifford parallels, and a proof that nothing like them can occur in hyperbolic geometry.

The book is well written, simple yet rigorous. On the other hand, it is slightly old-fashioned, long-winded in some places, and a little too much dominated by Hilbert. There is no history or bibliography; the names of Gauss and Bolyai, Cayley and Klein are seldom (if ever) mentioned.

H. S. M. COXETER

Tensor calculus. By J. L. Synge and A. Schild. (Mathematical Expositions, no 5.) University of Toronto Press, 1949. 12+324 pp. $6.00.

This book is an outgrowth of a series of lectures delivered by Professor Synge at the University of Toronto, Ohio State University, and Carnegie Institute of Technology. It is a general brief introduction to the calculus of tensors and its applications without the usual historical development of the subject. A short bibliography of some of the leading texts on the subject is given on page 319 and an index on pages 321–324.

There are eight chapters. The first four deal with the usual concepts of tensors, Riemannian spaces, Riemannian curvature, and spaces of constant curvature. The next three chapters are concerned with applications to classical dynamics, hydrodynamics, elasticity, electromagnetic radiation, and the theorems of Stokes and Green. In the final chapter, an introduction is given to non-Riemannian spaces including such subjects as affine, Weyl, and projective spaces. There are two appendices in which are discussed the reduction of a quadratic form and multiple integration. At the conclusion of each chapter, a summary of the more important formulas is given and also a set of exercises is included to illustrate the material of the chapter.

In the first two chapters the authors discuss the concepts of absolute tensors and Riemannian geometry. The equations of the geodesics are derived by the methods of the calculus of variations. After showing that the expressions for the covariant derivative of a vector form a tensor, the authors define parallel displacement of a vector along a curve by the vanishing of the absolute derivative. The elegant geometrical treatment of Levi-Civita of parallel displacement is not mentioned. Among other subjects treated are the Serret-Frenet formulas and the curvatures of a curve in a general Riemannian space. In the third chapter, the Riemannian curvature tensor is introduced by means of the commutation rule for the covariant second derivative. Parallel displacement around a closed circuit is discussed, and various formulas for the Riemannian curvature are obtained. In particular, it is proved that parallel displacement is independent of the path if and only if the space is flat. In the fourth chapter, the important theorem of Schur is established, namely, if the Riemannian curvature is locally constant, then it is constant throughout the whole space. The remaining subject matter is devoted to spaces of constant curvature and flat spaces. In a space of positive constant curvature, the geodesics through a point P meet again in a point P'. The space is antipodal (spherical) or polar (elliptic) according as $P' \neq P$ or $P' = P$. Flat spaces are studied by means of homogeneous coordinates which are cartesian coordinates for flat spaces with positive definite quadratic form. The theory of cartesian tensors is developed. In this connection, the special theory of relativity is briefly discussed.

In the fifth chapter, the dynamics of a particle are studied. The Lagrangian equations and, for a conservative field of force, the potential and action integral including Hamilton's principle of least action, are developed. Elsewhere these concepts have been extended so as to include not only dynamical trajectories, but also brachistochrones, catenaries, velocity systems, and other families of curves in a given field of force. Also studied are the equations of motion of a rigid body. In the sixth chapter, the Lagrangian method and the Euler method of describing the motion of a fluid are developed. After establishing the equation of continuity, the authors give the general equations of motion of a perfect fluid in general curvilinear coordinates. For an irrotational fluid, the integral of Bernoulli is found. In elasiticity, the stress and strain tensors are developed, and the equations of motion of a continuous medium are given. In the theory of electromagnetic radiation, Maxwell's equations are written in general curvilinear coordinates. Special types of solutions are given and various interesting geometrical properties of these solutions are obtained. The sixth chapter concludes with a brief discussion of the general theory of relativity.

In the seventh chapter, the authors discuss relative tensors and the nonmetrical concept of extension, originally due to Grassmann. A generalized Stokes theorem is established without the use of a metric. Finally a general form of Green's theorem is obtained. In the eighth and final chapter, an introduction to non-Riemannian geometries is given. After a discussion of a general affine space and the affine curvature tensor, the authors study the gauge transformations in a Weyl space and the gauge invariants. The final subject consists of projective geometry and the Weyl projective curvature tensor.

This tensor vanishes identically for a projective space of two dimensions. For spaces of three or more dimensions, the Weyl projective curvature tensor vanishes if and only if the given affine space is projectively equivalent to a flat space.

Although the book seems to lack geometrical motivation, the material is presented in a very understandable fashion. The ideas and concepts are given very concisely and thus a wide range of subjects is covered. This book is a very worthwhile introduction to the subject for a beginning student of the absolute differential calculus and its applications.

JOHN DECICCO

BRIEF MENTION

The hodograph method in gas dynamics. By A. G. Ghaffari. Teheran, Taban, 1950. 129 pp., 19 figs.

This book gives an introductory account of one of the most interesting methods in gas dynamics, the hodograph method. Particular emphasis has been given to approximate methods (Chapters IV, V, VII), including some of the author's own work. Chaplygin's method and its current developments are discussed briefly in Chapter III, while Chapter VI gives a rather detailed account of Bergman's method. A clear presentation of the standard material in gas dynamics is given in Chapters I, II, III, and VIII. The book contains a preface by Professor G. Temple which concludes with the statement: "This book can be confidently recommended to students of aerodynamics who desire an elementary survey of the theory and use of the Hodograph method."

C. C. LIN

Gelöste und ungelöste mathematische Probléme aus alter und neuer Zeit. By Heinrich Tietze. Munich, Biederstein, 1949; vol. 1, 20+256 pp.; vol. 2, 4+305 pp.

This is one of the deepest and at the same time the most charming of the popular books on mathematics that I have ever seen. In two moderate sized volumes Tietze presents fourteen lectures, on topics as diverse as digital representations of integers, distribution of primes, properties of geodesics on surfaces, dimension theory, the regular 17-gon, solution of equations by radicals, and the concept of infinity. The lectures are addressed to a lay audience. A mathematically untrained reader of the book (assuming that he has no difficulty with Tietze's picturesque German) can get a lot of understanding and enjoyment from the exposition. This is true even if the reader's training is so slight as not to include any trigonometry. At the other extreme, it is very likely that the reader who is a professional mathematician will encounter facts and tricks new to him. I do not mean to say that Tietze's book contains material that is not already available in the literature, but the chances are that only a few mathematicians are acquainted with *all* the interesting and special information that the book contains. Even the trite and altogether too famous problems (such as angle trisection, map coloring, circle squaring, and Fermat's last theorem) receive fresh treatment at Tietze's hands. In connection with the coloring problem, for instance, Tietze discusses the problem for maps in which a country need not be a connected set but is allowed to have several components. Here, as elsewhere, the book is documented in a careful, scholarly manner. About a fourth of each volume (the last fourth) is devoted to notes in which one can find sometimes the mathematical derivation of facts merely stated in the text, at other times indications of generalizations, and, at all times, precise references to the sources of the material. There are several photographs of famous mathematicians, and there is a detailed and usable index.

If the book were in English, it would be ideal for the bright undergraduate and for the interested but uninformed layman; translating it might well turn out to be a profitable venture for someone. As it is, it is at least an ideal source of inspiration for the mathematician who is asked to deliver a popular lecture; I think every mathematical library ought to contain a well-thumbed copy.

PAUL R. HALMOS

NOTES

The School of Mathematics of the Institute for Advanced Study will allocate a small number of grants-in-aid to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1952–1953. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, and are returnable by January 1, 1952.

The Army Ordnance Corps has established at Duke University an Office of Ordnance Research which will support a program of basic research in the scientific fields in which the Ordnance Corps has a potential interest. The field of mathematics is included. Inquiries and proposals from universities, colleges, and other research institutions, for research to be done under contract with the Office, should be addressed to the Office of Ordnance Research, 2127 Myrtle Drive, Duke University, Durham, North Carolina. Projects sponsored by the Office may be made a part of the research program of the institution and used in the training of research students.

Associate Professor E. S. Kennedy of the American University of Beirut has been appointed to a lectureship at the University of Tehran.

Dr. H. P. Mulholland has been appointed to a lectureship at the University of Birmingham, England.

Mr. C. M. Ablow has accepted a position as research engineer at the Boeing Airplane Company, Seattle, Washington.

Professor Nachman Aronszajn of Oklahoma Agricultural and Mechanical College has been appointed to a visiting professorship at the University of Kansas.

Mr. E. W. Banhagel of Northwestern University has accepted a position as mathematician at Bendix Aviation Corporation, Detroit, Michigan.

Mr. J. H. Barrett of the University of Texas has been appointed to an assistant professorship at the University of Delaware.

Assistant Professor W. R. Baum of the University of Tennessee has been appointed to an assistant professorship at Syracuse University.

Assistant Professor W. D. Berg of Kenyon College has accepted a position at the Mutual Life Insurance Company of New York.

Professor L. M. Blumenthal of the University of Missouri is on leave of absence and has accepted a position as mathematician at the Institute for Numerical Analysis, Los Angeles, California.

Mr. W. G. Brady has been appointed to an assistant professorship at Washington and Jefferson College.

Mr. H. H. Brown has accepted a position as research mathematician at the Armour Research Foundation, Illinois Institute of Technology.

Mr. H. D. Brunk of Rice Institute has accepted a position as mathematician at the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Eugenio Calabi of the Louisiana State University is on leave of absence and has accepted a position with the Office of Naval Research at Princeton University.

Dr. E. D. Cashwell of Ohio State University has accepted a position as staff member at the Sandia Corporation, Albuquerque, New Mexico.

Dr. Elsie T. Church of the University of Kentucky has been appointed to an associate professorship at Northwestern State College, Natchitoches, Louisiana.

Assistant Professor B. H. Colvin of the University of Wisconsin has accepted a position with the Boeing Airplane Company, Seattle Washington.

Professor A. H. Diamond of Oklahoma Agricultural and Mechanical College has been appointed to a visiting professorship at the University of Kansas.

Mr. G. P. Dinneen has accepted a position as senior engineer at the Goodyear Aircraft Corporation, Akron, Ohio.

Mr. J. L. Dolby has accepted a position as development engineer at General Electric Company, Schenectady, New York.

Mr. W. F. Donoghue has accepted a position with the Applied Physics Laboratory of the Johns Hopkins University.

Assistant Professor William H. Durfee of Dartmouth College has accepted a position with the Applied Mathematics Laboratories, National Bureau of Standards, Washington, D. C.

Mr. F. J. Dyson of the Institute for Advanced Study has been appointed to a professorship in physics at Cornell University.

Mr. R. E. Ekstrom of Westminster College has accepted a position as mathematician at the U. S. Naval Ordnance Plant, Indianapolis, Indiana.

Professor G. M. Ewing of the University of Missouri is on leave of absence and has accepted a position at the Sandia Corporation, Albuquerque, New Mexico.

Dr. Chester Feldman has been appointed to an assistant professorship at Antioch College.

Dr. D. T. Finkbeiner of Yale University has been appointed to an associate professorship at Kenyon College.

Dr. L. E. Fuller has accepted a position as senior development engineer at Goodyear Aircraft Corporation, Akron, Ohio.

Mr. W. R. Fuller has accepted a position as mathematician at the U. S. Naval Ordnance Plant, Indianapolis, Indiana.

Mr. J. W. Gaddum has accepted a position as mathematician with the Institute for Numerical Analysis, Los Angeles, California.

Mr. Leonard Geller has accepted a position as engineer at the Union Carbide and Carbon Corporation, Oak Ridge, Tennessee.

Dr. Abolghassem Ghaffari has been appointed a member of the Institute for Advanced Study.

Assistant Professor H. E. Goheen of Syracuse University has been appointed to an assistant professorship at the Moore School of Electrical Engineering, University of Pennsylvania.

Dr. Oscar Goldman of Harvard University has been appointed to an assistant professorship at Brandeis University.

Professor R. D. Gordon of Washington College has been appointed to an associate professorship at Hampton Institute.

Assistant Professor R. E. Graves of the University of Minnesota has accepted a position as mathematician at the White Sands Proving Ground, Las Cruces, New Mexico.

Associate Professor J. W. Green of the University of California, Los Angeles, is on leave of absence at the Institute for Advanced Study.

Mr. J. C. Gysbers of Oregon State College has accepted a position as research engineer at North American Aviation, Inc., Downey California.

Mr. F. R. Halpern of Bell Aircraft Company has accepted a position as assistant structural engineer at the Armour Research Foundation, Illinois Institute of Technology.

Mr. J. J. Hart of the Tennessee Polytechnic Institute has accepted a position as mathematician at the Redstone Arsenal, Huntsville, Alabama.

Mr. H. D. Hartstein has accepted a position as purchasing agent for Smith-Scharff Paper Company, St. Louis, Missouri.

Mr. Nickolas Heerema has been appointed to an assistant professorship at Florida State University.

Mr. Melvin Henriksen has been appointed to an assistant professorship at the University of Alabama.

Mr. E. H. Jacobs has accepted a position as mathematician at the Rand Corporation, Santa Monica, California.

Assistant Professor S. J. Jasper of Kent State University has been appointed to an assistant professorship at East Tennessee State College.

Mr. R. W. Jollensten has accepted a position as analyst with the Department of Defense, Washington, D. C.

Professor Joseph Kampé de Fériet of the University of Lille is on leave of absence and has been appointed to a visiting professorship at the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

Assistant Professor W. J. Klimczak of the University of Rochester has been appointed to an assistant professorship at Trinity College.

Assistant Professor C. E. Langenhop of Iowa State College of Agriculture and Mechanic Arts is on leave of absence and has accepted a position as acting director of the Analytic Research Group, Princeton University.

Miss Hilda M. Lucy has been appointed to an assistant professorship at Lincoln University.

Dr. Harold Luxenberg has accepted a position at Hughes Aircraft Company, Culver City, California.

Assistant Professor F. A. McMahon of Manhattan College has accepted a position at Sperry Gyroscope Company, Lake Success, New York.

Professor J. D. Mancill of the University of Alabama has accepted a position as consultant at the Redstone Arsenal, Huntsville, Alabama.

Assistant Professor Murray Mannos of the Johns Hopkins University has accepted a position as mathematician with the National Bureau of Standards, Washington, D. C.

Dr. F. I. Mautner of Pennsylvania State College has been appointed to an assistant professorship at the Johns Hopkins University.

Professor W. L. Miser of Vanderbilt University has retired and has been appointed a teacher at McKendree College.

Dr. E. E. Moise has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor E. F. Moore of the University of Illinois has accepted a position as staff member with Bell Telephone Laboratories, Inc., Morristown, New Jersey.

Associate Professor W. R. Murray of Wagner College has accepted a position as mathematician with the Armed Forces Security Agency, Department of Defense, Washington, D. C.

Mr. W. J. Nemerever of Oklahoma Agricultural and Mechanical

College has accepted a position as engineer at Chance Vought Aircraft Corporation, Dallas, Texas.

Mr. L. W. Neustadt has accepted a position as mathematician at Bell Aircraft Corporation, Buffalo, New York.

Associate Professor E. N. Oberg of the State University of Iowa is on leave of absence and has accepted a position as research engineer at North American Aviation, Inc., Downey, California.

Mr. H. W. Oliver has been appointed to an assistant professorship at Williams College.

Dr. Athanasios Papoulis of the University of Pennsylvania has been appointed to a professorship in electrical engineering at Union College.

Mr. F. D. Parker of the Case Institute of Technology has been appointed to an assistant professorship at St. Lawrence University.

Mr. G. M. Petersen has been appointed to a lectureship at Brandon College.

Assistant Professor D. H. Potts of Northwestern University has accepted a position as mathematician at the U. S. Navy Electronics Laboratory, San Diego, California.

Assistant Professor O. W. Rechard of the Ohio State University has been appointed a staff member at the Los Alamos Scientific Laboratory.

Professor P. R. Rider of Washington University is on leave of absence and has accepted a position as advisor to the Chief of the Office of Air Research, Wright-Patterson Air Force Base, Dayton, Ohio.

Assistant Professor C. L. Riggs of Kent State University has been appointed to an assistant professorship at East Texas State Teachers College.

Mr. B. V. Ritchie has accepted a position as mathematician at the U. S. Naval Ordnance Plant, Indianapolis, Indiana.

Mr. J. W. Rolley has accepted a position as research analyst at North American Aviation, Inc., Downey, California.

Associate Professor P. C. Rosenbloom of Syracuse University has been appointed to an associate professorship at the University of Minnesota.

Mr. H. L. Royden has been appointed to an assistant professorship at Stanford University.

Mr. Theodore Rubin has accepted a position at the Wright-Patterson Air Force Flight Research Laboratory, Dayton, Ohio.

Mr. I. R. Savage has accepted a position as mathematician at the

Statistical Engineering Laboratory of the National Bureau of Standards, Washington, D. C.

Mr. Leonard Schieber has accepted a position as design engineer at the International Business Machines Corporation, Johnson City, New York.

Dr. Raymond Sedney has accepted a position as aerodynamicist at the Douglas Aircraft Company, Santa Monica, California.

Mr. R. F. Shaw has accepted a position as senior engineer at the Electronic Computer Corporation, Brooklyn, New York.

Assistant Professor S. S. Shrikhande of the College of Science, Nagpur, India, has been appointed to a visiting assistant professorship at the University of Kansas.

Mr. J. A. Silva has been appointed research associate at the Johns Hopkins Institute for Cooperative Research, Baltimore, Maryland.

Dr. M. L. Slater has accepted a position as mathematician at the Cornell Aeronautical Laboratories, Inc., Buffalo, New York.

Mr. W. K. Smith has been appointed to an assistant professorship at Bucknell University.

Associate Professor Andrew Sobczyk of Boston University is on leave of absence and has accepted a position as staff member at the Los Alamos Scientific Laboratory.

Associate Professor T. H. Southard of Wayne University is on leave of absence and has accepted a position as mathematician at the Institute for Numerical Analysis, Los Angeles, California.

Dr. R. L. Sternberg of Northwestern University has accepted a position as mathematician at the Laboratory for Electronics, Boston, Massachusetts.

Professor W. T. Stratton of Kansas State College has retired with the title emeritus.

Dr. D. L. Thomsen has accepted a position as research engineer at the Jet Propulsion Laboratory of the California Institute of Technology.

Dr. F. H. Tingey has accepted a position as mathematical statistician at the General Electric Company, Richland, Washington.

Mr. Eugene Usdin has accepted a position as engineer at the Stanolind Oil and Gas Company, Tulsa, Oklahoma.

Dr. D. H. Wagner has accepted a position as scientific analyst at the Operations Evaluation Group, Massachusetts Institute of Technology, Washington, D. C.

Dr. Maria A. Weber has been appointed to a lectureship at the University of California, Los Angeles.

Mr. A. M. Wedel has been appointed to an assistant professorship at Bethel College.

Mr. Harry Weingarten of New York University has accepted a position as statistician with the Quality Control Division of the Bureau of Ordnance, Navy Department, Washington, D. C.

Dr. Anne M. Whitney has been appointed to an assistant professorship at Goucher College.

Dr. G. M. Wing of the University of California is on leave of absence and has accepted a position as staff member at the Los Alamos Scientific Laboratory.

Mr. G. N. Wollan has been appointed to an assistant professorship at Memphis State College.

The following promotions have been announced:

Joseph Andrushkiw, Seton Hall University, to an associate professorship.

Frances E. Baker, Vassar College, to a professorship.

Alice K. Bell, Fresno State College, to a professorship.

Dorothy L. Bernstein, University of Rochester, to an associate professorship.

Frederick Mosteller, Harvard University, to a professorship in mathematical statistics.

S. T. Parker, Kansas State College of Agriculture and Applied Science, to a professorship.

A. J. Penico, Tufts College, to an assistant professorship.

F. C. Smith, College of St. Thomas, to a professorship.

D. B. Sumner, Hamilton College, McMaster University, to an associate professorship.

Alexander Tartler, Lafayette College, to a professorship.

Bradford Tye, Bethany College, to an associate professorship.

C. T. West, Ohio State University, to an associate professorship.

A. L. Whiteman, University of Southern California, to an associate professorship.

P. M. Young, Kansas State College of Agriculture and Applied Science, to a professorship.

H. J. Zimmerburg, Rutgers University, to an associate professorship.

The following appointments to instructorships are announced: Brandeis University: Dr. Kathleen B. Whitehead; University of California: Mr. W. G. Bade, Dr. Harley Flanders; University of California, Los Angeles: Mr. Barrett O'Neill: University of Connecticut: Mr. G. B. Robison, Mr. Henry Zatzkis; Massachusetts Institute of Technology: Mr. A. H. Copeland, Jr., Mr. J. M. Stark;

University of Kansas: Mr. W. K. Moore; Michigan State College: Mr. J. B. Kelly; University of Michigan: Mr. A. B. Clarke; Northwestern University: Dr. J. S. MacNerney; Ohio State University: Mr. A. D. Ziebur; Princeton University: Mr. G. L. Thompson; University of Washington: Dr. M. G. Arsove; Western Reserve University: Mr. R. L. Shively; University of Wisconsin: Mr. C. W. Curtis; Xavier University: Dr. R. W. Allen.

Professor F. H. Hodge of Purdue University died on August 21, 1951 at the age of eight-one years. He had been a member of the Society for thirty-eight years.

Professor Emeritus Richard Morris of Rutgers University died on September 19, 1951 at the age of eighty-three years. He had been a member of the Society for forty-five years.

Dr. J. I. Vass of the University of Wisconsin died on March 31, 1951 at the age of fifty years. He had been a member of the Society for twenty-nine years.

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p. 921, equations (10), (11). For "$\sum_{m=1}^{\infty}$" read "$\sum_{m=0}^{\infty}$."
p. 921, equation (12). For "$\sum_{m=1}^{2N+1}$" read "$\sum_{m=0}^{2N+1}$."

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New Publications. p. 408, line 7. For "Trans. by F. Prihonský" read "Ed. by F. Prihonský. Trans. by D. A. Steele."

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